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Properties of additive functionals of Brownian motion with resetting

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We study the distribution of time-additive functionals of reset Brownian motion, a variation of normal Brownian motion in which the path is interrupted at a given rate according to a Poisson process and placed back to a given reset position. For three examples of functionals (occupation time, area, absolute area), we investigate the effect of the resetting by computing various moments and distributions, using a recent result that links the generating function with resetting to the generating function without reset. We also derive a general variational formula for the large deviation rate function, which we use to analyze two different large deviation regimes appearing in the presence of resetting.

Keywords: Brownian motion, resetting, catastrophes, large deviations

I. INTRODUCTION

In this paper we study a variation of Brownian motion (BM) that includes resetting events at random times. Let $(W_t)_{t\geq 0}$ be a BM on $\mathbb{R}$ and consider a Poisson process on $[0,\infty)$ with intensity $r \in (0,\infty)$ and law $\mathbb{P}$, producing $N(T)$ random points $\{\tau_i\}_{i=1}^{N(T)}$ in the time interval $[0,T]$ in such a way that $\mathbb{E}[N(T)] = rT$. From these two processes, we construct the *reset Brownian motion* (rBM), $(W_t^r)_{t\geq 0}$, by glueing together $N(T)$ independent trajectories of the BM, all starting from a reset position $x_* \in \mathbb{R}$ and evolving freely over the successive time lapses

$$\{\tau_{i+1} - \tau_i\}_{i=0}^{N(T)-1}, \tag{I.1}$$

with $\tau_0 = 0$. Without loss of generality, we assume that $x_* = 0$ and $W_0^r = 0$. We also denote by $\mathbb{P}_r$ the probability with respect to rBM with reset rate r.

The properties of rBM, and reset processes in general [25], have been the subject of several recent studies, related to random searches and randomized algorithms [6, 8, 15, 20] (which can be made more efficient by the addition of resetting [7]), queueing theory (where resetting accounts for the accidental clearing of queues or buffers), as well as birth-death processes [3, 4, 16, 22, 26, 27] (in which a population is drastically reduced as a result of natural disasters or catastrophes). In biology, the attachment, targeting, and transcription dynamics of enzymes, proteins and other bio-molecules can also be modelled with reset processes [2, 23, 28–30, 35].

Formally, resetting acts as a "confinement" around the reset position, which can bring the process from being non-stationary to being stationary. The simplest example is rBM, which has a stationary density ρ given by [6]

$$\rho(x) = \sqrt{\frac{r}{2}}\, e^{-\sqrt{2r}|x|}, \quad x \in \mathbb{R}. \tag{I.2}$$

The motivation for the present paper is to study the effect of the confinement on the distribution of additive functionals of rBM of the general form

$$F_T = \int_0^T f(W_t^r)\, \mathrm{d}t, \tag{I.3}$$

where f is a given $\mathbb{R}$-valued measurable function. We are especially interested to study the effect of resetting on the large deviation properties of these functionals, and to determine whether resetting is "strong enough" to bring about a large deviation principle (LDP) for the sequence of random variables $(T^{-1}F_T)_{T>0}$ whenever it does not have the LDP without resetting.

For this purpose, we use a recent result [23, 24] based on the renewal structure of reset processes that links the Laplace transform of the Feynman-Kac generating function of F_T with resetting to the same generating function without resetting. Additionally, we derive a variational formula for the large deviation rate function of $(T^{-1}F_T)_{T>0}$, obtained by combining the LDPs for the frequency of resets, the duration of the reset periods, and the value of F_T in between resets. This variational formula complements the result based on generating functions by providing insight into how a large deviation event is created in terms of the constituent processes. These two results are stated in Secs. II–III and apply in principle to any functional F_T of the type defined above. We illustrate them for three particular functionals:

$$A_T = \int_0^T 1_{[0,\infty)}(W_t^r)\,\mathrm{d}t, \qquad B_T = \int_0^T W_t^r\,\mathrm{d}t, \qquad C_T = \int_0^T |W_t^r|\,\mathrm{d}t, \tag{I.4}$$

i.e., the occupation time of the positive half-line, the area, and the absolute area (the latter can also be interpreted as the area of rBM reflected at the origin). These functionals are discussed in Secs. IV, V and VI, respectively.

It seems possible to extend part of our results to general diffusion processes with resetting, although we will not attempt to do so in this paper. The advantage of focusing on rBM is that we can do exact computations.

II. TWO KEY RESULTS

In this section we present two results that will be used to study the distributions (Theorem II.1) and large deviations (Theorem II.2) associated with additive functionals of rBM.

The first result is based on the generating function of F_T:

$$G_r(k,T) = \mathbb{E}_r\left[\mathrm{e}^{kF_T}\right], \qquad k\in\mathbb{R},\ T\in[0,\infty), \tag{II.1}$$

where $\mathbb{E}_r$ denotes the expectation with respect to rBM with rate r. The Laplace transform of this function is defined as

$$\tilde{G}_r(k,s) = \int_0^\infty \mathrm{d}T\,\mathrm{e}^{-sT}\,G_r(k,T), \qquad k\in\mathbb{R},\ s\in[0,\infty). \tag{II.2}$$

Both may be infinite for certain ranges of the variables. The same quantities are defined analogously for the reset-free process and are given the subscript 0. The following theorem expresses the reset Laplace transform in terms of the reset-free Laplace transform.

Theorem II.1. *If $r\tilde{G}_0(k,s+r)<1$, then*

$$\tilde{G}_r(k,s) = \frac{\tilde{G}_0(k,s+r)}{1-r\tilde{G}_0(k,s+r)}. \tag{II.3}$$

Theorem II.1 was proved in [23] with the help of a renewal argument relating the process with resetting to the one without reset. Its result can be used to study the effect of resetting on the distribution of F_T or its moments. As shown in [23], it can also be used to show that $(T^{-1}F_T)_{T>0}$ satisfies the LDP with resetting by studying the singularities of $\tilde{G}_0(k,s)$, under the assumption that $G_0(k,T)$ exists for k in some open neighbourhood of the origin. We recall that $(T^{-1}F_T)_{T>0}$ satisfies the LDP with speed T and rate function χ_r, formally, if

$$\frac{\mathbb{P}_r(T^{-1}F_T \in \mathrm{d}\phi)}{\mathrm{d}\phi} = \mathrm{e}^{-T\chi_r(\phi)+o(T)} \tag{II.4}$$

as $T \to \infty$. See [12, Chapter III] for the precise definition of the LDP. For background on large deviation theory the reader is referred to [12, 33].

When the assumption on the existence of $G_0(k,T)$ is not satisfied, we can use the following theorem, which provides a variational formula for the rate function of $(T^{-1}F_T)_{T>0}$ in terms of the rate functions of the three processes underlying this random variable with resetting, namely:

(1) The rate function for $(T^{-1}N(T))_{T>0}$, the number of resets per unit of time:

$$I_r(n) = n \log\left(\frac{n}{r}\right) - n + r, \qquad n \in [0,\infty). \tag{II.5}$$

(2) The rate function for $L_N = (N^{-1}\sum_{i=1}^N \delta_{\tau_i})_{N\in\mathbb{N}}$, the empirical distribution of the duration of the reset periods:

$$J_r(\mu) = h(\mu \mid \mathcal{E}_r), \qquad \mu \in \mathcal{P}([0,\infty)). \tag{II.6}$$

Here, $\mathcal{P}([0,\infty))$ is the set of probability distributions on $[0,\infty)$, $\mathcal{E}_r$ is the exponential distribution with mean $1/r$, and $h(\cdot \mid \cdot)$ denotes the relative entropy

$$h(\mu \mid \nu) = \int_0^\infty \mu(dx) \log\left[\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)\right], \qquad \mu,\nu \in \mathcal{P}([0,\infty)). \tag{II.7}$$

(3) The rate function for $(N^{-1}\sum_{i=1}^N F_{\tau,i})_{N\in\mathbb{N}}$, the empirical average of i.i.d. copies of the *reset-free* functional F_τ over a time τ:

$$K_\tau(u) = \sup_{v\in\mathbb{R}} \{uv - \phi_\tau(v)\}, \qquad u \in \mathbb{R},\ \tau \in [0,\infty). \tag{II.8}$$

Here, $\phi_\tau(v) = \log \mathbb{E}_0\left[\mathrm{e}^{vF_\tau}\right]$ is the cumulant generating function of F_τ without reset, and we require ϕ_τ to exist in a neighborhood of 0.

Theorem II.2. *The family $(\mathbb{P}_r(T^{-1}F_T \in \cdot))_{T>0}$ satisfies for $r > 0$ the LDP on $\mathbb{R}$ with speed T and with rate function χ_r given by*

$$\chi_r(\phi) = \inf_{(n,\mu,w)\in\Phi(\phi)} \left\{I_r(n) + nJ_r(\mu) + n\int_0^\infty \mu(\mathrm{d}t)\, K_t(w(t))\right\}, \qquad \phi \in \mathbb{R}, \tag{II.9}$$

where

$$\Phi(\phi) = \left\{(n,\mu,w) \in [0,\infty) \times \mathcal{P}([0,\infty)) \times \mathcal{B}(\mathbb{R})\colon\ n\int_0^\infty \mu(\mathrm{d}t)\, w(t) = \phi\right\} \tag{II.10}$$

with $\mathcal{B}(\mathbb{R})$ the set of Borel-measurable functions on $\mathbb{R}$.

Proof. The LDP for $(T^{-1}F_T)_{T>0}$ follows by combining the LDPs for the constituent processes and using the contraction principle [12, Chapter III].

First, recall that $N(T)$ is the number of reset events in the time interval $[0,T]$. By Cramér's Theorem [12, Chapter I], $(T^{-1}N(T))_{T>0}$ satisfies the LDP on $[0,\infty)$ with speed T and with rate function I_r in (II.5), because resetting occurs according to a Poisson process with intensity r. This rate function has a unique zero at $n = r$ and takes the value r at $n = 0$.

Next, consider the empirical distribution of the reset periods,

$$L_m = \frac{1}{m} \sum_{i=1}^{m} \delta_{\tau_i}. \tag{II.11}$$

By Sanov's Theorem [12, Chapter II], $(L_m)_{m\in\mathbb{N}}$ satisfies the LDP on $\mathcal{P}([0,\infty))$, the space of probability distributions on $[0,\infty)$, with speed m and with rate function J_r in (II.6). This rate function has a unique zero at $\mu = \mathcal{E}_r$.

Finally, consider the empirical average of N independent trials $\{F_{\tau,i}\}_{i=1}^{N}$ of the reset-free process of length τ,

$$m_N = \frac{1}{N} \sum_{i=1}^{N} F_{\tau,i}. \tag{II.12}$$

By Cramér's Theorem, $(m_N)_{N\in\mathbb{N}}$ satisfies the LDP on $[0,\infty)$ with speed N and with rate function K_τ in (II.8). This rate function has a unique zero at $u = \mathbb{E}_0(T_\tau)$.

Now, the probability that $nt\,\mu(\mathrm{d}\tau)$ excursion times of length τ contribute an amount $u\,nt\,\mu(\mathrm{d}\tau)$ to the integral equals approximately

$$\mathrm{e}^{-nt\,\mu(\mathrm{d}\tau)\,K_\tau(u)} \tag{II.13}$$

for any $u \in \mathbb{R}$. If we condition on $N(T) = nT$ and $L_{N(T)} = \mu$, and pick $w \in \mathcal{B}(\mathbb{R})$, then the probability that nT duration times contribute an amount $\phi\,nT$ to the integral with

$$\phi = n \int_0^\infty \mu(\mathrm{d}t)\, w(t) \tag{II.14}$$

equals approximately

$$\mathrm{e}^{-nT \int_0^\infty \mu(\mathrm{d}t) K_t(w(t))}. \tag{II.15}$$

Therefore, by the contraction principle [12, Chapter III],

$$\frac{\mathbb{P}_r(T^{-1}F_T \in \mathrm{d}\phi)}{\mathrm{d}\phi} = \mathrm{e}^{-T\chi_r(\phi)+o(T)}, \tag{II.16}$$

where $\chi_r(\phi)$ is given the variational formula in (II.9). □

III. PROPERTIES OF THE RATE FUNCTION

The variational formula (II.9) can be used to derive general properties of the rate function under resetting. In this section, we show that the rate function is flat beyond the mean with resetting *when the mean without resetting diverges*, and that it is quadratic locally below the mean with resetting. Both properties will be illustrated in Sec. VI for the absolute area of rBM.

A. Zero rate function above the mean

Theorem III.1. *If* $\lim_{T\to\infty} \mathbb{E}_0[T^{-1}F_T] = \infty$, *then*

$$\chi_r(\phi) = 0 \quad \forall\, \phi \geq \phi_r^*, \qquad r > 0, \tag{III.1}$$

with $\phi_r^* = \lim_{T\to\infty} \mathbb{E}_r[T^{-1}F_T]$.

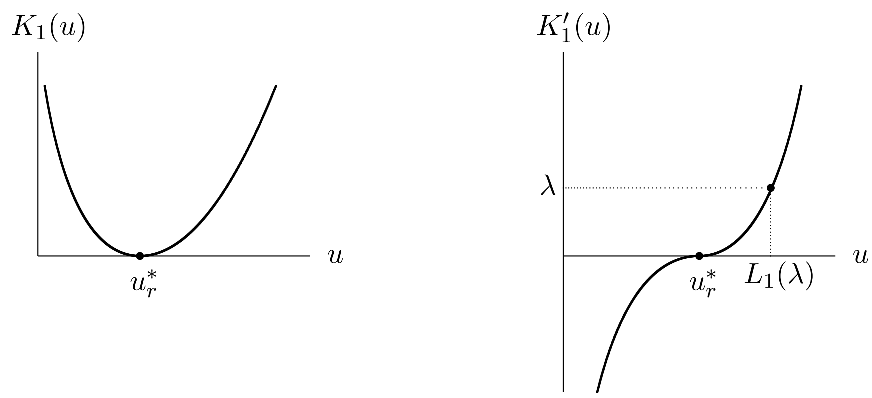

FIG. 1. Qualitative plot of $u \mapsto K_1(u)$ and $u \mapsto K_1'(u)$.

Proof. The calculation of the rate function in Theorem II.1 is a constrained functional optimization problem that requires knowledge of the function $K_t(w(t))$. We use the method of Lagrange multipliers. For fixed n, μ, the Lagrangian reads

$$\mathcal{L}(w(\cdot)) = I_r(n) + nJ_r(\mu) + n\int_0^\infty \mu(\mathrm{d}t)\, K_t(w(t)) - \lambda n \int_0^\infty \mu(\mathrm{d}t)\, w(t), \tag{III.2}$$

where λ is the Lagrange multiplier that enforces the constraint

$$n\int_0^\infty \mu(\mathrm{d}t) K_t(w(t)) = \phi. \tag{III.3}$$

We look for solutions $w_\lambda(\cdot)$ of the equation $\frac{\partial \mathcal{L}}{\partial w(t)}(\cdot) = 0$ for all $t \geq 0$, i.e.,

$$K_t'(w_\lambda(t)) = \lambda, \qquad t \geq 0, \tag{III.4}$$

where $w_\lambda(\cdot)$ must satisfy the constraint $n\int_0^\infty \mu(\mathrm{d}t)\, K_t(w_\lambda(t)) = \phi$. To that end, let $L_t(\cdot)$ be the inverse of $K_t'(\cdot)$, so that (III.4) becomes

$$w_\lambda(t) = L_t(\lambda), \qquad t \geq 0. \tag{III.5}$$

Then

$$\chi_r(\phi) = \inf_{n\in[0,\infty),\, \mu\in\mathcal{P}([0,\infty))} \Big\{ I_r(n) + nJ_r(\mu) + n\int_0^\infty \mu(\mathrm{d}t)\, K_t(L_t(\lambda)) \Big\}, \tag{III.6}$$

where $\lambda = \lambda(n, \mu)$ must be chosen such that

$$n\int_0^\infty \mu(\mathrm{d}t)\, L_t(\lambda) = \phi. \tag{III.7}$$

Our task is to show that the rate function of $(T^{-1}F_T)_{T>0}$ with resetting is zero to the right of $\phi_r^* = \mathbb{E}_r[F_1]$, provided that the asymptotic mean of this random variable diverges without resetting, i.e.,

$$\lim_{T\to\infty} \mathbb{E}_0[T^{-1}F_T] = \infty. \tag{III.8}$$

To do so, we perturb $\chi_r(\phi)$ around ϕ_r^*. To see how, we first rescale time. The proper rescaling depends on how F_T scales with T without resetting. Namely, we assume that there exists an $\alpha \in (1,\infty)$ such that

$$T^{-\alpha} F_T \overset{d}{=} F_1 \qquad \forall\, T > 0, \tag{III.9}$$

where $\overset{d}{=}$ means equality in distribution. Then

$$K_t(u) = K_1(ut^{-\alpha}), \qquad u \in (0,\infty),\ t \geq 0. \tag{III.10}$$

For example, for the area and the absolute area we have $\alpha = \frac{3}{2}$, while for the occupation time we have $\alpha = 1$ (which does not qualify because (III.8) fails). The rescaling changes the constraint in (III.3) to

$$n \int_0^\infty \mu(\mathrm{d}t)\, K_1(w(t))\, t^\alpha = \phi \tag{III.11}$$

and the constraint in (III.7) to

$$n \int_0^\infty \mu(\mathrm{d}t)\, L_1(\lambda)\, t^\alpha = \phi \tag{III.12}$$

(see Fig. 1).

We claim that, for every fixed n, it is possible to find a minimising sequence of probability distributions $(\mu_m)_{m\in\mathbb{N}}$ such that $\lambda = \lambda(n,\mu_m) = 0$ for all $m \in \mathbb{N}$, μ_m converges to $\mathcal{E}_r$ pointwise and in the L^1-norm as $m \to \infty$, but the convergence does not hold in the L^α-norm. As we will see, this implies that the rate function is zero for $\phi > \phi_r^*$. We will construct the sequence by perturbing $\mathcal{E}_r$ slightly, adding a small probability mass near some large time and taking the same probability mass away near time 0.

Let u_r^* be such that $K_1(u_r^*) = 0$, i.e.,

$$r \int_0^\infty \mathcal{E}_r(\mathrm{d}t)\, u_r^*\, t^\alpha = \phi_r^* \tag{III.13}$$

(recall that $\mathcal{E}_r(\mathrm{d}t) = r\mathrm{e}^{-rt}\,\mathrm{d}t$). Since $u_r^* = L_1(0)$, if we require the probability distributions μ over which we minimise to satisfy

$$n \int_0^\infty \mu(\mathrm{d}t)\, u_r^*\, t^\alpha = \phi, \tag{III.14}$$

then the optimisation problem reduces to

$$\inf_{n\in[0,\infty)} \Big\{ I_r(n) + n \inf_{\mu\in\mathcal{P}([0,\infty))} J_r(\mu) \Big\}. \tag{III.15}$$

Our goal is to prove that this infimum is zero for all $\phi > \phi_r^*$.

We get an upper bound by picking $n = r$ and

$$\mu_m(\mathrm{d}t) = \mathcal{E}_r(\mathrm{d}t) + \nu_m(\mathrm{d}t) \tag{III.16}$$

with

$$\nu_m(\mathrm{d}t) = -\epsilon_m \delta_0(\mathrm{d}t) + \epsilon_m \delta_{\theta_m}(\mathrm{d}t), \tag{III.17}$$

where ϵ_m, θ_m will be chosen later such that $\lim_{m\to\infty} \epsilon_m = 0$ and $\lim_{m\to\infty} \theta_m = \infty$. Substituting this perturbation into (III.14) and using (III.13), we get

$$ru_r^* \, \epsilon_m(\theta_m)^\alpha = \phi - \phi_r^*, \tag{III.18}$$

which places a constraint on our choice of ϵ_m, θ_m. On the other hand, substituting the perturbation into the expression for $J_r(\mu)$, we obtain

$$\begin{aligned} J_r(\mu_m) &= \int_0^\infty (\mathcal{E}_r - \epsilon_m\delta_0 + \epsilon_m\delta_\theta)(\mathrm{d}t)\, \log\Big(\frac{\mathcal{E}_r - \epsilon_m\delta_0 + \epsilon_m\delta_\theta}{\mathcal{E}_r}\Big)(t) \\ &= \int_0^\infty \mathcal{E}_r(\mathrm{d}t)\, \log\Big(1 + \frac{-\epsilon_m\delta_0 + \epsilon_m\delta_\theta}{\mathcal{E}_r}\Big)(t) \\ &\quad - \epsilon_m \int_0^\infty \delta_0(\mathrm{d}t)\, \log\Big(1 + \frac{-\epsilon_m\delta_0 + \epsilon_m\delta_\theta}{\mathcal{E}_r}\Big)(t) \\ &\quad + \epsilon_m \int_0^\infty \delta_{\theta_m}(\mathrm{d}t)\, \log\Big(1 + \frac{-\epsilon_m\delta_0 + \epsilon_m\delta_\theta}{\mathcal{E}_r}\Big)(t). \end{aligned} \tag{III.19}$$

For a proper computation, δ_0 and δ_θ must be approximated by $\eta^{-1}\,1_{[0,\eta]}$ and $\eta^{-1}\,1_{[\theta,\theta+\eta]}$, followed by $\eta \downarrow 0$. After we perform the integrals, the three terms in the right-hand side of (III.19) become

$$\begin{aligned} & r\eta \log\Big(\frac{1 - \epsilon_m/\eta}{r}\Big) + r\mathrm{e}^{-r\theta_m}\eta \log\Big(1 + \frac{\epsilon_m/\eta}{r\mathrm{e}^{-r\theta_m}}\Big), \\ & - \epsilon_m \log\Big(1 - \frac{\epsilon_m/\eta}{r}\Big), \\ & + \epsilon_m \log\Big(1 + \frac{\epsilon_m/\eta}{r\mathrm{e}^{-r\theta_m}}\Big). \end{aligned} \tag{III.20}$$

For all of these terms to vanish as $m \to \infty$ followed by $\eta \downarrow 0$, it suffices to pick ϵ_m and θ_m such that $\lim_{m\to\infty} \epsilon_m = 0$, $\lim_{m\to\infty} \theta_m = \infty$ and $\lim_{m\to\infty} \theta_m\epsilon_m = 0$. Clearly, this can be done while matching the constraint in (III.18) for any $\phi > \phi_r^*$, because $\alpha \in (1, \infty)$, and so we conclude that indeed the infimum in (III.15) is zero.

It is easy to check that the same argument works when, instead of (III.9), there exists a $T \mapsto L(T)$ with $\lim_{T\to\infty} L(T) = \infty$ such that

$$(TL(T))^{-1} F_T \overset{d}{=} F_1 \qquad \forall\, T > 0. \tag{III.21}$$

Then the constraint in (III.13) becomes $ru_r^* \, \epsilon_m\theta_m L(\theta_m) = \phi - \phi_r^*$, which can be matched too. It is also not necessary that the scaling in (III.9) and (III.21) holds for all $T > 0$. It clearly suffices that they hold asymptotically as $T \to \infty$. Hence, all that is needed is that $T^{-1}F_T$ without resetting diverges in mean as $T \to \infty$. □

The interpretation of the above approximation is as follows. Shifting a tiny amount of probability mass into the tail of the probability distribution μ has a negligible cost on the exponential scale. The shift produces a small fraction of reset periods that are longer than typical. In these reset periods large contributions occur at a negligible cost, since the growth without reset is faster than linear. In this way we can produce any ϕ that is larger than ϕ_r^* at zero cost on the scale T of the LDP.

B. Quadratic rate function below the mean

Theorem III.2. *For $\phi \uparrow \phi_r^*$,*

$$\chi_r(\phi) \sim C_r(\phi_r^* - \phi)^2, \tag{III.22}$$

with $C_r \in (0,\infty)$ a constant that is given by the variational formula in (III.29)–(III.31) below. (The symbol $\sim$ means that the quotient of the left-hand side and the right-hand side tends to 1.)

Proof. We perturb (II.9) around its zero by taking

$$n = r + m\epsilon, \qquad \mu(\mathrm{d}t) = \mathcal{E}_r(\mathrm{d}t)\,[1+\nu(t)\epsilon], \qquad w(t) = u_r^* + v(t)\epsilon, \tag{III.23}$$

subject to the constraint $\int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,\nu(t) = 0$, with $\nu(\cdot), v(\cdot)$ Borel measurable. This gives

$$I_r(r+m\epsilon) = F_r^*(m)\epsilon^2 + O(\epsilon^3) \tag{III.24}$$

with $F_r^*(m) = \frac{m^2}{2r}$. Next, we have

$$J_r(\mu) = \int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,[1+\nu(t)\epsilon]\log[1+\nu(t)\epsilon]. \tag{III.25}$$

Expanding the logarithm in powers of ϵ and using the normalisation condition, we obtain

$$J_r(\mu) = \epsilon^2 G_r^*(\nu) + O(\epsilon^3) \tag{III.26}$$

with $G_r^*(\nu) = \frac{1}{2}\int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,\nu^2(t)$. Lastly, we know that (see Fig. 1)

$$K_1(u_r^* + v(t)\epsilon) \sim \tfrac{1}{2}\epsilon^2\, v(t)^2\, K_1''(u_r^*), \tag{III.27}$$

so that the last term in the variational formula becomes

$$(r+m\epsilon)\int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,[1+\nu(t)\epsilon]\,K_1(u_r^*+v(t)\epsilon) = \epsilon^2 H_r^*(v) + O(\epsilon^3) \tag{III.28}$$

with $H_r^*(v) = \frac{r}{2}K_1''(u_r^*)\int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,v(t)^2$. It follows that

$$\chi(\phi_r^* + \epsilon) = C_r\epsilon^2 + O(\epsilon^3) \tag{III.29}$$

with

$$C_r = \inf_{(m,\nu,v)\in\Phi}\Big\{F_r^*(m) + G_r^*(\nu) + H_r^*(v)\Big\} \tag{III.30}$$

where

$$\Phi = \left\{(m,\nu,v)\colon\ \int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,\nu(t) = 0,\ r\int_0^\infty \mathcal{E}_r(\mathrm{d}t)\,\left[\frac{m}{r} + \nu(t) + v(t)\right]\,t^\alpha = 1\right\}. \tag{III.31}$$

The last constraint arises from (III.13)–(III.14) after inserting (III.23) and letting $\epsilon \downarrow 0$, all under assumption (III.9). Finally, it is easy to check again that the same argument works when (III.9) is replaced by (III.21). In that case, t^α in (III.31) becomes $tL(t)$.

Note that the perturbation is only possible for $\epsilon < 0$ (below ϕ_r^*), since there is no minimiser to expand around for $\epsilon > 0$ above ϕ_r^*. Also note that $C_r > 0$ because the choice $m = 0$, $\nu(\cdot) \equiv 0$, $v(\cdot) \equiv 0$ does not match the last constraint. □

IV. OCCUPATION TIME

We illustrate the results of the previous sections for rBM, starting with its positive occupation time, defined as

$$A_T = \int_0^T 1_{[0,\infty)}(W_t^r)\,\mathrm{d}t. \tag{IV.1}$$

This random variable has a density with respect to the Lebesgue measure, which we denote by $p_r^A(a)$, so that

$$p_r^A(a) = \frac{\mathbb{P}_r(A_T \in \mathrm{d}a)}{\mathrm{d}a}, \qquad a \in (0,T). \tag{IV.2}$$

Without resetting, this density is

$$p_0^A(a) = \frac{1}{\pi\sqrt{a(T-a)}}, \qquad a \in (0,T), \tag{IV.3}$$

which is the derivative of the famous arcsine law found by Lévy [19]. The next theorem shows how this result is modified under resetting.

Theorem IV.1. *The occupation time of rBM has the probability density*

$$p_r^A(a) = \frac{r}{T}\,\mathrm{e}^{-rT}\,W\big(r\sqrt{a(T-a)}\,\big), \qquad a \in (0,T), \tag{IV.4}$$

where

$$W(x) = \frac{1}{x}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma(\frac{j+1}{2})^2} = I_0(2x) + \frac{1}{x\pi}\,{}_1F_2\big(\{1\},\{\tfrac{1}{2},\tfrac{1}{2}\},x^2\big), \qquad x \in (0,\infty), \tag{IV.5}$$

with $I_0(y)$ the modified Bessel function of the first kind with index 0 and ${}_1F_2(\{a\},\{b,c\},y)$ the generalized hypergeometric function [1, Section 9.6, Formula 15.6.4]*.*

Proof. In what follows, the regions of convergence of the generating functions will be obvious, so we do not specify them.

The non-reset generating function in (II.1) for the occupation time started at $X_0 = 0$ is known to be [21]

$$\tilde{G}_0(k,s) = \frac{1}{\sqrt{s(s-k)}}. \tag{IV.6}$$

This can be explicitly inverted to obtain the density in (IV.3).

To find the Laplace transform of the reset generating function, we use Theorem II.1. Inserting (IV.6) into (II.3), we find

$$\tilde{G}_r(k,s) = \frac{1}{\sqrt{(s+r)(s+r-k)} - r}. \tag{IV.7}$$

This can be explicitly inverted to obtain the density in (IV.4), as follows. Write

$$p_r^A(a) = \mathrm{e}^{-rT}H(aT,(1-a)T), \tag{IV.8}$$

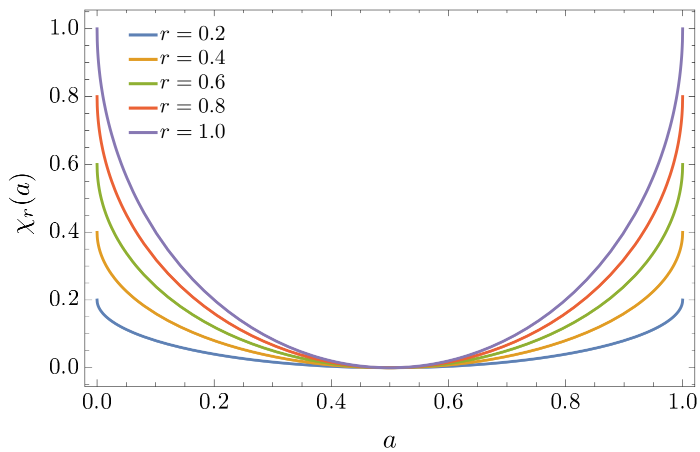

FIG. 2. Rate function $\chi_r^A(a)$ for the positive occupation of rBM.

where H is to be determined. Substituting this form into (II.2), we get

$$\tilde{G}_r(k,s) = \int_0^\infty \mathrm{d}T \int_0^1 \mathrm{d}a\ \mathrm{e}^{kTa}\, \mathrm{e}^{-(s+r)T}\, H(aT,(1-a)T). \tag{IV.9}$$

Performing the change of variable $t_1 = aT$ and $t_2 = (1-a)T$, we get

$$\tilde{G}_r(k,s) = \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\ \mathrm{e}^{-(r+s-k)t_1}\mathrm{e}^{-(r+s)t_2} H(t_1,t_2). \tag{IV.10}$$

Let $\lambda_1 = r+s-k$ and $\lambda_2 = r+s$. Then (IV.10), along with the right-hand side of (IV.7), gives

$$\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\ \mathrm{e}^{-\lambda_1 t_1 - \lambda_2 t_2} H(t_1,t_2) = \frac{1}{\sqrt{\lambda_1\lambda_2} - r}. \tag{IV.11}$$

To invert the Laplace transform in (IV.11), we expand the right-hand side in r,

$$\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\ \mathrm{e}^{-\lambda_1 t_1 - \lambda_2 t_2} H(t_1,t_2) = \sum_{j=0}^\infty \frac{r^j}{(\lambda_1\lambda_2)^{(j+1)/2}}, \tag{IV.12}$$

and invert term by term using the identity

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}t\ t^{\alpha-1}\mathrm{e}^{-\lambda t} = \frac{1}{\lambda^\alpha}, \qquad \alpha > 0. \tag{IV.13}$$

This leads us to the expression

$$H(t_1,t_2) = \sum_{j=0}^\infty \frac{r^j}{\Gamma(\frac{j+1}{2})^2}(t_1t_2)^{(j-1)/2} = r\sum_{j=0}^\infty \frac{(r\sqrt{t_1t_2})^{j-1}}{\Gamma(\frac{j+1}{2})^2}. \tag{IV.14}$$

Substituting this expression into (IV.8), we find the result in (IV.4)–(IV.5). □

The arcsine density in (IV.3) is recovered in the limit $r \to 0$ by noting that $W(x) \sim (\pi x)^{-1}$ as $x \to 0$. On the other hand, we have

$$W(x) \sim \frac{1}{2\sqrt{\pi x}}\, e^{2x}, \qquad x \to \infty \tag{IV.15}$$

As a result,

$$T\,p_r^A(aT) \sim \frac{\sqrt{r}}{2\sqrt{\pi T}\;\big(a(1-a)\big)^{1/4}}\,\mathrm{e}^{-r\,T\left(1-2\sqrt{a(1-a)}\right)}, \qquad a\in(0,1), \quad T\to\infty. \tag{IV.16}$$

Keeping only the exponential term, we thus find that $(T^{-1}A_T)_{T>0}$ satisfies the LDP with speed T and with rate function χ_r^A given by

$$\chi_r^A(a) = r\left(1-2\sqrt{a(1-a)}\right), \qquad a\in[0,1]. \tag{IV.17}$$

The same rate function, plotted in Fig. 2, can be obtained following [23] by noting that the largest real pole of $\tilde{G}(k,s)$ in the s-complex plane is

$$\lambda_r(k) = \frac{1}{2}\left(k-2r+\sqrt{k^2+4r^2}\right), \qquad k\in\mathbb{R}, \tag{IV.18}$$

which defines the scaled cumulant generating function of A_T as $T\to\infty$ (see Eq. (VI.25) below). Since this function is differentiable for all $k\in\mathbb{R}$, we can use the Gärtner-Ellis Theorem [12, Chapter V] to find the rate function of $(T^{-1}A_T)_{T>0}$ as the Legendre transform of $\lambda_r(k)$.

Note that the occupation time does not satisfy the LDP when $r=0$, since $p_0^A(a)$ is not exponential in T and does not concentrate as $T\to\infty$. Resetting is thus “strong enough” here to force concentration of $T^{-1}A_T$ on the value $\frac{1}{2}$, with fluctuations around this value that are determined by the LDP and the rate function χ_r^A above. In particular, since $\chi_r^A(0)=\chi_r^A(1)=r$, the probability that rBM always stays positive or always stays negative is determined on the large deviation scale by the probability e^{-rT} of having no reset up to time T.

V. AREA

We next consider the area of rBM, defined as

$$B_T = \int_0^T W_t^r\,\mathrm{d}t. \tag{V.1}$$

Its density with respect to the Lebesgue measure is denoted by $p_r^B(b)$, $b\in\mathbb{R}$. We are able to obtain a central limit theorem for this random variable, prompted by the following exact results for its moments.

Theorem V.1. *The area of rBM for $r\geq 0$ has vanishing odd moments and non-vanishing even moments. The first four even moments are*

$$\begin{aligned}
\mathbb{E}_r[B_T^2] &= \frac{2}{r^3}\left(rT-2+\mathrm{e}^{-rT}(2+rT)\right),\\
\mathbb{E}_r[B_T^4] &= \frac{1}{r^6}\left(12(rT)^2+120rT-840+\mathrm{e}^{-rT}[9(rT)^4+68(rT)^3+288(rT)^2+720\,rT+840]\right),\\
\mathbb{E}_r[B_T^6] &= \frac{3}{7r^9}\Big(280(rT)^3+10080(rT)^2+262080rT-2755200+\mathrm{e}^{-rT}\big[87(rT)^7\\
&\quad+1176(rT)^6+10220(rT)^5+65800(rT)^4+317800(rT)^3+1105440(rT)^2\\
&\quad+2493120\,rT+2755200\big]\Big),\\
\mathbb{E}_r[B_T^8] &= \frac{1}{10r^{12}}\Big(16800(rT)^4+1276800(rT)^3+79833600(rT)^2+3965068800rT-53794540800\\
&\quad+\mathrm{e}^{-rT}\big[1641(rT)^{10}+32150(rT)^9+425250(rT)^8+4445200(rT)^7+38124800(rT)^6\\
&\quad+269115840(rT)^5+1539384000(rT)^4+6902112000(rT)^3+22852368000(rT)^2\\
&\quad+49829472000\,rT+53794540800\big]\Big).
\end{aligned} \tag{V.2}$$

Proof. The result follows directly from the renewal formula (II.3) and the Laplace transform of the generating function of B_T without resetting:

$$\tilde{Q}_0(k,s) = \int_0^\infty \mathrm{d}T\ \mathrm{e}^{-sT}\mathbb{E}_0[\mathrm{e}^{kB_T}] = \int_0^\infty \mathrm{d}T\ \mathrm{e}^{\frac{1}{6}k^2T^3 - sT}, \tag{V.3}$$

knowing that B_T is a Gaussian random variable with mean 0 and variance $T^3/3$. Expanding the exponential in k above and using (II.3), we obtain the following expansion for the Laplace transform of the characteristic function with resetting:

$$\begin{aligned}\tilde{Q}_r(k,s) = \frac{1}{s} + \frac{1}{s^2(r+s)^2}k^2 + \frac{(r+10s)}{s^3(r+s)^5}k^4 + \frac{r^2+20rs+280s^2}{s^4(r+s)^8}k^6 \\ + \frac{r^3+30r^2s+660rs^2+15400s^3}{s^5(r+s)^{11}}k^8 + O(k^{10}).\end{aligned} \tag{V.4}$$

Taking the inverse Laplace transform, we then find that the odd moments are all zero, since there are no odd powers of k, and that the even moments are given by the inverse Laplace transforms ($\mathcal{L}^{-1}$) of the corresponding even powers of k. Thus

$$\begin{aligned}\mathbb{E}_r[B_T^2] &= \mathcal{L}^{-1}\Big[\frac{2!}{s^2(r+s)^2}\Big] \\ \mathbb{E}_r[B_T^4] &= \mathcal{L}^{-1}\Big[\frac{4!(r+10s)}{s^3(r+s)^5}\Big] \\ \mathbb{E}_r[B_T^6] &= \mathcal{L}^{-1}\Big[\frac{6!(r^2+20rs+280s^2)}{s^4(r+s)^8}\Big] \\ \mathbb{E}_r[B_T^8] &= \mathcal{L}^{-1}\Big[\frac{8!(r^3+30r^2s+660rs^2+15400s^3)}{s^5(r+s)^{11}}\Big],\end{aligned} \tag{V.5}$$

which yields the results shown in (V.2). □

The second moment, which gives the variance, shows that there is a crossover in time from a *reset-free regime* characterized by

$$\mathbb{E}_r[B_T^2] \sim \tfrac{1}{3}T^3, \qquad T \downarrow 0, \tag{V.6}$$

which is the variance obtained for $r = 0$, to a *reset regime* characterized by

$$\mathbb{E}_r[B_T^2] \sim \frac{2T}{r^2}, \qquad T \to \infty. \tag{V.7}$$

The crossover time T where the two regimes meet is given by $T = \sqrt{6}/r$, which is proportional to the mean reset time. This gives, as illustrated in Fig. 3, a rough estimate of the time needed for the variance to become linear in T because of resetting.

From this result, we anticipate that the small fluctuations of B_T of order $\sqrt{T}$ around the origin are Gaussian-distributed or, equivalently, that $B_T/\sqrt{T}$ has a Gaussian stationary distribution in the long-time limit. This is confirmed by noting that the even moments of B_T scale like

$$\mathbb{E}_r[B_T^n] \sim \frac{(2n)!}{n!}\left(\frac{\sqrt{T}}{r}\right)^n, \quad T \to \infty, \tag{V.8}$$

so that

$$\mathbb{E}_r\left[\left(\frac{B_T}{\sqrt{T}}\right)^n\right] \sim \frac{(2n)!}{n!r^n} \tag{V.9}$$

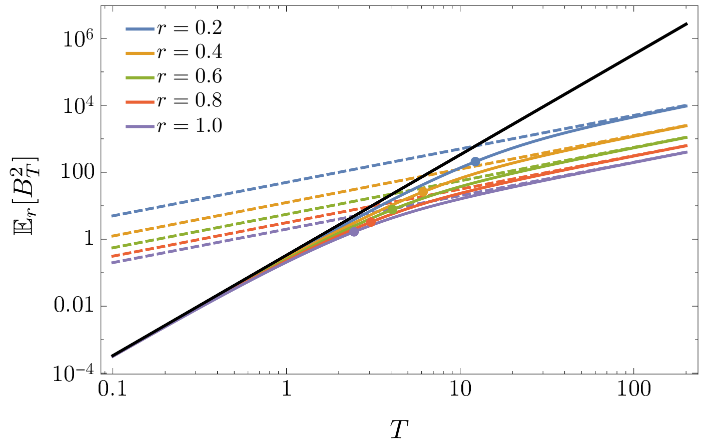

FIG. 3. Log-log plot of the variance of the area B_T of rBM, showing the crossover from the T^3-scaling (black line) to the T-scaling (dashed lines) for various values of r. The filled circles show the location of the crossover time $T = \sqrt{6}/r$.

for n even. This implies that the cumulants all asymptotically vanish, except for the variance. Indeed, it can be verified that

$$\kappa_2 = \lim_{T\to\infty} \mathbb{E}_r[T^{-1}B_T^2] = \frac{2}{r^2}, \tag{V.10}$$

while

$$\begin{aligned}
\kappa_4 &= \lim_{T\to\infty} \mathbb{E}_r[T^{-2}B_T^4] - 3\mathbb{E}_r[T^{-1}B_T^2]^2 = \frac{12}{r^4} - 3\left(\frac{2}{r^2}\right)^2 = 0,\\
\kappa_6 &= \lim_{T\to\infty} \mathbb{E}_r[T^{-3}B_T^6] - 15\mathbb{E}_r[T^{-2}B_T^4]\mathbb{E}_r[T^{-1}B_T^2] + 30\mathbb{E}_r[T^{-1}B_T^2]^3\\
&= \frac{120}{r^6} - 15\frac{12}{r^4}\frac{2}{r^2} + 30\Big(\frac{2}{r^2}\Big)^3 = 0,\\
\kappa_8 &= \lim_{T\to\infty} \mathbb{E}_r[T^{-4}B_T^8] - 7\mathbb{E}_r[T^{-1}B_T^2]\mathbb{E}_r[T^{-3}B_T^6] = \frac{1680}{r^8} - 7\frac{2}{r^2}\frac{120}{r^6} = 0,
\end{aligned} \tag{V.11}$$

and similarly for all higher even cumulants. Accordingly, there is a central limit theorem for $T^{-1/2}B_T$, which we formulate next.

Theorem V.2. *The area of rBM satisfies the central limit theorem,*

$$\lim_{T\to\infty} \sigma\sqrt{T}\; p_r^B\left(\frac{b}{\sigma\sqrt{T}}\right) = N(0,1) \tag{V.12}$$

with $N(0,1)$ the standard Gaussian distribution and $\sigma = 2/r^2$.

Proof. The result follows from the cumulants computed above. Alternatively, we can start from the Laplace inversion formula of the renewal formula,

$$p_r^B(b) = \mathrm{e}^{-rT}\int_{\mathbb{R}} \frac{dk}{2\pi}\mathrm{e}^{-ikb}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\mathrm{e}^{sT}\,\frac{\tilde{Q}_0(k,s)}{1-r\tilde{Q}_0(k,s)}, \tag{V.13}$$

where c is a value in the region of convergence of $\tilde{Q}_0(k,s)$ in the s complex plane. Rescaling b by $b = \bar{b}\sqrt{T}$, as is standard in proofs of the central limit theorem, we obtain

$$p_r^B(\bar{b}\sqrt{T}) = \frac{\mathrm{e}^{-rT}}{\sqrt{T}}\int_{\mathbb{R}} \frac{dl}{2\pi}\mathrm{e}^{-il\bar{b}}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\mathrm{e}^{sT}\,\frac{\tilde{Q}_0(l/\sqrt{T},s)}{1-r\tilde{Q}_0(l/\sqrt{T},s)}, \tag{V.14}$$

where $l = k/\sqrt{T}$. Given a fixed l and letting $T \to \infty$, we then use the known expression of $\mathbb{E}_0[\mathrm{e}^{ikB_T}]$ in (V.3) to Taylor-expand $\tilde{Q}_0(k,s)$ around $k=0$,

$$\tilde{Q}_0(k,s) = \frac{1}{s} - \frac{k^2}{s^4} + O(k^4), \tag{V.15}$$

to obtain

$$\frac{\tilde{Q}_0(l/\sqrt{T},s)}{1 - r\tilde{Q}_0(l/\sqrt{T},s)} = \frac{1 + O(l^2/T)}{s - r + \frac{rl^2}{s^3 T} + O(l^4/T^2)}. \tag{V.16}$$

This expression has a simple pole at

$$s^* = r - \frac{l^2}{r^2 T} + O(l^4/T^2), \tag{V.17}$$

so that, deforming the Bromwich contour through that pole, we get

$$\sqrt{T}\, p_r^B(\bar{b}\sqrt{T}) = \mathrm{e}^{-rT} \int_{\mathbb{R}} \frac{dl}{2\pi} \mathrm{e}^{-il\bar{b}} \mathrm{e}^{s^* T} = \int_{\mathbb{R}} \frac{dl}{2\pi} \mathrm{e}^{-il\bar{b}} \mathrm{e}^{-l^2/r^2 + O(l^4/T)}. \tag{V.18}$$

As $T \to \infty$, only the quadratic term remains in the exponential, yielding a Gaussian distribution with variance $2/r^2$. □

The convergence to the Gaussian distribution can be much slower than the mean reset time, as can be seen in Fig. 3, especially for small reset rates. From simulations we have found that the distribution of $T^{-1/2}B_T$ is well approximated by a Gaussian distribution near the origin. However, the tails are strongly non-Gaussian, even at long times, indicating that there are important finite-time corrections to the central limit theorem, related to rare events involving few resets and, therefore, to large Gaussian excursions characterised by the T^3 variance.

These corrections can be analysed, in principle, by going beyond the dominant scaling in time of the moments shown in (V.8), so as to obtain corrections to the cumulants, which do not vanish for $T < \infty$. It also seems possible to obtain information about the tails by performing a saddle-point approximation of the combined Laplace–Fourier inversion formula for values of B_T scaling with $T^{3/2}$. We have attempted such an approximation, but have found no results supported by numerical simulations performed to estimate $p_r^B(b)$. More work is therefore needed to find the tail behavior of this density in the intermediate regime where $T^{1/2} \lesssim b \lesssim T^{3/2}$.

At this point, we can only establish that $(T^{-1}B_T)_{T>0}$ does not satisfy the LDP with speed T, so that $p_r^B(b)$ does not decay exponentially with T on the scale $B_T \sim T$. This follows from the following upper bound:

$$\chi_r(\phi) \le \chi_0(\phi) + r, \tag{V.19}$$

found in [24]. Here, we have $\chi_0^B(b) = 0$ for all $b \in \mathbb{R}$ at the speed T, so that $\chi_r^B(b) \le r$. Since rate functions are expected to be convex, the latter equality can then only mean that $\chi_r^B(b) = 0$ for all $b \in \mathbb{R}$. Note, incidentally, that this bound is satisfied by the rate function $\chi_r^A(a)$ of the mean occupation time, defined for $a \in [0,1]$ (see (IV.17) and Fig. 2).

VI. ABSOLUTE AREA

We finally consider the absolute area of rBM, defined as

$$C_T = \int_0^T |W_t^r|\, \mathrm{d}t, \tag{VI.1}$$

which can also be seen as the area of an rBM reflected at the origin. We denote, as before, the density of this random variable by

$$p_r^C(c) = \frac{\mathbb{P}_r(C_T \in \mathrm{d}c)}{\mathrm{d}c}, \qquad c \in [0, \infty). \tag{VI.2}$$

This density was studied for pure BM ($r = 0$) by Kac [14] and Takács [31] (see also [32]). It satisfies the LDP with speed T, when C_T is rescaled by T, but with a divergent mean, which translates into the rate function having its zero at infinity. The effect of resetting is to bring the mean of $T^{-1}C_T$ to a finite value. Below the mean, we find that the LDP holds at the speed T with a non-trivial rate function derived from Theorem II.1, whereas above the mean we find that the rate function vanishes, in agreement with Theorem III.1. This indicates that the upper tail of $T^{-1}C_T$ decays slower than exponentially in T.

As a prelude to these results, we show how the mean and variance of C_T are affected by resetting. We do not know the full distribution, and also the scaling remains elusive.

Theorem VI.1. *The absolute area of rBM has a mean and a variance given by*

$$\mathbb{E}_r[C_T] = T^{3/2} f_1(rT), \qquad \mathbb{V}\mathrm{ar}_r[C_T] = T^3 f_2(rT), \qquad r > 0, \tag{VI.3}$$

where

$$f_1(\rho) = \frac{1}{\sqrt{2\pi}} \left[\frac{\mathrm{e}^{-\rho}}{\rho} + \frac{\sqrt{\pi}}{2(\rho)^{3/2}} (2\rho - 1)\, \mathrm{erf}[\sqrt{\rho}]\right] \tag{VI.4}$$

and

$$f_2(\rho) = \frac{1}{8\pi(\rho)^3} \left[2\pi \left(2\rho^2 + \rho - 6 + (5\rho + 6)\mathrm{e}^{-\rho}\right) - \left(2\sqrt{\rho}\,\mathrm{e}^{-\rho} + \sqrt{\pi}(2\rho - 1)\,\mathrm{erf}[\sqrt{\rho}]\right)^2\right]. \tag{VI.5}$$

Proof. The absolute area of pure BM ($r = 0$) is known to scale as $T^{3/2}$, so it is convenient to rescale C_T as

$$C_T = T^{3/2} \int_0^1 \mathrm{d}t\, |W_t^r| = T^{3/2}\, D, \tag{VI.6}$$

which defines a new random variable D. Expanding (II.2) in terms of k, we get

$$\begin{aligned} \tilde{G}_0(k, s) &= \int_0^\infty \mathrm{d}T \mathrm{e}^{-sT} \left[1 + kT^{3/2}\, \mathbb{E}_0[D] + \tfrac{1}{2}k^2 T^3\, \mathbb{E}_0[D^2] + O(k^3)\right] \\ &= \frac{1}{s} + \frac{\mathbb{E}_0[D]\, \Gamma(\frac{5}{2})\, k}{s^{5/2}} + \frac{3\,\mathbb{E}_0[D^2]\, k^2}{s^4} + O(k^3). \end{aligned} \tag{VI.7}$$

Abbreviate $a = \mathbb{E}_0[D]\, \Gamma(\frac{5}{2})$ and $b = \mathbb{E}_0[D^2]$ [13]. Inserting (VI.7) into (II.3), we find

$$\tilde{G}_r(k, s) = \frac{\frac{1}{s+r} + \frac{ak}{(s+r)^{5/2}} + \frac{3bk^2}{(s+r)^4} + O(k^3)}{1 - r\left[\frac{1}{s+r} + \frac{ak}{(s+r)^{5/2}} + \frac{3bk^2}{(s+r)^4} + O(k^3)\right]} = \frac{\frac{1}{s}\left[1 + \frac{ak}{(s+r)^{3/2}} + \frac{3bk^2}{(s+r)^3} + O(k^3)\right]}{1 - \frac{rak}{s(s+r)^{3/2}} - \frac{3rbk^2}{s(s+r)^3} + O(k^3)}. \tag{VI.8}$$

Using $(1 + ck + dk^2)^{-1} = 1 - ck + (c^2 - d)k^2 + O(k^3)$, we obtain

$$\tilde{G}_r(k, s) = \frac{1}{s} + \frac{a}{s^2(s + r)^{1/2}} k + \left(\frac{b}{s^2(s + r)^2} + \frac{ra^2}{s^3(s + r)^2}\right) k^2 + O(k^3). \tag{VI.9}$$

We can also expand $\tilde{G}_r(k,s)$ directly from its definition:

$$\tilde{G}_r(k,s) = \frac{1}{s} + k\int_0^\infty \mathrm{d}T\ \mathrm{e}^{-sT}\,\mathbb{E}_r[C_T] + \frac{k^2}{2}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{-sT}\,\mathbb{E}_r[C_T^2] + O(k^3). \tag{VI.10}$$

Comparing (VI.8) and (VI.10), we then find

$$\begin{aligned}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{-sT}\,\mathbb{E}_r[C_T] &= \frac{a}{s^2(s+r)^{1/2}}\\ \frac{1}{2}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{-sT}\,\mathbb{E}_r[C_T^2] &= \frac{b}{s^2(s+r)^2} + \frac{ra^2}{s^3(s+r)^2}.\end{aligned} \tag{VI.11}$$

To calculate the first and the second moment, we simply need to invert the Laplace transforms. For the mean, we find

$$\mathbb{E}_r[C_T] = T^{3/2} f_1(rT) \tag{VI.12}$$

where we use that $\mathbb{E}_0[D] = \frac{4}{3\sqrt{2\pi}}$ by [31, Table 3]. For the second moment, we use $\mathbb{E}_0[D^2] = b = 3/8$ from the same reference to find

$$\mathbb{E}_r[C_T^2] = T^3 f_3(rT) \tag{VI.13}$$

with

$$f_3(rT) = \frac{1}{4(rT)^3}\Big[2(rT)^2 + rT - 6 + (5rT+6)\mathrm{e}^{-rT}\Big]. \tag{VI.14}$$

The variance is then found to be

$$\mathbb{V}\mathrm{ar}_r[C_T] = T^3 f_3(rT) - T^3 f_1^2(rT) = T^3 f_2(rT). \tag{VI.15}$$

□

The result for the mean converges to $\mathbb{E}_0[D]$ when $rT \to 0$ and scales like $\frac{3}{4}\mathbb{E}_0[D]\sqrt{\frac{\pi}{rT}}$ when $rT \to \infty$. Therefore,

$$\lim_{T\to\infty} \mathbb{E}_r[T^{-1}C_T] = \frac{1}{\sqrt{2r}}. \tag{VI.16}$$

The same analysis for the variance yields

$$\lim_{T\to\infty} T^{-1}\mathbb{V}\mathrm{ar}_r[C_T] = \lim_{T\to\infty} T\,\mathbb{V}\mathrm{ar}_r[T^{-1}C_T] = \frac{3}{4r^2}. \tag{VI.17}$$

These two results suggest that $(T^{-1}C_T)_{T>0}$ satisfies the LDP. To describe the corresponding rate function, we define the function

$$H(x) = -2^{1/3}\frac{\mathrm{AI}(x)}{\mathrm{Ai}'(x)}, \tag{VI.18}$$

where

$$\mathrm{AI}(x) = \int_x^\infty \mathrm{Ai}(t)\,\mathrm{d}t \tag{VI.19}$$

is the integral Airy function and $\mathrm{Ai}(x)$ is the Airy function [1, Section 10.4] defined, for example, by

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right)\,\mathrm{d}t. \tag{VI.20}$$

The next theorem gives an explicit representation of the rate function of $(T^{-1}C_T)_{T>0}$ for values below its mean.

Theorem VI.2. *Let $c_r^* = 1/\sqrt{2r}$ be the asymptotic mean of $(T^{-1}C_T)_{T>0}$, and let s_k^* be the largest real root in s of the equation*

$$\frac{r}{(-k)^{2/3}} H\left(\frac{2^{1/3}(s+r)}{(-k)^{2/3}}\right) = 1, \qquad k < 0. \tag{VI.21}$$

Then $(T^{-1}C_T)_{T>0}$ satisfies the LDP on $(0, c_r^*)$ with speed T and with rate function given by the Legendre transform of s_k^*.

Proof. With the same rescaling as in (VI.6), the generating function for C_T can be written as

$$G_0(k,T) = \mathbb{E}_0[\mathrm{e}^{kT^{3/2}D}]. \tag{VI.22}$$

Using [13, Eq. (173)] we have

$$\int_0^\infty \mathrm{e}^{-sT}\mathbb{E}_0[\mathrm{e}^{-\sqrt{2}T^{3/2}\xi C_T}]\mathrm{d}T = -\frac{\mathrm{AI}[\xi^{-2/3}s]}{\xi^{2/3}\mathrm{Ai}'[\xi^{-2/3}s]}, \quad \xi > 0, \tag{VI.23}$$

so that the Laplace transform of $G_0(k,T)$ has the explicit expression

$$\tilde{G}_0(k,s) = \frac{1}{(-k)^{2/3}} H\left(\frac{2^{1/3}s}{(-k)^{2/3}}\right), \quad k < 0, \tag{VI.24}$$

where $H(x)$ is the function defined in (VI.18).

With this result, we follow the method detailed in [23]: we insert the expression of $\tilde{G}_0(k,s)$ into (II.3) to find the expression of $\tilde{G}_r(k,s)$ and locate the largest real pole of that function, which is known to determine the *scaled cumulant generating function* (SCGF) of C_T, defined as

$$\lambda_r(k) = \lim_{T\to\infty} \frac{1}{T}\log G_r(k,T). \tag{VI.25}$$

Due to the form of $\tilde{G}_r(k,s)$ in (II.3), this pole must be given by the largest real root of the equation $r\tilde{G}_0(k, s+r) = 1$, which yields the equation shown in (VI.21). From there we apply the Gärtner-Ellis Theorem [12] by noting that $\lambda_r(k) = s_k^*$ is finite and differentiable for all $k < 0$. Consequently, the rate function is given by the Legendre transform

$$\chi_r^C(c_k) = kc_k - \lambda_r(k), \tag{VI.26}$$

where $c_k = \lambda_r'(k)$ for all $k < 0$. It can be verified that $\lambda_r'(k) \to 0$ as $k \to -\infty$ and $\lambda_r'(k) \to c_r^*$ as $k \uparrow 0$. Thus, the rate function obtained is valid for $c \in (0, c_r^*)$. □

The plot on the left in Fig. 4 shows the SCGF $\lambda_r(k)$, while the plot on the right shows the rate function $\chi_r^C(c)$ obtained by solving the equation in (VI.21) numerically and by computing the Legendre transform (VI.26). The rate function is compared with the rate function without resetting, which is given by

$$\chi_0^C(c) = \frac{2|\zeta_0'|^3}{27\,c^2}, \tag{VI.27}$$

where ζ_0' is the first zero of the derivative of the Airy function. The derivation of χ_0^C also follows from the Gärtner-Ellis Theorem and is given in Appendix A.

Comparing the two rate functions, we see that $T^{-1}C_T$ has a finite mean c_r^* with resetting. Above this value, it is not possible to obtain $\chi_r^C(c)$ from $G_r(k,T)$, since the latter function is not defined

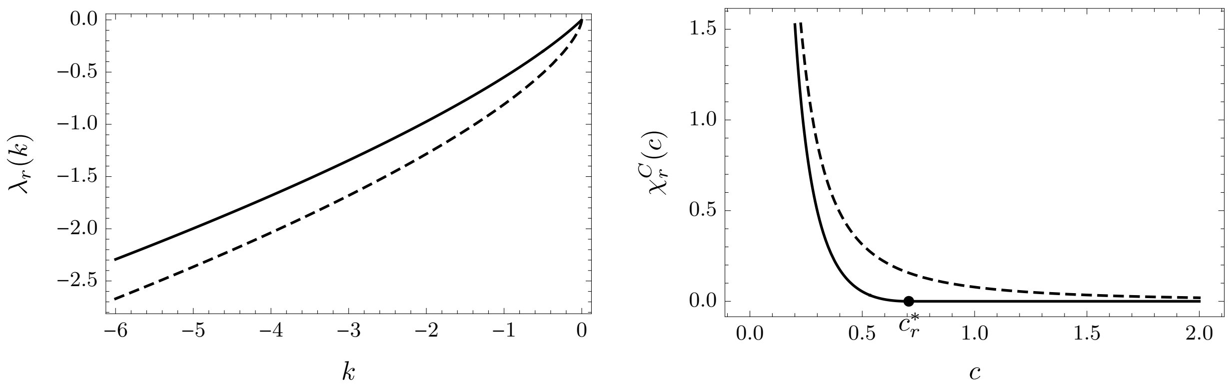

FIG. 4. Left: SCGF of the absolute area of rBM as a function of k for $r = 1$ (full line) and $r = 0$ (dashed line). Right: Corresponding rate function obtained by Legendre transform for $r = 1$ (full line) and $r = 0$ (dashed line). Above the mean $c_r^* = 1/\sqrt{2r}$, $\chi_r^C(c)$ is flat.

for $k > 0$, which indicates that $\chi_r^C(c)$ is either non-convex or has a zero branch for $c > c_r^*$ (see [33, Sec. 4.4]). Since this is a special case of Theorem III.1, the second alternative applies, i.e., $\chi_r^C(c) = 0$ for all $c > c_r^*$, which implies that the right tail of $T^{-1}C_T$ decays slower than e^{-T}. Similar rate functions with zero branches also arise in stochastic collision models [9, 17], as well as in non-Markovian random walks [10], and are related to a speed in the LDP that grows slower than T. For the absolute area of rBM, we do not know what the exact decay of the density of $T^{-1}C_T$ is above the mean or whether, in fact, the density satisfies the LDP. This is an open problem.

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Appendix A: Rate function of the absolute area for BM

The SCGF, defined in (VI.25), is known to be given for BM without resetting by the principal eigenvalue of the following differential operator:

$$\mathcal{L}_k = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + k|x|, \quad x \in \mathbb{R}, \tag{A.1}$$

called the tilted generator, so that

$$(\mathcal{L}_k\psi_k)(x) = \lambda(k)\psi_k(x), \tag{A.2}$$

where $\psi_k(x)$ is the associated eigenfunction satisfying the natural (Dirichlet) boundary conditions $\psi(x) \to 0$ as $x \to \pm\infty$ [34]. Since $|W_t|$ has the same distribution as BM reflected at zero, we can also obtain $\lambda(k)$ as the principal eigenvalue of

$$\mathcal{L}_k = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + kx, \quad x \geq 0, \tag{A.3}$$

with the Neumann boundary condition $\psi_k'(0) = 0$, which accounts for the fact that there is no current at the reflecting barrier, and the Dirichlet boundary condition $\psi_k(\infty) = 0$.

The solution $\psi_k(x)$ of both eigenvalue problems is given in terms of the Airy function, $\mathrm{Ai}(\zeta)$, with

$$\zeta = \Big(\frac{-2k}{\sigma^2}\Big)^{1/3}\Big(x - \frac{\lambda(k)}{k}\Big). \tag{A.4}$$

Imposing the boundary conditions yields a discrete eigenvalue spectrum, given by

$$\lambda^{(i)}(k) = \Big(\frac{\sigma^2}{2}\Big)^{1/3}(-k)^{2/3}\zeta_i', \tag{A.5}$$

where ζ_i' is the ith zero of $\mathrm{Ai}'(x)$.

The largest eigenvalue $\lambda^{(0)}(k)$ corresponds to the SCGF $\lambda_0(k)$ without resetting (see Fig. 4), which yields the rate function χ_0^C, shown in (VI.27), after applying the Legendre transform shown in (VI.26). The function $\lambda_0(k)$ is defined only for $k \leq 0$, but since it is steep at $k = 0$, the Gärtner-Ellis Theorem can be applied in this case.

Note that this spectral method can also be used to find the rate function χ_r^C of the absolute area of rBM, following the method explained in [23]. However, the expression of the generating function $\tilde{G}_0(k, s)$ in this case is explicit, so it is more convenient to use it, as done in the proof of Theorem VI.1, with the renewal formula of Theorem II.1.

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