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Abstract

In previous work by Avena and den Hollander [2], a model of a one-dimensional random walk in a dynamic random environment was proposed where the random environment is resampled from a given law along a growing sequence of deterministic times. In the regime where the increments of the resampling times diverge, which is referred to as the cooling regime, a weak law of large numbers and certain fluctuation properties were derived under the annealed measure. In the present paper we show that a strong law of large numbers and a quenched large deviation principle hold as well. In the cooling regime, the random walk can be represented as a sum of independent variables, distributed as the increments of a random walk in a static random environment over increasing periods of time. Our proofs require suitable multi-layer decompositions of sums of random variables controlled by moments bounds and concentration estimates. Along the way we derive two results of independent interest, namely, a concentration inequality for the cumulants of the displacement in the static random environment and an ergodic theorem that deals with limits of sums of triangular arrays representing the structure of the cooling regime. We close by discussing our present understanding of homogenisation effects as a function of the speed of divergence of the increments of the resampling times.

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1 Introduction, main results and discussion

Random walk in random environment is a model for a particle moving in an inhomogeneous potential. When the random environment is *static* this model exhibits striking features. Namely, there are regions where the random walk remains *trapped* for a long time. The presence of these traps leads to a local slow-down of the random walk in comparison to a homogeneous random walk, and may result in anomalous scaling, especially in low dimensions. At present, these slow-down phenomena have been fully understood only in dimension one (see Zeitouni [13], and references therein).

The situation where the random environment is *dynamic* has seen major progress in the last ten years. While the random environment evolves over time, it remains inhomogeneous but dissolves existing traps and creates new traps. Depending on the choice of the dynamics, the random walk behaviour can either be similar to that in the static model or be similar to that in the homogeneous model. Up to now, most dynamic models require strong space-time mixing conditions, guaranteeing negligible trapping effects and resulting in scaling properties similar to those of a homogeneous random walk (see Avena, Blondel and Faggionato [1], and references therein).

In Avena and den Hollander [2], a new random walk model was introduced, called *Random Walk in Cooling Random Environment* (RWCRE). This has a dynamic random environment, but differs from other dynamic models in that it allows for an explicit control of the time mixing in the environment. Namely, at time zero an i.i.d. random environment is generated, and this is *fully resampled* along an increasing sequence of deterministic times. If the resampling times increase rapidly enough, then we expect to see a behaviour close to that of the static model. Conversely, if the resampling times increase slowly enough, then we expect to see a behaviour that is close to the homogeneous model. Thus, RWCRE allows for different scenarios as a function of the speed of growth of the resampling times. The name "cooling" is used because the static model is sometimes called "frozen".

In order to advance our understanding of RWCRE, we need to acquire detailed knowledge of fluctuations and large deviations for the classical one-dimensional Random Walk in Random Environment (RWRE). Some of this knowledge is available from the literature, but other parts are not and need to be developed along the way. A few preliminary results were proved in Avena and den Hollander [2] under the *annealed* law. In the simplest scenario where the increments of the resampling times stay bounded, which is referred to as the *no-cooling regime*, full homogenisation takes place, and both a classical Strong Law of Large Numbers (SLLN) and a classical Central Limit Theorem (CLT) hold. Moreover, it was shown that as soon as the increments of the resampling times diverge, which is referred to as the *cooling regime*, a Weak Law of Large Numbers (WLLN) holds with an asymptotic speed that is the *same* as for the corresponding RWRE [2, Theorem 1.5]. As far as fluctuations are concerned, for the case where the RWRE is in the so-called Sinai regime (recurrent, subdiffusive, non-standard limit law; see Sinai [11], Kesten [9]), it was shown that RWCRE exhibits Gaussian fluctuations with a scaling that *depends* on the speed of divergence of the increments of the resampling times [2, Theorem 1.6]. The proof of this fact requires that the convergence to the limit law for the corresponding RWRE is in L^p for some $p > 2$. In [2, Appendix C] it was shown that the convergence is in L^p for all $p > 0$.

In the present paper we pursue a more refined investigation of RWCRE. We focus on the cooling regime and aim for a deeper understanding of homogenisation effects. In particular, we derive a SLLN and a *quenched* Large Deviation Principle (LDP), with a limiting speed and a rate function that are the *same* as for the corresponding RWRE (Theorems 1.10 and 1.11 below). Both results are not unexpected, but at the same time are far from obvious. As we will see, they lead to some subtle surprises, which we discuss below. A crucial ingredient in both proofs is a general limit property we call *cooling ergodic theorem*, which is needed to control certain variables representing the structure of the cooling regime (Theorem 1.12 below). This theorem not only is a key tool in our proofs, it will also be useful to address other questions not investigated here. To prove the SLLN and the LDP we also need certain *concentration inequalities* for the corresponding RWRE (Theorem 1.13 below).

Outline. In Section 1.1 we define one-dimensional RWRE and recall some basic facts that are used throughout the paper. In Section 1.2 we define RWCRE. In Section 1.3 we state our four main theorems and provide some insight into their proofs. In Section 1.4 we discuss what is known about RWCRE, explain how the results derived so far relate to each other, and state a number of open problems. The remainder of the paper is devoted to the proofs: Section 2 for the cooling ergodic theorem of RWCRE, Section 3 for the concentration inequalities of RWRE, and Section 4 for the SLLN and the LDP of RWCRE.

1.1 RWRE: some basic facts

Throughout the paper we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $\mathbb{N} = \{1, 2, \dots\}$. The classical one-dimensional static model is defined as follows. Let $\omega = \{\omega(x)\colon\ x \in \mathbb{Z}\}$ be an i.i.d. sequence with probability distribution

$$\mu = \alpha^{\mathbb{Z}} \tag{1.1}$$

for some probability distribution α on $(0, 1)$. We write $\langle \cdot \rangle$ to denote the expectation w.r.t. α.

Definition 1.1 (RWRE). Let ω be an environment sampled from μ. We call *Random Walk in Random Environment* the Markov chain $Z = (Z_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{Z}$ and transition probabilities

$$P^{\omega}(Z_{n+1} = x + e \mid Z_n = x) = \begin{cases} \omega(x) & \text{if } e = 1, \\ 1 - \omega(x) & \text{if } e = -1, \end{cases} \qquad n \in \mathbb{N}_0. \tag{1.2}$$

We denote by $P^{\omega}_x(\cdot)$ the *quenched* law of the Markov chain identified by the transitions in (1.2) starting from $x \in \mathbb{Z}$, and by

$$P^{\mu}_x(\cdot) = \int P^{\omega}_x(\cdot)\, \mu(\mathrm{d}\omega),$$

the corresponding *annealed* law.

The understanding of one-dimensional RWRE is well developed, both under the quenched and the annealed law. For a general overview, we refer the reader to the lecture notes by Zeitouni [13]. Here we collect some basic facts and definitions that will be needed throughout the paper.

The asymptotic properties of RWRE are controlled by the distribution of the ratio of the transition probabilities to the left and to the right at the origin, i.e.,

$$\rho = \frac{1 - \omega(0)}{\omega(0)}. \tag{1.3}$$

We will impose that the support of α is contained in an interval of the form $[\mathfrak{c}, 1-\mathfrak{c}]$ for some $\mathfrak{c} > 0$. This corresponds to a *uniform ellipticity* condition on μ, meaning that

$$\mu\big(\omega\colon\ 0 < \mathfrak{c} \le \omega(x) \le 1-\mathfrak{c} < 1,\, \forall\, x \in \mathbb{Z}\big) = 1. \tag{1.4}$$

Let $\rho_{\max}$ and $\rho_{\min}$ denote the maximum and minimum of ρ over the support of α. We will further impose that

$$\rho_{\min} < 1 < \rho_{\max}. \tag{1.5}$$

The inequalities in (1.5) ensures that we are in the *nested* situation, i.e., at some sites the random walk prefers to go to the right while at other sites it prefers to go to the left.

Definition 1.2 (Basic environment distribution). We call a probability distribution μ on $(0,1)^{\mathbb{Z}}$ *basic* (= i.i.d., uniform elliptic, nested) if (1.1), (1.4) and (1.5) hold.

The following proposition due to Solomon [12] characterises recurrence versus transience and asymptotic speed. To state the result in a simple form we may assume without loss of generality that

$$\langle \log \rho \rangle \le 0. \tag{1.6}$$

The case where $\langle \log \rho \rangle > 0$ follows by a reflection argument. Indeed, define $\widetilde{\omega}$ by $\widetilde{\omega}(x) = 1-\omega(x)$, $x \in \mathbb{Z}$. From (1.2) we see that $P_0^{\omega}(-Z_n \in \cdot) = P_0^{\widetilde{\omega}}(Z_n \in \cdot)$. Therefore, statements for the left of the origin can be obtained from statements for the right of the origin in the reflected environment and so (1.6) is assumed for convenience.

Proposition 1.3 (Recurrence, transience and speed of RWRE [12]).
Suppose that μ is basic and that (1.6) holds. Then:

- *Z is recurrent when $\langle \log \rho \rangle = 0$.*
- *Z is transient to the right when $\langle \log \rho \rangle < 0$.*
- *For μ-a.e. ω, P_0^{ω}-a.s.,*

$$\lim_{n\to\infty} \frac{Z_n}{n} = v_\mu = \begin{cases} 0, & \text{if } \langle \rho \rangle \ge 1, \\ \frac{1-\langle \rho \rangle}{1+\langle \rho \rangle} > 0, & \text{if } \langle \rho \rangle < 1. \end{cases} \tag{1.7}$$

The above proposition shows that the speed of RWRE is a deterministic function of μ (or of α; recall (1.1)). Note that for α such that $\langle \log \rho \rangle < 0$ and $\langle \rho \rangle \ge 1$, the random walk is transient to the right with zero speed. In this regime Z diverges, but only sublinearly due to the presence of *traps*, i.e., local regions of the environment pushing the random walk against its global drift.

Similar trapping effects give rise to other anomalous behaviour for fluctuations and large deviations. In order to state the latter, we recall that a family of probability measures $(P_n)_{n\in\mathbb{N}}$ defined on the Borel sigma-algebra of a topological space $(\mathcal{S}, \mathcal{T})$ is said to satisfy the LDP with rate n and with rate function $I\colon\ \mathcal{S} \to [0,\infty]$ when

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log P_n(\mathcal{O}) &\ge -\inf_{x\in\mathcal{O}} I(x) \qquad \forall\, \mathcal{O} \subset \mathcal{S} \text{ open}, \\ \limsup_{n\to\infty} \frac{1}{n} \log P_n(\mathcal{C}) &\le -\inf_{x\in\mathcal{C}} I(x) \qquad \forall\, \mathcal{C} \subset \mathcal{S} \text{ closed}, \end{aligned} \tag{1.8}$$

I has compact level sets and $I \not\equiv \infty$ (see e.g. den Hollander [8, Chapter III]). The following proposition due to Greven and den Hollander [7] identifies the LDP for the empirical speed under the *quenched* law.

Proposition 1.4 (**Quenched LDP for RWRE displacements** [7]). *Suppose that μ is basic. Then, for μ-a.e. ω, $(Z_n/n)_{n\in\mathbb{N}}$ under P_0^ω satisfies the LDP on $\mathbb{R}$ with rate n and with a convex and deterministic rate function $\mathcal{I} = \mathcal{I}_\mu$.*

See [7] for a representation of $\mathcal{I}$ in terms of random continued fractions and Fig. 2 for the qualitative behaviour of $\mathcal{I}$ on different regimes.

In the sequel we will need refined results about the cumulant generating function of Z_n/n. For that we need to introduce the hitting times to the right

$$H_n = \inf\{\, m \in \mathbb{N} \colon Z_m = n \,\}, \quad n \in \mathbb{N}, \tag{1.9}$$

state the *weak* LDP for H_n/n, which was derived in Comets, Gantert and Zeitouni [5], and show its relation with the LDP for Z_n/n. See also den Hollander [8, Chapter VII]. We recall that for the weak LDP the second line in (1.8) is only required to hold for compact sets, and the rate function is only required to be lower semi-continuous.

Proposition 1.5 (**Quenched LDP for RWRE hitting times** [5]). *Suppose that μ is basic. Then, for μ-a.e. ω, $(H_n/n)_{n\in\mathbb{N}}$ under P_0^ω satisfies the weak LDP on $\mathbb{R}$ with rate n and with a convex and deterministic weak rate function $\mathcal{J} = \mathcal{J}_\mu$ given by (see Fig. 1)*

$$\mathcal{J}(x) = \sup_{\lambda\in\mathbb{R}} \big[\lambda x - \mathcal{J}^*(\lambda)\big], \qquad x \in \mathbb{R}, \tag{1.10}$$

where

$$\mathcal{J}^*(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log E_0^\omega\big[e^{\lambda H_n}\big] \quad \omega-a.s., \qquad \lambda \in \mathbb{R}. \tag{1.11}$$

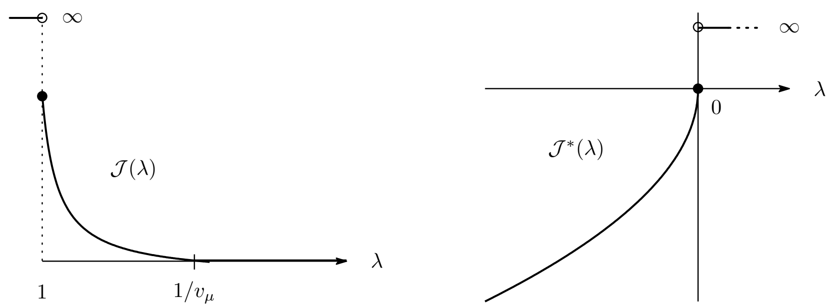

Figure 1: *Left:* Graph of $\mathcal{J}$, the quenched rate function of RWRE hitting times in (1.10). *Right:* Graph of $\mathcal{J}^*$, the scaled cumulant generating function of RWRE hitting times in (1.11).

For the hitting times to the left, defined by (1.9) with $n \in -\mathbb{N}$, we have the weak rate function $\widetilde{\mathcal{J}} = \widetilde{\mathcal{J}}_\mu$:

$$\widetilde{\mathcal{J}}(x) = \mathcal{J}(x) - \langle \log \rho \rangle, \qquad x \in \mathbb{R}. \tag{1.12}$$

Moreover, the following relation between $\mathcal{J}$ and $\mathcal{I}$ holds (see [8, Chapter VII]):

$$\mathcal{I}(x) = \begin{cases} x\mathcal{J}(1/x), & x \in (0,1], \\ 0, & x = 0, \\ (-x)\mathcal{J}(1/(-x)), & x \in [-1,0). \end{cases} \tag{1.13}$$

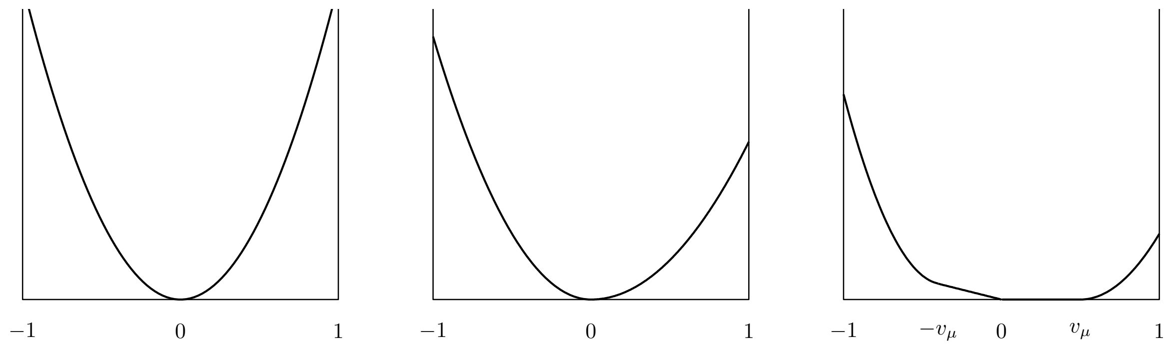

Figure 2: Graph of $\mathcal{I}$, the quenched rate function of RWRE displacements in (1.13). Three cases are shown from left to right: recurrent, transient with zero speed, transient with positive speed.

The empirical speed of RWRE also satisfies the LDP under the annealed law.

Proposition 1.6 (**Annealed LDP for RWRE displacements** [5]). *Suppose that μ is basic. Then $(Z_n/n)_{n\in\mathbb{N}}$ under P_0^μ satisfies the LDP on $\mathbb{R}$ with rate n and with a convex rate function $\mathcal{I}^{ann} = \mathcal{I}_\mu^{ann}$.*

As shown in [5], the annealed and the quenched rate function are related through the following variational principle

$$\mathcal{I}^{\mathrm{ann}}(\theta) = \mathcal{I}^{\mathrm{ann}}_\mu(\theta) = \inf_\nu \big[\mathcal{I}_\nu(\theta) + |\theta|\, h(\nu \mid \mu)\big], \tag{1.14}$$

where $\mathcal{I}_\nu$ is the quenched rate function associated with a random environment that has law ν, $h(\nu \mid \mu)$ denotes the relative entropy of ν with respect to μ, and the infimum runs over the set of probability measures on $(0,1)^{\mathbb{Z}}$ endowed with the weak topology (see [5] for more details). In particular, $\mathcal{I}^{\mathrm{ann}}$ is qualitatively similar to $\mathcal{I}$ in Fig. 2, in the sense that $\mathcal{I}^{\mathrm{ann}}$ is strictly decreasing on $[-1,0]$, zero on $[0,v_\mu]$, and strictly increasing on $[v_\mu,1]$. The presence of the flat piece $[0,v_\mu]$ in the positive speed case makes our analysis more delicate, and we will need the following large deviation bound characterising the right decay when zooming in on the flat piece:

Proposition 1.7 (**Refined annealed large deviations in the flat piece** [6]). *Suppose that μ is basic and that $\langle\rho\rangle < 1$. Then there is a unique $s > 1$ satisfying $\langle\rho^s\rangle = 1$ such that, for any $\mathcal{O} \subset (0,v_\mu)$ open and separated from v_μ,*

$$\lim_{n\to\infty} \frac{1}{\log n} \log P_0^\mu\left(\frac{Z_n}{n} \in \mathcal{O}\right) = 1 - s. \tag{1.15}$$

1.2 RWCRE: Cooling

The cooling random environment is the *space-time* random environment built by partitioning $\mathbb{N}_0$, and assigning independently to each piece an environment sampled from μ in (1.1) (see Fig. 3). Formally, let $\tau\colon \mathbb{N}_0 \to \mathbb{N}_0$ be a strictly increasing function with $\tau(0) = 0$, referred to as the *cooling map.* The cooling map determines a sequence of *refreshing* times $(\tau(k))_{k\in\mathbb{N}_0}$ that we use to construct the dynamic random environment.

Definition 1.8 (Cooling Random Environment). Given a cooling map τ, let $\Omega = (\omega_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of random variables with law μ in (1.1). The *cooling random environment* is built from the pair (Ω, τ) by assigning the environment ω_k to the k-th interval I_k defined by

$$I_k = [\tau(k-1), \tau(k)), \qquad k \in \mathbb{N}. \tag{1.16}$$

In the present paper we consider the *cooling regime*, i.e., we consider τ such that the length of I_k in (1.16) diverges:

$$T_k = \tau(k) - \tau(k-1), \qquad \lim_{k\to\infty} T_k = \infty. \tag{1.17}$$

The role of this assumption is clarified in Section 1.4.

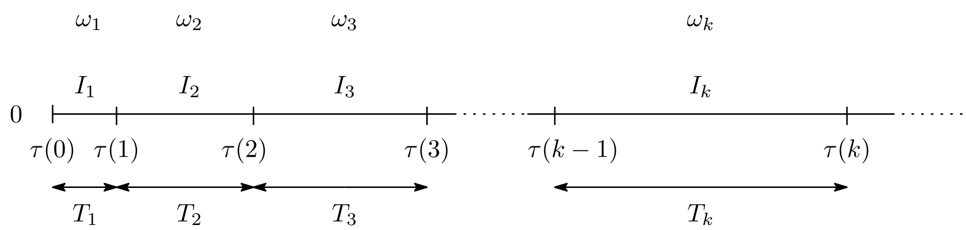

Figure 3: Structure of the cooling random environment (Ω, τ).

Definition 1.9 (RWCRE). Let τ be a cooling map and Ω an environment sequence sampled from $\mu^{\mathbb{N}}$. We call *Random Walk in Cooling Random Environment* the Markov chain $X = (X_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{Z}$ and transition probabilities

$$P^{\Omega,\tau}(X_{n+1} = x + e \mid X_n = x) = \begin{cases} \omega_{\ell(n)}(x), & e = 1, \\ 1 - \omega_{\ell(n)}(x), & e = -1, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.18}$$

where

$$\ell(n) = \inf\{k\colon\ \tau(k) > n\} \tag{1.19}$$

is the index of the interval n belongs to. Similarly to Definition 1.1, we denote by

$$P_x^{\Omega,\tau}(\cdot) \quad \textit{and} \quad P_x^{\mu,\tau}(\cdot) = \int P_x^{\Omega,\tau}(\cdot)\, \mu^{\mathbb{N}}(\mathrm{d}\Omega), \tag{1.20}$$

the corresponding *quenched* and *annealed* laws, respectively.

In words, RWCRE moves according to a given environment sampled from μ, until the next refreshing time $\tau(k)$, when a new environment is sampled from μ. Equivalently, the random walk trajectory is independent across the intervals, and during each interval I_k moves like a RWRE in the environment ω_k. In view of assumption (1.17), the environment is resampled along a diverging sequence of time increments. Our goal is to understand in what way this makes RWCRE behave similarly as RWRE (see Section 1.4 below).

The position X_n of RWCRE admits the following key decomposition into pieces of RWRE. Define the *refreshed increments* and the *boundary increment* as

$$Y_k = X_{\tau(k)} - X_{\tau(k-1)}, \quad k \in \mathbb{N}, \qquad \bar{Y}^n = X_n - X_{\tau(\ell(n)-1)}, \tag{1.21}$$

and the *running time at the boundary* as

$$\bar{T}^n = n - \tau(\ell(n) - 1). \tag{1.22}$$

Note that, by (1.17),

$$\sum_{k=1}^{\ell(n)-1} T_k + \bar{T}^n = n. \tag{1.23}$$

By construction, we can write X_n as the sum

$$X_n = \sum_{k=1}^{\ell(n)-1} Y_k + \bar{Y}^n, \quad n \in \mathbb{N}_0. \tag{1.24}$$

This decomposition shows that, in order to analyse X, we must analyse the vector

$$(Y_1, \cdots, Y_{\ell(n)-1}, \bar{Y}^n) \tag{1.25}$$

consisting of independent components, each distributed as an increment of Z (defined in Section 1.1) over a given time length determined by τ and n. Fig. 4 illustrates this piece-wise decomposition of X_n. More precisely, for any measurable function $f\colon \mathbb{Z} \to \mathbb{R}$, any Ω sampled from $\mu^{\mathbb{N}}$ and any τ,

$$E_0^{\Omega,\tau}\left[f(Y_k)\right] = E_0^{\omega_k}\left[f(Z_{T_k})\right], \qquad E_0^{\Omega,\tau}\left[f(\bar{Y}^n)\right] = E_0^{\omega_{\ell(n)}}\left[f(Z_{\bar{T}^n})\right]. \tag{1.26}$$

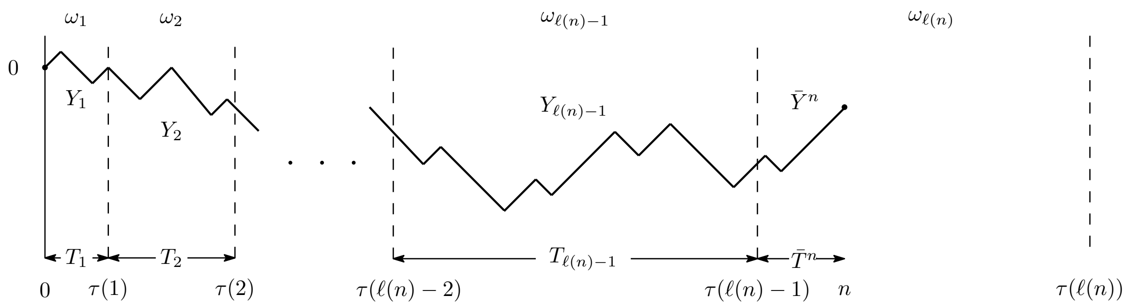

Figure 4: The decomposition of RWCRE in pieces of RWRE as presented in (1.24).

1.3 Main results

We can now state our main results for the asymptotic behaviour of RWCRE.

Theorem 1.10 (SLLN for RWCRE displacements). *Suppose that μ is basic and that τ satisfies* (1.17). *Then, for $\mu^{\mathbb{N}}$- a.e. Ω,*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_\mu \qquad P_0^{\Omega,\tau} - a.s. \tag{1.27}$$

with v_μ as in (1.7).

Theorem 1.11 (**Quenched LDP for RWCRE displacements**). *Suppose that μ is basic and that τ satisfies* (1.17)*. Then, for $\mu^{\mathbb{N}}$-a.e. Ω, $(X_n/n)_{n\in\mathbb{N}}$ under $P_0^{\Omega,\tau}$ satisfies the LDP on $\mathbb{R}$ with rate n and with the same rate function $\mathcal{I} = \mathcal{I}_\mu$ as in Proposition 1.4.*

Both theorems will be discussed in Section 1.4 and will be proved in Section 4. Their derivation will be based on the following general limit property tailored to RWCRE.

Theorem 1.12 (**Cooling Ergodic Theorem**). *Let $(\psi_n^{(k)})_{n,k\in\mathbb{N}}$ be an array of real-valued random variables with law $\mathbb{P}$ such that the following assumptions hold:*

$(A1)$ *For all $k, k' \in \mathbb{N}$ with $k \neq k'$, $(\psi_n^{(k)})_{n\in\mathbb{N}}$ and $(\psi_n^{(k')})_{n\in\mathbb{N}}$ are independent.*

$(A2)$ *For all $k \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} |\psi_{n+1}^{(k)} - \psi_n^{(k)}| \leq C$ for some $C > 0$.*

$(A3)$ *There exist $L \in \mathbb{R}$, $\delta > 0$, such that for all $\varepsilon > 0$, there is a $C' = C'(\varepsilon)$ for which*

$$\mathbb{P}\left(\left|\frac{\psi_n^{(k)}}{n} - L\right| > \varepsilon\right) < \frac{C'}{n^\delta}. \tag{1.28}$$

Then, for any cooling map τ satisfying (1.17),

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{\ell(n)-1} \psi_{T_k}^{(k)} + \psi_{\bar{T}^n}^{(\ell(n))}\right) = L \qquad \mathbb{P}-a.s. \tag{1.29}$$

with $\ell(n)$ as in (1.19)*, T_k as in* (1.17) *and $\bar{T}^n$ as in* (1.22)*.*

Theorem 1.12 is useful for controlling limits of sums of the form appearing in (1.29). Its proof is presented in Section 2 and is based on moments bounds and concentration estimates, applied to a further decomposition into what we call refreshed, boundary and deterministic terms, respectively. Theorem 1.12 is a key ingredient in our paper.

To check Assumption $(A3)$ is a challenge. In itself, $(A3)$ is only a mild decay requirement, but it forces us to derive concentration inequalities for RWRE, which is a non-trivial task. For the SLLN in Theorem 1.10, the required concentration inequalities are already at our disposal, since they are encoded in the annealed large deviation bound recalled in Proposition 1.7. However, for Theorem 1.11 a concentration inequality for the cumulants of the displacements of RWRE is needed which, to the best of our knowledge, is not available in the literature. Therefore we state such a result, which is of interest in itself.

Theorem 1.13 (**Concentration of cumulants for RWRE displacements**). *Suppose that μ is basic. Then, for any $\lambda \in \mathbb{R}$, $\delta \in (0,1)$ and $\varepsilon > 0$ there are $C, c \in (0,\infty)$ (depending on $\mu, \lambda, \delta, \varepsilon$) such that*

$$\mu\left(\omega\colon \left|\frac{1}{n}\log E_0^\omega\left[e^{\lambda Z_n}\right] - \mathcal{I}^*(\lambda)\right| > \varepsilon\right) \leq C\, e^{-cn^{1-\delta}} \qquad \forall\, n \in \mathbb{N}, \tag{1.30}$$

where $\mathcal{I}^*$ is the Legendre transform of the rate function $\mathcal{I}$ in Proposition 1.4, i.e.,

$$\mathcal{I}^*(\lambda) = \sup_{x\in\mathbb{R}} \left[\lambda x - \mathcal{I}(x)\right], \qquad \lambda \in \mathbb{R}. \tag{1.31}$$

The proof of Theorem 1.13 is given in Section 3. The idea is to prove an analogous concentration inequality for the hitting times, and then transfer this to displacements via successive approximations. As is usual in the context of RWRE, hitting times are easier to handle and their concentration will follow from an adaptation of an argument presented in the proof of Lemma 3.4.10 in Zeitouni [13].

1.4 Discussion

No-cooling: bounded time increments. The regime where assumption (1.17) does not hold and the increments T_k in (1.17) are of order one has been investigated in [2]. Due to the fast resampling, no trapping effects enter the game and full homogenisation takes place. In fact, the decomposition in (1.24) gives us a sum of almost i.i.d. random variables and the resulting behaviour is as if X were a homogeneous Markov chain: [2, Theorem 1.4] shows a corresponding classical SLLN and classical CLT under the annealed law.

Weak and strong law of large numbers. Theorem 1.10 states that, as soon as the cooling is effective, i.e., assumption (1.17) is in force, the asymptotic speed exists a.s., is deterministic and is equal to the one for RWRE. The same statement has been derived in weak form in [2, Theorem 1.5]. The strong form presented here requires a much more involved proof, based on RWRE concentration inequalities. In fact, Theorem 1.10 is far from trivial because the cooling map allows for fluctuations that could in principle hamper the almost sure convergence. As the proof reveals, the fact that this is not the case comes from a non-trivial averaging due to the cooling resampling mechanism. Roughly speaking, the slower the cooling, the stronger are the fluctuations of the constituent pieces in the sum in (1.24), but these fluctuations average out, as will be shown with the help of the moments bounds and the concentration estimates mentioned earlier.

Large deviations and fluctuations. As soon as the increments between the resampling times diverge, the rate function in the LDP for RWCRE in Theorem 1.11 is the same as for RWRE. In words, the cost to deviate from the typical speed is determined by the trapping in a fixed environment and the resampling has *no further homogenising effect*. This is true when we look on an exponential scale, but we may expect RWRE and RWCRE to show different large deviation behaviour when we zoom in on the flat piece $[0, v_\mu]$ when $v_\mu > 0$ (see Fig. 2). Theorem 1.11 deserves further comments because, when we look at fluctuations, RWRE and RWCRE actually give rise to *different scaling limits*. This has been proved for recurrent RWRE, which exhibits non-standard fluctuations after scaling by $\log^2 n$. Indeed, [2, Theorem 1.6] shows that, for certain τ's under the annealed law, RWCRE exhibits Gaussian fluctuations after scaling by a factor that *grows faster than* $\log^2 n$ and *depends on the cooling map* τ. Similar scenarios, and even the presence of a crossover, have been conjectured to hold for RWCRE in other regimes (see Table 1). This may all sound paradoxical, because it is folklore to expect that the zeros of the rate function in an LDP encode information on the order of the fluctuations. However, the latter is only true when the rate function is smooth near its zeros. This is not the case for the rate functions in Fig. 2, and so the paradox is explained.

Relaxing the i.i.d. assumption on μ. It is worth mentioning that the i.i.d. assumption on μ made in (1.1) can in principle be relaxed to the assumption that μ is stationary and ergodic with respect to translations. The reader is invited to check that all the steps in the proofs below work in this more general setting. On the other hand, we will make use of certain known properties of RWRE some of which require further technical assumptions to guarantee local product structure (see e.g. [13, Theorem 2.4.3, p. 236] for the extension of Proposition 1.7).

Comparison and open problems. We conclude by summarising our present understanding of RWCRE based on the results derived here and in [2]. Let us stress again that the RWCRE model can be seen as a model that interpolates between the classical static model (i.e., $\tau(1) = \infty$) and the model with i.i.d. resamplings every unit of time (i.e., $\tau(n) = n$). The latter reduces to a homogeneous nearest-neighbour random walk under the annealed measure, but even under the quenched law the independent space-time structure leads to a strong homogenising scenario for which e.g. a classical CLT holds (see e.g. Boldrighini, Minlos and Pellegrinotti [3]). The interesting features therefore appear as we explore different cooling regimes, which allow for a competition between the effect of traps in the static environment and the effect of homogenisation coming from the resampling. Table 1 gives a qualitative comparison for RWRE, RWCRE and standard homogeneous nearest-neighbour random walk, abbreviated as RW. In view of the discussion above, the no-cooling regime $\tau(n) \sim n$ is in the same "universality class" as homogeneous random walk, which is why it is put in the same column as RW.

Model	RW $\simeq$ No-Cooling	RWCRE	RWRE
Medium	Homogeneous	Cooling	Static
Recurrence	local drift $= 0$	**global, depending on** (τ, μ)**?**	$\langle \log \rho \rangle = 0$, global
Speed	local drift	v_μ (non-local)	v_μ (non-local)
LDP rate n	Cramér-analytic rate fn	non-analytic rate fn $\mathcal{I}$	non-analytic rate fn $\mathcal{I}$
Fluctuations $\langle \log \rho \rangle = 0$	CLT	$\log \tau(n) \leq cn$: Gaussian $\log \tau(n) \geq cn$**: Kesten-Sinai?**	Kesten-Sinai scale: $\log^2 n$
Fluctuations $\langle \log \rho \rangle < 0$		**???**	Kesten-Kozlov-Spitzer $s < 2$ stable law $s > 2$ CLT

Table 1: Comparison among standard RW, RWCRE and RWRE. Marked in boldface are what we consider challenging open problems.

Let us comments on the most relevant items in Table 1.

- *Recurrence vs Transience:* While for a homogenous RW we know that it is recurrent if and only if the corresponding *local* drift is zero, for RWRE the recurrence criterion is encoded in the condition $\langle \log \rho \rangle = 0$ (recall Proposition 1.3). In particular, it can happen that the local drift is non-zero, but still the above condition holds and the random walk is recurrent. In fact, a random walk in a non-homogeneous environment builds up non-negligible correlations over time, and its long-time behaviour is a truly *global* feature. For RWCRE we expect some subtle surprises related to the fluctuations of the corresponding RWRE. In particular, we expect a non-local criterion as for RWRE, but controlled by a delicate interplay between the environment law μ and the cooling map τ.

- *Asymptotic Speed:* As for the recurrence criterion, the asymptotic speed of a homogeneous RW is given by its local drift, while for RWRE it is influenced by the presence of the traps. Theorem 1.10 shows that for any cooling map subject to (1.17), RWCRE has the same speed as RWRE. In other words, on scale $1/n$ the long-time behaviours in the two models are equivalent and emerge as global features.

- *Large Deviations:* Concerning large deviations of order n, for homogeneous RW displacements Cramér's theorem tells us that their probabilities decay exponentially fast and are determined by a smooth rate function (see e.g. [8, Chapter I]). On the other hand, as we saw in Section 1.1, large deviations for RWRE are drastically different, both under the quenched and the annealed measure. In particular, both rate functions are non-analytic, not strictly convex when $\langle \log \rho \rangle \neq 0$, and contain an interval of zeros when $v_\mu > 0$. As previously discussed, Theorem 1.11 says that RWCRE satisfies the same LDP at rate n under the quenched law. Still, we may expect differences between quenched large deviations for RWRE and RWCRE when zooming in on the flat piece, i.e., when considering decays that are slower than the rate n encoded in the LDP. This constitutes yet another interesting open problem. Let us further note that we have not looked at the annealed LDP for RWCRE. We expect no surprises, namely, we believe that the annealed rate function for RWCRE is the same as the one for RWRE in Proposition 1.6. In fact, the proof presented in Section 4.2 could be easily adapted (and even significantly simplified) if we had existence and convexity in the annealed setting. In the quenched setting, existence and convexity are guaranteed by a general result derived in Campos *et. al* [4].

- *Fluctuations:* We conclude with what we consider to be the most challenging open problem, namely, to characterise the fluctuations for RWCRE, both under the quenched and the annealed measure. Some noteworthy results in this direction were derived in [2], where an annealed CLT for the no-cooling regime was shown [2, Theorem 1.4] and, for $\langle \log \rho \rangle = 0$ and τ growing either polynomially or exponentially, the annealed centered RWCRE displacement was shown to converge to a Gaussian law after an appropriate scaling that depends on τ [2, Theorem 1.6]. We expect that for sufficiently fast cooling a crossover occurs when $\langle \log \rho \rangle = 0$, namely, we expect to see the Kesten [9] limit law just as in the static case. What happens when $\langle \log \rho \rangle \neq 0$ seems to be even more intricate and remains fully unexplored. In this case for RWRE, Kesten, Kozlov and Spitzer [10] proved that annealed fluctuations can be Gaussian or can be characterised by proper stable law distributions, depending on the value of the root $s \in (1, \infty)$ defined in Proposition 1.7. It is reasonable to expect a rich pallet of behaviour depending on the interplay between τ and s. The quenched fluctuations seem even more difficult to analyse in view of the corresponding more delicate results for RWRE (see Zeitouni [13]).

2 Cooling ergodic theorem

In this section we prove Theorem 1.12.

Proof. We represent the sum in (1.29) as the convex combination

$$\frac{1}{n}\left(\sum_{k=1}^{\ell(n)-1} \psi^{(k)}_{T_k} + \psi^{(\ell(n))}_{\bar{T}^n}\right) = \sum_{k=1}^{\ell(n)-1} \frac{T_k}{n}\frac{\psi^{(k)}_{T_k}}{T_k} + \frac{\bar{T}^n}{n}\frac{\psi^{(\ell(n))}_{\bar{T}^n}}{\bar{T}^n}, \tag{2.1}$$

and use the abbreviations

$$\gamma_{k,n} = \frac{T_k}{n}\mathbb{1}_{\{k \leq \ell(n)-1\}}, \quad \bar{\gamma}^n = \frac{\bar{T}^n}{n}. \tag{2.2}$$

To prove (1.29), we subtract L from (2.1) and center each term in (2.1):

$$
\begin{aligned}
&\sum_{k\in\mathbb{N}} \gamma_{k,n} \frac{\psi^{(k)}_{T_k}}{T_k} + \bar{\gamma}^n \frac{\psi^{(\ell(n))}_{\bar{T}^n}}{\bar{T}^n} - L \\
&= \underbrace{\left(\sum_{k\in\mathbb{N}} \gamma_{k,n} \mathcal{C}_k\right)}_{R_n} + \underbrace{\bar{\gamma}^n \bar{\mathcal{C}}^n}_{B_n} + \underbrace{\left(\sum_{k\in\mathbb{N}} \gamma_{k,n}(L_k - L) + \bar{\gamma}^n(\bar{L}^n - L)\right)}_{D_n}
\end{aligned}
\tag{2.3}
$$

where

$$
\mathcal{C}_k = \frac{\psi^{(k)}_{T_k}}{T_k} - L_k, \quad \bar{\mathcal{C}}^n = \frac{\psi^{(\ell(n))}_{\bar{T}^n}}{\bar{T}^n} - \bar{L}^n, \qquad L_k = \mathbb{E}\left[\frac{\psi^{(k)}_{T_k}}{T_k}\right], \quad \bar{L}^n = \mathbb{E}\left[\frac{\psi^{(\ell(n))}_{\bar{T}^n}}{\bar{T}^n}\right]. \tag{2.4}
$$

The terms R_n, B_n and D_n correspond to *refreshed*, *boundary* and *deterministic* increments, respectively. In Sections 2.1, 2.2 and 2.3 we treat each of these terms separately, and show that they are asymptotically vanishing.

2.1 Refreshed term R_n

In this section we show that

$$
\limsup_{n\to\infty} |R_n| = 0 \quad \mathbb{P}-\text{a.s.} \tag{2.5}
$$

In view of Assumption $(A3)$, we split the increments of the resampling times according to a growth parameter $\gamma > 0$ such that $\gamma\delta > 1$,

$$
\begin{aligned}
\sum_{k\in\mathbb{N}} \gamma_{k,n}\mathcal{C}_k &= \underbrace{\sum_{k\in\mathbb{N}} \gamma_{k,n}\mathcal{C}_k \mathbb{1}_{\{T_k \geq k^\gamma\}}}_{R_n^L} + \underbrace{\sum_{k\in\mathbb{N}} \gamma_{k,n}\mathcal{C}_k \mathbb{1}_{\{T_k < k^\gamma\}}}_{R_n^S},
\end{aligned}
\tag{2.6}
$$

which corresponds to the sum of large and small increments, respectively. The goal is to bound both $\limsup_{n\to\infty} |R_n^L|$ and $\limsup_{n\to\infty} |R_n^S|$.

We first treat the sum of large increments R_n^L. By Assumption $(A3)$, if $\gamma\delta > 1$, then

$$
\sum_{k\in\mathbb{N}} \mathbb{P}\left(|\mathcal{C}_k| \mathbb{1}_{\{\, T_k \geq k^\gamma \,\}} > \varepsilon\right) < \infty. \tag{2.7}
$$

Applying the Borel-Cantelli lemma, we get

$$
\limsup_{k\to\infty} \mathcal{C}_k \mathbb{1}_{\{\, T_k \geq k^\gamma \,\}} \leq \varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.8}
$$

Since $\lim_{n\to\infty} \gamma_{k,n} = 0$ for fixed $k \in \mathbb{N}$ and $\sum_{k\in\mathbb{N}} \gamma_{k,n} \leq 1$, we obtain

$$
\limsup_{n\to\infty} |R_n^L| \leq \varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.9}
$$

To deal with the sum of small increments R_n^S, we apply the Markov inequality:

$$
\mathbb{P}\left(|R_n^S| > \varepsilon\right) \leq \frac{1}{\varepsilon^{2N}} \mathbb{E}\left[\left(\sum_{k=1}^n \gamma^S_{k,n} \mathcal{C}_k\right)^{2N}\right], \qquad \gamma^S_{k,n} = \gamma_{k,n} \mathbb{1}_{\{T_k < k^\gamma\}}. \tag{2.10}
$$

Since the $\mathcal{C}_k$'s are independent, zero-mean and bounded random variables, when we expand the $2N$-th power, all terms with first moment disappear. Therefore, we can estimate the moments as

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{n}\gamma_{k,n}^{S}\mathcal{C}_k\right)^{2N}\right]\\
&=\sum_{m=1}^{2N}\sum_{\substack{p_1+\cdots+p_m=2N\\ p_1,\cdots,p_m\geq 1}}\binom{2N}{p_1\cdots p_m}\sum_{k_1>\ldots>k_m}\mathbb{E}\left[\left(\gamma_{k_1,n}^{S}\mathcal{C}_{k_1}\right)^{p_1}\times\cdots\times\left(\gamma_{k_m,n}^{S}\mathcal{C}_{k_m}\right)^{p_m}\right]\\
&\leq\sum_{m=1}^{N}\sum_{\substack{p_1+\cdots+p_m=2N\\ p_1,\cdots,p_m\geq 2}}\binom{2N}{p_1\cdots p_m}\sum_{k_1>\ldots>k_m}\left(C\,\frac{k_1^{\gamma}}{n}\right)^{p_1}\times\cdots\times\left(C\,\frac{k_m^{\gamma}}{n}\right)^{p_m}\\
&\leq\sum_{m=1}^{N}\binom{2N-m-1}{m-1}n^m\,C^{2N}\,n^{2N(\gamma-1)}\leq c_N n^{N(2\gamma-1)},
\end{aligned}
\tag{2.11}
$$

where in the third line we use independence (Assumption (A1)) and bound each $\gamma_{k_i,n}^{S}\mathcal{C}_{k_i}$ by $C\,k_i^{\gamma}\,n^{-1}$ (Assumption (A2)). The right-hand side of (2.11) is summable in n as long as $\gamma<\frac{1}{2}$, because we can choose N arbitrarily large. This suggests that we need to further separate the argument.

Case $\delta>2$: single split. If $\delta>2$, then we can pick $\gamma<\frac{1}{2}$. From (2.10) and (2.11), we obtain

$$\sum_{n\in\mathbb{N}}\mathbb{P}\left(|R_n^S|>\varepsilon\right)<\infty. \tag{2.12}$$

Hence, by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty}|R_n^S|\leq\varepsilon\qquad\mathbb{P}-\text{a.s.} \tag{2.13}$$

Combining (2.6), (2.9) and (2.13), we get that for $\delta>2$,

$$\limsup_{n\to\infty}|R_n|\leq 2\varepsilon\qquad\mathbb{P}-\text{a.s.} \tag{2.14}$$

Case $\delta<2$: multi-layer split. If $\delta<2$, then we must pick $\gamma>\frac{1}{2}$ to satisfy $\gamma\delta>1$. Here we can no longer use the previous argument to obtain (2.12). To overcome this difficulty, we implement a multi-layer scheme distinguishing small and large increments according to a growth parameter. Take $M\in\mathbb{N}$ such that $\frac{M}{3}\delta>1$. Similarly to (2.6), define the first split:

$$
\begin{aligned}
R_n=\sum_{k\in\mathbb{N}}\gamma_{k,n}\mathcal{C}_k&=\underbrace{\sum_{k\in\mathbb{N}}\gamma_{k,n}^{1,L}\mathcal{C}_k}+\underbrace{\sum_{k\in\mathbb{N}}\gamma_{k,n}^{1,S}\mathcal{C}_k}\\
&=\qquad R_n^{1,L}\qquad+\qquad R_n^{1,S},
\end{aligned}
\tag{2.15}
$$

where

$$\gamma_{k,n}^{1,S}=\gamma_{k,n}\mathbb{1}_{\{T_k<k^{1/3}\}},\qquad\gamma_{k,n}^{1,L}=\gamma_{k,n}\mathbb{1}_{\{T_k\geq k^{1/3}\}}. \tag{2.16}$$

1. To estimate $R_n^{1,S}$, as in (2.10) and (2.11) we apply the Markov inequality, estimate the moments and obtain

$$\mathbb{P}\left(|R_n^{1,S}| > \varepsilon\right) \le c_N n^{-N/3}. \tag{2.17}$$

Since we can choose $N > 3$ in (2.17), we conclude that $\mathbb{P}(|R_n^{1,S}| > \varepsilon)$ is summable in n and therefore, by the Borel-Cantelli lemma,

$$\limsup_{n\to\infty} |R_n^{1,S}| \le \varepsilon \qquad \mathbb{P} - \text{a.s.} \tag{2.18}$$

Multi-layer Split	Moments	Concentration
$R_n = \underline{R_n^{1,L}} + R_n^{1,S}$	$\limsup_n R_n^{1,S} \le \varepsilon$	
$\downarrow$		
$R_n^{1,L} = \underline{R_n^{2,L}} + R_n^{2,S}$	$\limsup_n R_n^{2,S} \le \varepsilon$	
$\vdots$	$\vdots$	
$R_n^{M-1,L} = \boxed{R_n^{M,L}} + R_n^{M,S}$	$\limsup_n R_n^{M,S} \le \varepsilon$	$\limsup_n R_n^{M,L} \le \varepsilon$

Table 2: Splitting scheme.

2. To estimate $R_n^{1,L}$, the idea is to it decompose iteratively, as we did with R_n in (2.15), and control the small increments with moment bounds until we can apply concentration estimates. The resulting scheme is summarised in Table 2.

2a. To build the second split, we relabel the terms in $R_n^{1,L}$, i.e., we choose an ordered subsequence $(k_j^1)_{j\in\mathbb{N}}$ such that

$$\{k_1^1, k_2^1, \dots\} = \{j \in \mathbb{N} \colon T_j \ge j^{1/3}\}. \tag{2.19}$$

Denoting by $J(1;n)$ the cardinality of $\{k_j^1 \colon \tau(k_j^1) \le n\}$, we define the second split:

$$\begin{aligned} R_n^{1,L} = \sum_{j=1}^{J(1;n)} \gamma_{k_j^1,n} \mathcal{C}_{k_j^1} &= \underbrace{\sum_{j=1}^{J(1;n)} \gamma_{k_j^1,n}^{2,L} \mathcal{C}_{k_j^1}} + \underbrace{\sum_{j=1}^{J(1;n)} \gamma_{k_j^1,n}^{2,S} \mathcal{C}_{k_j^1}} \\ &= \qquad R_n^{2,L} \qquad + \qquad R_n^{2,S}, \end{aligned} \tag{2.20}$$

where

$$\gamma_{k_j^1,n}^{2,S} = \gamma_{k_j^1,n} \mathbb{1}_{\{T_{k_j^1} < j^{2/3}\}}, \qquad \gamma_{k_j^1,n}^{2,L} = \gamma_{k_j^1,n} \mathbb{1}_{\{T_{k_j^1} \ge j^{2/3}\}}. \tag{2.21}$$

Next, we abbreviate $n(1;J) = \inf\{\, n \colon J(1;n) = J \,\}$. Then, since

$$\limsup_{n\to\infty} |R_n^{2,S}| = \limsup_{J\to\infty} |R_{n(1;J)}^{2,S}|, \tag{2.22}$$

it suffices to show that $\limsup_{J\to\infty} |R_{n(1;J)}^{2,S}| \le \varepsilon$ $\mathbb{P}$-a.s. Note that, since $T_{k_j^1} \ge j^{1/3}$, we have a lower bound on $n(1;J)$:

$$n(1;J) \ge \sum_{j=1}^{J} T_{k_j^1} \ge \sum_{j=1}^{J} j^{1/3} \ge c\, J^{4/3}, \tag{2.23}$$

which yields

$$\gamma^{2,S}_{k^1_j,n(1;J)} \le \frac{j^{2/3}}{cJ^{4/3}} \le \frac{1}{cJ^{2/3}}. \tag{2.24}$$

Similarly to the first split, we apply the Markov inequality

$$\mathbb{P}\left(\left|R^{2,S}_{n(1;J)}\right| > \varepsilon\right) \le \frac{1}{\varepsilon^{2N}}\,\mathbb{E}\left[\left(\sum_{j=1}^{J} \gamma^{2,S}_{k^1_j,n}\mathcal{C}_{k^1_j}\right)^{2N}\right], \tag{2.25}$$

and estimate moments

$$\mathbb{E}\left[\left(\sum_{j=1}^{J} \gamma^{2,S}_{k^1_j,n(1;J)}\mathcal{C}_{k^1_j}\right)^{2N}\right] \le c_N\, J^{-N/3}. \tag{2.26}$$

Once we choose $N > 3$, this becomes summable in J. Therefore, by the Borel-Cantelli lemma and (2.22), we obtain

$$\limsup_{n\to\infty} |R^{2,S}_n| \le \varepsilon \quad \mathbb{P}-\text{a.s.} \tag{2.27}$$

2b. We continue the induction steps. For any $i < M$, after bounding $\limsup_{n\to\infty}|R^{i,S}_n|$, we relabel the terms in $R^{i,L}_n$ and define

$$\{k^i_1, k^i_2, \ldots\} = \{j \in \mathbb{N} \colon T_{k^{i-1}_j} \ge j^{i/3}\}. \tag{2.28}$$

Denoting by $J(i;n)$ the cardinality of $\{k^i_j \colon \tau(k^i_j) \le n\}$, we define the $(i+1)$-st split:

$$\begin{aligned} R^{i,L}_n = \sum_{j=1}^{J(i;n)} \gamma_{k^i_j,n}\mathcal{C}_{k^i_j} &= \underbrace{\sum_{j=1}^{J(i;n)} \gamma^{i+1,L}_{k^i_j,n}\mathcal{C}_{k^i_j}} + \underbrace{\sum_{j=1}^{J(i;n)} \gamma^{i+1,S}_{k^i_j,n}\mathcal{C}_{k^i_j}} \\ &= \qquad R^{i+1,L}_n \qquad + \qquad R^{i+1,S}_n, \end{aligned} \tag{2.29}$$

where

$$\gamma^{i+1,S}_{k^i_j,n} = \gamma_{k^i_j,n}\mathbb{1}_{\{T_{k^i_j} < j^{(i+1)/3}\}}, \qquad \gamma^{i+1,L}_{k^i_j,n} = \gamma_{k^i_j,n}\mathbb{1}_{\{T_{k^i_j} \ge j^{(i+1)/3}\}}. \tag{2.30}$$

Let $n(i;J) = \inf\{\, n \colon J(i;n) = J\,\}$. Then, by a similar computation as in (2.23) and (2.24), we have the following bounds:

$$n(i;J) \ge c\, J^{1+i/3}, \qquad \gamma^{i+1,S}_{k^i_j,n(i;J)} \le \frac{1}{cJ^{2/3}}. \tag{2.31}$$

Using the Markov inequality and moments bounds, we obtain

$$\sum_{J\in\mathbb{N}} \mathbb{P}\left(\left|R^{i+1,S}_{n(i;J)}\right| > \varepsilon\right) < \infty. \tag{2.32}$$

Therefore we conclude that

$$\limsup_{n\to\infty} |R^{i+1,S}_n| \le \varepsilon \quad \mathbb{P}-\text{a.s.} \tag{2.33}$$

2c. Once we bound $\limsup_{n\to\infty} |R_n^{M,S}|$, we are left with the term $R_n^{M,L}$. Since

$$R_n^{M,L} = \sum_{j\in\mathbb{N}} \gamma_{k_j^M,n} \mathbb{1}_{\{T_{k_j^M} > jM/3\}} \mathcal{C}_{k_j^M} \tag{2.34}$$

and $\frac{M}{3}\delta > 1$, we apply Assumption $(A3)$ to obtain

$$\sum_{j\in\mathbb{N}} \mathbb{P}\left(|\mathcal{C}_{k_j^M}| \mathbb{1}_{\{T_{k_j^M} > jM/3\}} > \varepsilon \right) < \infty. \tag{2.35}$$

Hence, by the Borel-Cantelli lemma,

$$\limsup_{j\to\infty} |\mathcal{C}_{k_j^{M-1}}| \mathbb{1}_{\{ T_{k_j^{M-1}} > jM/3 \}} \leq \varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.36}$$

Since $\lim_{n\to\infty} \gamma_{k,n} = 0$ for fixed k and $\sum_{k\in\mathbb{N}} \gamma_{k,n} \leq 1$, we obtain

$$\limsup_{n\to\infty} |R_n^{M,L}| \leq \varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.37}$$

3. Combining (2.37) and (2.33) for $i \in \{ 0, \ldots, M-1 \}$, we conclude that

$$\limsup_{n\to\infty} |R_n| \leq (M+1)\varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.38}$$

and (2.5) follows since $\varepsilon > 0$ is arbitrary.

2.2 Boundary term B_n

We next show that

$$\limsup_{n\to\infty} |B_n| = 0 \qquad \mathbb{P}-\text{a.s.} \tag{2.39}$$

Let $V_k = \sup\{ \bar{\gamma}_n |\bar{\mathcal{C}}^n| \colon n \in I_k \}$. Because $\cup_{k\in\mathbb{N}} I_k \supset \mathbb{N}_0$, we have $\limsup_{n\to\infty} \bar{\gamma}_n |\bar{\mathcal{C}}^n| = \limsup_{k\to\infty} V_k$. It therefore suffices to show that for arbitrary $\varepsilon > 0$,

$$\limsup_{k\to\infty} V_k \leq \varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.40}$$

If $\bar{\gamma}^n \leq \varepsilon$, then using Assumption $(A2)$ we can bound $|B_n| \leq C\,\varepsilon$. Therefore we only need to consider $\bar{\gamma}^n > \varepsilon$ (see Fig. 5), in which case we see that

$$n > \frac{\tau(k-1)}{1-\varepsilon} = N_{k,\varepsilon}. \tag{2.41}$$

If $\tau(k) \leq N_{k,\varepsilon}$, then the interval I_k can be ignored. Defining

$$\{ k_1, k_2, \ldots \} = \{ k \in \mathbb{N} \colon \tau(k) \geq N_{k,\varepsilon} \}, \tag{2.42}$$

our task reduces to showing that

$$\limsup_{j\to\infty} V_{k_j} \leq C\varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.43}$$

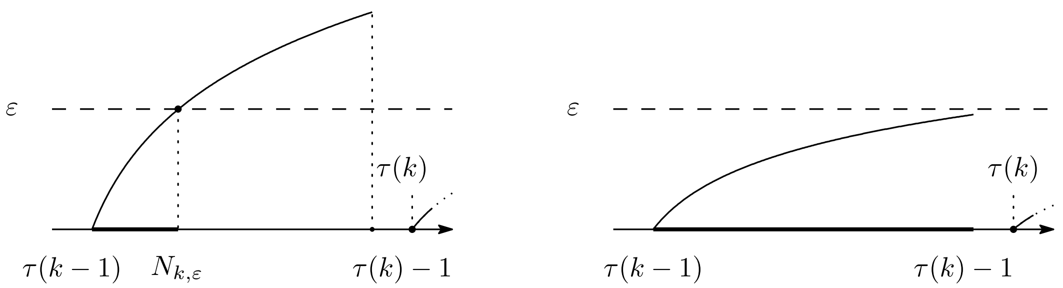

Figure 5: *Left:* If $N_{k,\varepsilon} < \tau(k)$, then $\bar{\gamma}_n \leq \varepsilon$ for $n \in [\tau(k-1), N_{k,\varepsilon})$ and $\bar{\gamma}_n > \varepsilon$ for $n \in [N_{k,\varepsilon}, \tau(k))$. *Right:* If $N_{k,\varepsilon} \geq \tau(k)$, then $\bar{\gamma}_n \leq \varepsilon$ for all $n \in I_k$.

Note that the subsequence $(\tau(k_j))_{j\in\mathbb{N}}$ grows at least exponentially fast once

$$\tau(k_j) \geq N_{k_j,\varepsilon} > (1+\varepsilon)\tau(k_j - 1) \geq (1+\varepsilon)\tau(k_{j-1}) \geq (1+\varepsilon)^{j-1}\tau(k_1). \tag{2.44}$$

Since $|B_n| \leq C\varepsilon$ for $n \in [\tau(k_j-1), N_{k_j,\varepsilon})$ and $\bar{\gamma}^n \leq 1$, by letting $m = \bar{T}^n = n - \tau(k_j - 1)$ and noting that $(1+\varepsilon)\tau(k_j-1) \leq N_{k,\varepsilon}$, we obtain

$$\begin{aligned} \mathbb{P}\left(V_{k_j} > C\varepsilon\right) &= \mathbb{P}\left(\sup_{N_{k,\varepsilon}\leq n<\tau(k_j)} \bar{\gamma}_n|\bar{C}^n| > C\varepsilon\right) \leq \mathbb{P}\left(\sup_{N_{k,\varepsilon}\leq n<\tau(k_j)} |\bar{C}^n| > C\varepsilon\right) \\ &\leq \mathbb{P}\left(\sup_{\varepsilon\tau(k_j-1)\leq m<T_{k_j}} \left|\frac{\psi_m^{(k_j)}}{m} - \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m}\right]\right| > C\varepsilon\right). \end{aligned} \tag{2.45}$$

By the union bound applied to (2.45), we arrive at

$$\mathbb{P}\left(V_{k_j} > C\varepsilon\right) \leq \sum_{m=\varepsilon\tau(k_j-1)}^{T_{k_j}} \mathbb{P}\left(\left|\frac{\psi_m^{(k_j)}}{m} - \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m}\right]\right| > C\varepsilon\right). \tag{2.46}$$

By Assumption $(A3)$, $\lim_{m\to\infty}\mathbb{E}[\psi_m^{(k)}/m] = L$ uniformly in k, and in particular, from (2.46), we get

$$\mathbb{P}\left(V_{k_j} > C\varepsilon\right) \leq \sum_{m=\varepsilon\tau(k_j-1)}^{T_{k_j}} \frac{\widetilde{C}}{m^\delta}, \tag{2.47}$$

for some $\widetilde{C} > 0$ not depending on k_j.

Case $\delta > 1$**.** From (2.47) we see that

$$\mathbb{P}\left(V_{k_j} > C\varepsilon\right) \leq \frac{C''}{\varepsilon\tau(k_j-1)^{\delta-1}}. \tag{2.48}$$

If $\delta > 1$, then together with (2.44) this implies that (2.48) is summable in j. Hence, by the Borel-Cantelli lemma, we obtain

$$\limsup_{j\to\infty} V_{k_j} \leq C\varepsilon \qquad \mathbb{P}-\text{a.s.} \tag{2.49}$$

Case $\delta < 1$**.** We need a more refined argument to prove (2.39). To control the boundary term on the interval I_{k_j}, we construct a sequence of times $(J_i)_{i\in\mathbb{N}_0}$ such that $J_0 = \varepsilon\tau(k_j - 1)$ and $J_i = (1+\varepsilon)J_{i-1}$. For $m \in (J_i, J_{i+1})$,

$$\left| \frac{\psi_m^{(k_j)}}{m} - \frac{\psi_{J_i}^{(k_j)}}{J_i} \right| \le \frac{1}{J_i} \left| \psi_m^{(k_j)} - \psi_{J_i}^{(k_j)} \right| \le C\varepsilon, \tag{2.50}$$

where we use Assumption $(A2)$ in the last inequality. Hence

$$\begin{aligned} \left| \frac{\psi_m^{(k_j)}}{m} - \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m} \right] \right| &\le \left| \frac{\psi_m^{(k_j)}}{m} - \frac{\psi_{J_i}^{(k_j)}}{J_i} \right| + \left| \frac{\psi_{J_i}^{(k_j)}}{J_i} - \mathbb{E}\left[\frac{\psi_{J_i}^{(k_j)}}{J_i} \right] \right| + \left| \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m} - \frac{\psi_{J_i}^{(k_j)}}{J_i} \right] \right| \\ &\le \left| \frac{\psi_{J_i}^{(k_j)}}{J_i} - \mathbb{E}\left[\frac{\psi_{J_i}^{(k_j)}}{J_i} \right] \right| + 2C\varepsilon. \end{aligned} \tag{2.51}$$

Therefore

$$\left\{ \sup_{\varepsilon\tau(k_j-1)\le m < T_{k_j}} \left| \frac{\psi_m^{(k_j)}}{m} - \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m} \right] \right| > 3C\varepsilon \right\} \subset \left\{ \sup_{i\in\mathbb{N}_0} \left| \frac{\psi_{J_i}^{(k_j)}}{J_i} - \mathbb{E}\left[\frac{\psi_{J_i}^{(k_j)}}{J_i} \right] \right| > C\varepsilon \right\}. \tag{2.52}$$

Hence, arguing as in (2.45) and using (2.52), the union bound and Assumption $(A3)$, we can estimate

$$\begin{aligned} \mathbb{P}\left(V_{k_j} > 3C\varepsilon \right) &= \mathbb{P}\left(\sup_{\varepsilon\tau(k_j-1)\le m < T_{k_j}} \left| \frac{\psi_m^{(k_j)}}{m} - \mathbb{E}\left[\frac{\psi_m^{(k_j)}}{m} \right] \right| > 3C\varepsilon \right) \\ &\le \mathbb{P}\left(\exists\, i \in \mathbb{N}_0 \colon \left| \frac{\psi_{J_i}^{(k_j)}}{J_i} - \mathbb{E}\left[\frac{\psi_{J_i}^{(k_j)}}{J_i} \right] \right| > C\varepsilon \right) \\ &\le \sum_{i\in\mathbb{N}_0} \frac{C'}{J_i^\delta} = \sum_{i\in\mathbb{N}_0} \frac{C'}{(1+\varepsilon)^i J_0^\delta} \le \frac{C'''}{J_0^\delta} = \frac{C'''}{\left(\varepsilon\tau(k_j-1)\right)^\delta}, \end{aligned} \tag{2.53}$$

which is summable in j due to (2.44). By the Borel-Cantelli lemma, we conclude that

$$\limsup_{n\to\infty} |B_n| \le 3C\varepsilon \qquad \mathbb{P} - \text{a.s.} \tag{2.54}$$

Since $\varepsilon > 0$ is arbitrary, (2.54) and (2.49) imply (2.39).

2.3 Deterministic term D_n

To conclude the proof of Theorem 1.12, it remains to show a.s. convergence to zero of D_n in (2.3). We make use of the following statement derived in [2, Lemma 3.1], which is a variant of the so-called Toeplitz lemma tailored to RWCRE.

Lemma 2.1. *Let* $(\gamma_{k,n})_{k,n\in\mathbb{N}}$, $\bar{\gamma}^n$ *be as in* (2.2) *and* $\bar{T}^n$ *be as in* (1.22). *Let* $(z_k)_{k\in\mathbb{N}}$ *be a real-valued sequence such that* $\lim_{k\to\infty} z_k = z^*$ *for some* $z^* \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \left(\sum_{k\in\mathbb{N}} \gamma_{k,n} z_k + \bar{\gamma}^n z_{\bar{T}^n} \right) = z^*. \tag{2.55}$$

Recall that $L = \lim_{k\to\infty} L_k$. By Lemma 2.1 with $z_k = L_k - L$ and $z^* = 0$, we conclude that

$$\limsup_{n\to\infty} D_n = 0 \qquad \mathbb{P} - a.s. \tag{2.56}$$

Combining (2.5), (2.39) and (2.56), we get the claim in (1.29). □

3 Concentration of cumulants for RWRE displacements

The proof of Theorem 1.13 will be divided into four steps, organised in Sections 3.1–3.4. The basic idea is to derive a concentration inequality for the hitting times of RWRE, which are easier to analyse, and then transfer this to a concentration inequality for the displacements of RWRE.

Here is the analogue of Theorem 1.13 for the hitting times defined in (1.9).

Proposition 3.1 (Concentration of cumulants for RWRE hitting times). *Suppose that μ is basic. Then, for any $\lambda \in \mathbb{R}$, $\delta \in (0,1)$ and $\varepsilon > 0$ there are $C, c \in (0,\infty)$ (depending on $\mu, \lambda, \delta, \varepsilon$) for which*

$$\mu\left(\omega\colon \left|\frac{1}{n}\log E_0^\omega\left[e^{\lambda H_n}\right] - \mathcal{J}^*(\lambda)\right| > \varepsilon\right) \le Ce^{-cn^{1-\delta}} \tag{3.1}$$

where $\mathcal{J}^*$ is the Legendre transform of the rate function $\mathcal{J}$ in Proposition 1.5.

The proof of Proposition 3.1 is given in Section 3.1 and is based on an adaptation of an argument presented in [13]. We will prove Theorem 1.13 by using Proposition 3.1 as follows:

- Section 3.2: Partition $[-1,1]$ into blocks and show that Theorem 1.13 follows from a concentration result for each block of the partition.
- Section 3.3: Show that concentration on half-lines implies concentration on blocks of the partition.
- Section 3.4: Prove concentration on half-lines by using Proposition 3.1 and the relation between hitting times and displacements for RWRE.

3.1 Concentration for hitting times

In this section we prove Proposition 3.1. The proof is based on [13, Lemma 3.4.10, p. 291].

Define, for fixed $K \in (0,\infty)$,

$$g^{\delta,n}(\omega) = \log E_0^\omega\left[e^{\lambda H_n}\mathbb{1}_{\{H_n<Kn\}}\mathbb{1}_{\{N^n<n^{\delta/2}\}}\right], \tag{3.2}$$

where $N^n = \sup_{x\in\mathbb{Z}} N_x^n$ and N_x^n is the number of visits at x before H_n. Note that $g^{\delta,n}(\omega)$ is a function of the environment coordinates $(\omega_i\colon\ |i| \le Kn)$. For $i \in \mathbb{N}$, define

$$\begin{aligned}
&\mathcal{F}_0 = \sigma\{\,\emptyset\,\}, \quad \mathcal{F}_1 = \sigma\{\,\omega_0\,\}, \quad \mathcal{F}_2 = \sigma\{\,\omega_0,\omega_1\,\}, \quad \mathcal{F}_3 = \sigma\{\,\omega_0,\omega_1,\omega_{-1}\,\},\\
&\quad\vdots\\
&\mathcal{F}_i = \sigma\{\,\omega_j\colon\ j \in (-\lceil i/2\rceil, \lfloor i/2\rfloor] \cap \mathbb{Z}\,\},
\end{aligned} \tag{3.3}$$

and denote by E^μ expectation with respect to μ. Then

$$E^\mu\left[g^{\delta,n}\mid\mathcal{F}_{2Kn}\right]=g^{\delta,n},\qquad E^\mu\left[g^{\delta,n}\mid\mathcal{F}_0\right]=E_0^\omega\left[g^{\delta,n}\right]. \tag{3.4}$$

Rewrite

$$g^{\delta,n}(\omega)-E^\mu\left[g^{\delta,n}\right]=\sum_{i=1}^{2Kn}d_i(\omega),$$

with

$$d_i(\omega)=E^\mu\left[g^{\delta,n}\mid\mathcal{F}_i\right](\omega)-E^\mu\left[g^{\delta,n}\mid\mathcal{F}_{i-1}\right](\omega).$$

Let $X_0=E^\mu[g^{\delta,n}]$ and $X_m=\sum_{i=1}^m d_i(\omega)$. Since $E^\mu\left[d_i\mid\mathcal{F}_m\right]=0$ for $i>m$, $\{\,X_m\,\}_{m\in\mathbb{N}_0}$ is a martingale. We obtain a bound for $d_i(\omega)$ by writing

$$d_{i+1}(\omega)=E^\mu\left[g^{\delta,n}\mid\mathcal{F}_{i+1}\right](\omega)-E^\mu\left[g^{\delta,n}\mid\mathcal{F}_i\right](\omega)\le\sup_{\omega^i}\left[g^{\delta,n}(\omega^i)-g^{\delta,n}(\omega)\right]=:|d_i|_\infty, \tag{3.5}$$

where $\omega^i_x=\omega_x$ for all $x\in\mathbb{Z}$, except for

$$x_i=\begin{cases}-i/2, & \text{if } i \text{ is even},\\ \lceil i/2\rceil, & \text{if } i \text{ is odd}.\end{cases} \tag{3.6}$$

To compute the difference in (3.5), we use a bound on the derivative of $g^{\delta,n}(\omega)$. From the computations in [13, p. 291] we have that, for any $\delta\in(0,1)$,

$$|d_i|_\infty\le\left|\frac{\partial g^{\delta,n}(\omega)}{\partial\omega_{x_i}}\right|\le\sqrt{K}\,\frac{n^{\delta/2}}{\mathfrak{c}}. \tag{3.7}$$

Applying the Azuma-Hoeffding inequality, we obtain

$$\mu\left(\omega\colon\ |X_{2Kn}-X_0|>u\right)\le 2\exp\left(-\frac{u^2}{2\sum_{i=1}^{2Kn}|d_i|_\infty^2}\right). \tag{3.8}$$

Since $X_{2Kn}=g^{\delta,n}(\omega)$ and $X_0=E^\mu[g^{\delta,n}]$, we obtain

$$\mu\left(\omega\colon\ \left|g^{\delta,n}(\omega)-E^\mu\left[g^{\delta,n}\right]\right|>un\right)\le 2\exp\left(-\frac{u^2n^2}{Cn^{1+\delta}}\right)\le 2\exp\left(-\frac{u^2}{C}n^{1-\delta}\right). \tag{3.9}$$

To conclude the proof of Proposition 3.1, we write

$$\begin{aligned}\mu\left(\omega\colon\ \left|\frac{1}{n}\log E_0^\omega\left[e^{\lambda H_n}\right]-\mathcal{J}^*(\lambda)\right|>\varepsilon\right)&\le\mu\left(\omega\colon\ \frac{1}{n}\left|\log E_0^\omega\left[e^{\lambda H_n}\right]-g^{\delta,n}(\omega)\right|>\tfrac13\varepsilon\right)\\ +\,\mu\left(\omega\colon\ \frac{1}{n}\left|g^{\delta,n}(\omega)-E^\mu\left[g^{\delta,n}\right]\right|>\tfrac13\varepsilon\right)&+\mu\left(\omega\colon\ \left|\frac{1}{n}E^\mu\left[g^{\delta,n}\right]-\mathcal{J}^*(\lambda)\right|>\tfrac13\varepsilon\right).\end{aligned} \tag{3.10}$$

We will estimate the second term in the right hand side by (3.9). Let us first show that the first and the third term vanish as $n\to\infty$. For the first term in (3.10), the ellipticity condition implies that for large n,

$$\frac{1}{n}\left|\log E_0^\omega\left[e^{\lambda H_n}\right]-g^{\delta,n}(\omega)\right|<\tfrac13\varepsilon. \tag{3.11}$$

Indeed, from the argument in the proof of [13, Lemma 3.4.10], specifically the computations just prior to the statement of [13, Lemma 3.4.14], we obtain the following estimate. For $K = K(\lambda)$ and n large enough,

$$E_0^\omega \left[e^{\lambda H_n} \mathbb{1}_{\{H_n < Kn\}} \mathbb{1}_{\{N < n^{\delta/2}\}} \right] \geq \tfrac{1}{2} E_0^\omega \left[e^{\lambda H_n} \right]. \tag{3.12}$$

Since

$$1 \leq \frac{E_0^\omega \left[e^{\lambda H_n} \right]}{E_0^\omega \left[e^{\lambda H_n} \mathbb{1}_{\{H_n < Kn\}} \mathbb{1}_{\{N < n^{\delta/2}\}} \right]} \leq 2, \tag{3.13}$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n} \left(\log E_0^\omega \left[e^{\lambda H_n} \right] - g^{\delta,n}(\omega) \right) = 0, \tag{3.14}$$

which implies (3.11). Furthermore, since $\lim_{n\to\infty} \frac{1}{n} \log E_0^\omega [e^{\lambda H_n}] = \mathcal{J}^*(\lambda)$, (3.11) also implies that for n large enough the third term in (3.10) is zero. We conclude that for n large enough,

$$\mu \left(\omega \colon \left| \frac{1}{n} \log E_0^\omega \left[e^{\lambda H_n} \right] - \mathcal{J}^*(\lambda) \right| > \varepsilon \right) \leq \mu \left(\omega \colon \frac{1}{n} \left| g^{\delta,n}(\omega) - E_0^\omega \left[g^{\delta,n} \right] \right| > \tfrac{1}{3}\varepsilon \right). \tag{3.15}$$

Picking $u = \frac{1}{3}\varepsilon$ in (3.9), we obtain (3.1).

3.2 Block decomposition

Note that $\frac{Z_n}{n} \in [-1,1]$. Consider the following block decomposition (see Fig. 6):

$$\Delta_i^N = \begin{cases} [-1, -1 + \frac{1}{N}], & \text{if } i = -N, \\ (\frac{i}{N}, \frac{i+1}{N}], & \text{if } i \in \{-N+1, \dots, N-1\}. \end{cases} \tag{3.16}$$

To deal with the flat piece of the rate function $\mathcal{I}$ in the positive-speed case, we define the following interval $\Delta_0^{*,N}$ containing $(0, v_\mu]$ (see Fig. 6):

$$\Delta_0^{*,N} = \left(0, \frac{\lfloor v_\mu N + 1 \rfloor}{N} \right] = \bigcup_i \left\{ \Delta_i^N \colon \mathcal{I}\left(\frac{i-1}{N} \right) = 0 \right\}, \qquad \Delta_i^{*,N} = \Delta_i^N \setminus \Delta_0^{*,N}. \tag{3.17}$$

By the intermediate value theorem, we have

$$E_0^\omega \left[e^{n\lambda \frac{Z_n}{n}} \mathbb{1}_{\{\frac{Z_n}{n} \in \Delta_i^{*,N}\}} \right] = e^{n\lambda u_i^*} P_0^\omega \left(\frac{Z_n}{n} \in \Delta_i^{*,N} \right) \tag{3.18}$$

for some $u_i^* \in \Delta_i^{*,N}$. Now,

$$\begin{aligned} \frac{1}{n} \log E_0^\omega \left[e^{\lambda Z_n} \right] - \mathcal{I}^*(\lambda) &= \frac{1}{n} \log e^{-n\mathcal{I}^*(\lambda)} \sum_i E_0^\omega \left[e^{n\lambda \frac{Z_n}{n}} \mathbb{1}_{\{\frac{Z_n}{n} \in \Delta_i^{*,N}\}} \right] \\ &= \frac{1}{n} \log \sum_i e^{n\lambda u_i^* - n\mathcal{I}_n^\omega(\Delta_i^{*,N}) - n\mathcal{I}^*(\lambda)}, \end{aligned} \tag{3.19}$$

where $\mathcal{I}_n^\omega(\Delta) = -\frac{1}{n} \log P_0^\omega \left(\frac{Z_n}{n} \in \Delta \right)$. Since $\mathcal{I}_n^\omega(\Delta)$ converges to $\mathcal{I}(\Delta)$ as $n \to \infty$, we define the block error

$$o(n, \Delta, \omega) = \mathcal{I}(\Delta) - \mathcal{I}_n^\omega(\Delta) \tag{3.20}$$

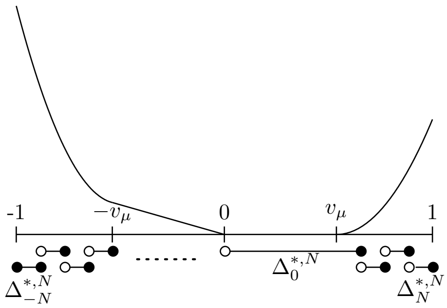

Figure 6: Block decomposition of $[-1, 1]$. Black and white circles indicate closed and open boundaries of the intervals, respectively. All the intervals are of length $1/N$, except possibly $\Delta_0^{*,N}$, which contains the flat piece $(0, v_\mu]$.

and obtain

$$\frac{1}{n}\log E_0^\omega\left[e^{\lambda Z_n}\right] - \mathcal{I}^*(\lambda) = \frac{1}{n}\log\sum_i e^{n\left(\lambda u_i^* - \mathcal{I}(\Delta_i^{*,N}) - \mathcal{I}^*(\lambda) + o(n,\Delta_i^{*,N},\omega)\right)}. \tag{3.21}$$

To estimate (3.21), we will need the following lemma.

Lemma 3.2 (Reduction to the worst block). *Given $\varepsilon > 0$, there is an N_0 such that, for $N > N_0$ and $n > n_0(N)$,*

$$\left|\frac{1}{n}\log E_0^\omega\left[e^{\lambda Z_n}\right] - \mathcal{I}^*(\lambda)\right| \le \tfrac{1}{2}\varepsilon + \max_i\left|o(n,\Delta_i^{*,N},\omega)\right| \tag{3.22}$$

with $o(n,\Delta_i^{*,N},\omega)$ as in (3.20)*.*

Proof. Since $\mathcal{I}$ is uniformly continuous in $[-1, 1]$, we have

$$\delta_N = \sup_{|s-t|\le 1/N}|\mathcal{I}(s) - \mathcal{I}(t)| \to 0, \qquad N \to \infty. \tag{3.23}$$

Let $\delta_i = \mathcal{I}(u_i^*) - \mathcal{I}(\Delta_i^{*,N})$, and note that $0 < \delta_i \le \delta_N$.

Upper bound. Since

$$\mathcal{I}^*(\lambda) = \sup_{u\in\mathbb{R}}\left[\lambda u - \mathcal{I}(u)\right] \ge \lambda u_i^* - \mathcal{I}(u_i^*), \tag{3.24}$$

we get the bound

$$\lambda u_i^* - \mathcal{I}(\Delta_i) - \mathcal{I}^*(\lambda) \le \lambda u_i^* - \left[\mathcal{I}(u_i^*) - \delta_i\right] - \left[\lambda u_i^* - \mathcal{I}(u_i^*)\right] = \delta_i < \delta_N. \tag{3.25}$$

Let N_0 be such that $\delta_{N_0} < \frac{1}{2}\varepsilon$. For $N > N_0$, let $n_0(N)$ be such that $\delta_{N_0} + \frac{\log 2N}{n_0} < \frac{1}{2}\varepsilon$. For $n > n_0(N)$, we have

$$\begin{aligned}\frac{1}{n}\log E_0^\omega\left[e^{\lambda Z_n}\right] - \mathcal{I}^*(\lambda) &\le \frac{1}{n}\log\sum_i e^{n\left(\delta_N + o(n,\Delta_i^{*,N},\omega)\right)}\\ &\le \delta_N + \frac{\log 2N}{n} + \max_i o(n,\Delta_i^{*,N},\omega)\\ &\le \tfrac{1}{2}\varepsilon + \max_i o(n,\Delta_i^{*,N},\omega).\end{aligned} \tag{3.26}$$

Lower bound. Let $\hat{x}$ be such that $\mathcal{I}^*(\lambda) = \lambda\hat{x} - \mathcal{I}(\hat{x})$. Then there exists a $\hat{\imath}$ such that $\hat{x} \in \Delta_{\hat{\imath}}^{*,N}$. For large N, we see that $\hat{x} \notin \Delta_0^{*,N}$. Indeed, if $\lambda < 0$, then $\hat{x} \leq 0$ and therefore $\hat{x} \notin \Delta_0^{*,N}$. On the other hand, if $\lambda > 0$, then $\hat{x} > v_\mu$. Pick N large enough so that $v_\mu + 1/N < \hat{x}$. Since $\Delta_0^{*,N} \subset (0, v_\mu + 1/N]$, we conclude that $\hat{x} \notin \Delta_0^{*,N}$.

Since $\hat{x} \in \Delta_{\hat{\imath}}^{*,N}$, it follows that $|\hat{x} - u_{\hat{\imath}}^*| < 1/N$. Choosing N_0 so that $\left|\frac{\lambda}{N_0} + \delta_{N_0}\right| < \frac{1}{2}\varepsilon$, we obtain

$$\begin{aligned} \frac{1}{n} \log E_0^\omega \left[e^{\lambda Z_n} \right] - \mathcal{I}^*(\lambda) &\geq \frac{1}{n} \log e^{n\left([\lambda u_{\hat{\imath}}^* - \mathcal{I}(\Delta_{\hat{\imath}}^{*,N})] - [\lambda\hat{x} - \mathcal{I}(\hat{x})] + o(n, \Delta_{\hat{\imath}}^{*,N}, \omega) \right)} \\ &\geq -\left(\frac{\lambda}{N} + \delta_N \right) + o(n, \Delta_{\hat{\imath}}^{*,N}, \omega) \\ &\geq -\tfrac{1}{2}\varepsilon + o(n, \Delta_{\hat{\imath}}^{*,N}, \omega). \end{aligned} \tag{3.27}$$

The claim in (3.22) follows from (3.26) and (3.27). □

In view of Lemma 3.2, we can bound

$$\begin{aligned} \mu\left(\omega\colon \left| \frac{1}{n} \log E_0^\omega \left[e^{\lambda Z_n} \right] - \mathcal{I}^*(\lambda) \right| > \varepsilon \right) &\leq \mu\left(\omega\colon \max_i \left| o(n, \Delta_i^{*,N}, \omega) \right| > \tfrac{1}{2}\varepsilon \right) \\ &\leq \sum_i \mu\left(\omega\colon \left| o(n, \Delta_i^{*,N}, \omega) \right| > \tfrac{1}{2}\varepsilon \right). \end{aligned} \tag{3.28}$$

Since there are only finitely many terms in the sum, (1.30) follows once we prove the following:

Lemma 3.3 (**Concentration of empirical speed on an interval).** *Let μ be basic, $\delta > 0$, and let $\Delta = (a, b]$ or $\Delta = [a, b]$ with $a \neq b$.*

- *If $0 \leq a$, then for every $\varepsilon > 0$ there are positive constants C, c such that*

$$\mu\left(\omega\colon \left| \frac{1}{n} \log P_0^\omega \left(\frac{Z_n}{n} \in \Delta \right) + \mathcal{I}(a) \right| > \varepsilon \right) \leq C\, e^{-cn^{1-\delta}}. \tag{3.29}$$

- *If $b \leq 0$, then for every $\varepsilon > 0$ there are positive constants C, c such that*

$$\mu\left(\omega\colon \left| \frac{1}{n} \log P_0^\omega \left(\frac{Z_n}{n} \in \Delta \right) + \mathcal{I}(b) \right| > \varepsilon \right) \leq C\, e^{-cn^{1-\delta}}. \tag{3.30}$$

The proof of Lemma 3.3 will be given in the next section as a corollary to the following concentration result for the half-line:

Lemma 3.4 (**Concentration of empirical speed on a half-line).** *Let μ be basic and $\delta > 0$.*

- *If $0 \leq a$, then for every $\varepsilon > 0$ there are positive constants C, c such that*

$$\mu\left(\omega\colon \left| \frac{1}{n} \log P_0^\omega \left(\frac{Z_n}{n} > a \right) + \mathcal{I}(a) \right| > \varepsilon \right) \leq C\, e^{-cn^{1-\delta}}. \tag{3.31}$$

- *If $b \leq 0$, then for every $\varepsilon > 0$ there are positive constants C, c such that*

$$\mu\left(\omega\colon \left| \frac{1}{n} \log P_0^\omega \left(\frac{Z_n}{n} \leq b \right) + \mathcal{I}(b) \right| > \varepsilon \right) \leq C\, e^{-cn^{1-\delta}}. \tag{3.32}$$

3.3 Concentration: from half-lines to intervals

In this section we prove that Lemma 3.3 follows from Lemma 3.4.

Proof. We will split the proof for $\Delta = (a, b]$ in two cases. The proof for $\Delta = [a, b]$ is similar.

Case $0 \leq a$**.** We start from the equation

$$P_0^\omega\left(\frac{Z_n}{n} \in \Delta\right) = P_0^\omega\left(\frac{Z_n}{n} > a\right) - P_0^\omega\left(\frac{Z_n}{n} > b\right). \tag{3.33}$$

Define $e(\omega, u, n) = \frac{1}{n} \log P_0^\omega(\frac{Z_n}{n} > u) + \mathcal{I}(u)$. Since $\mathcal{I}(b) - \mathcal{I}(a) = \eta > 0$, we obtain

$$\frac{P_0^\omega\left(\frac{Z_n}{n} > a\right)}{P_0^\omega\left(\frac{Z_n}{n} > b\right)} = e^{n(\mathcal{I}(b)-\mathcal{I}(a)+e(\omega,a,n)-e(\omega,b,n))} = e^{n(\eta+e(\omega,a,n)-e(\omega,b,n))}. \tag{3.34}$$

When $|e(\omega, a, n) - e(\omega, b, n)| < \frac{1}{2}\eta$, we have

$$P_0^\omega\left(\frac{Z_n}{n} > b\right) \leq e^{-n\frac{\eta}{2}}\, P_0^\omega\left(\frac{Z_n}{n} > a\right). \tag{3.35}$$

For large enough n, as soon as $e^{-n\frac{\eta}{2}} < \frac{1}{2}$ we get

$$\tfrac{1}{2} P_0^\omega\left(\frac{Z_n}{n} > a\right) \leq P_0^\omega\left(\frac{Z_n}{n} \in \Delta\right) \leq P_0^\omega\left(\frac{Z_n}{n} > a\right), \tag{3.36}$$

which implies that

$$\left|\frac{1}{n} \log P_0^\omega\left(\frac{Z_n}{n} \in \Delta\right) - \frac{1}{n} \log P_0^\omega\left(\frac{Z_n}{n} > a\right)\right| < \frac{1}{n} \log 2. \tag{3.37}$$

Therefore

$$\begin{aligned} &\mu\left(\omega\colon \left|\frac{1}{n} \log P_0^\omega\left(\frac{Z_n}{n} \in \Delta\right) - \frac{1}{n} \log P_0^\omega\left(\frac{Z_n}{n} > a\right)\right| > \frac{1}{n} \log 2\right) \\ &\leq \mu\left(\omega\colon\ |e(\omega, a, n) - e(\omega, b, n)| > \tfrac{1}{2}\eta\right) \\ &\leq \mu\left(\omega\colon\ |e(\omega, a, n)| > \tfrac{1}{4}\eta\right) + \mu\left(\omega\colon\ |e(\omega, b, n)| > \tfrac{1}{4}\eta\right). \end{aligned} \tag{3.38}$$

Since the concentration (3.31) in Lemma 3.4 bounds both terms in (3.38), after we pick n large enough so that $\frac{\log 2}{n} < \varepsilon$, we obtain (3.29).

Case $b \leq 0$**.** In this case we have the following equation:

$$P_0^\omega\left(\frac{Z_n}{n} \in \Delta\right) = P_0^\omega\left(\frac{Z_n}{n} \leq b\right) - P_0^\omega\left(\frac{Z_n}{n} \leq a\right). \tag{3.39}$$

Similarly, we define $\tilde{e}(\omega, u, n) = \frac{1}{n} \log P_0^\omega(\frac{Z_n}{n} \leq u) + \mathcal{I}(u)$. Since $\mathcal{I}(a) - \mathcal{I}(b) = \eta > 0$, we obtain

$$\frac{P_0^\omega\left(\frac{Z_n}{n} \leq b\right)}{P_0^\omega\left(\frac{Z_n}{n} \leq a\right)} = e^{n(\mathcal{I}(a)-\mathcal{I}(b)-\tilde{e}(\omega,a,n)+\tilde{e}(\omega,b,n))} = e^{n(\eta+\tilde{e}(\omega,a,n)-\tilde{e}(\omega,b,n))}. \tag{3.40}$$

When $|\tilde{e}(\omega,a,n)-\tilde{e}(\omega,b,n)|<\frac{1}{2}\eta$,

$$P_0^\omega\left(\frac{Z_n}{n}\le a\right)\le e^{-n\frac{\eta}{2}}\,P_0^\omega\left(\frac{Z_n}{n}\le b\right). \tag{3.41}$$

As we did in (3.35)–(3.37), for large n as soon as $e^{-n\frac{\eta}{2}}<\frac{1}{2}$ we conclude that

$$\begin{aligned}&\mu\left(\omega\colon\left|\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\in\Delta\right)-\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\le b\right)\right|>\tfrac{1}{n}\log 2\right)\\&\le\mu\left(\omega\colon\ |\tilde{e}(\omega,a,n)-\tilde{e}(\omega,b,n)|>\tfrac{1}{2}\eta\right)\\&\le\mu\left(\omega\colon\ |\tilde{e}(\omega,a,n)|>\tfrac{1}{4}\eta\right)+\mu\left(\omega\colon\ |\tilde{e}(\omega,b,n)|>\tfrac{1}{4}\eta\right).\end{aligned} \tag{3.42}$$

Since the concentration (3.32) in Lemma 3.4 bounds both terms in (3.42), after we pick n large enough so that $\frac{\log 2}{n}<\varepsilon$, we obtain (3.30) and Lemma 3.3 follows. □

3.4 Concentration: from hitting times to half-lines

In this section we prove Lemma 3.4.

Proof. Once we prove (3.31), the proof of (3.32) follows from a reflection argument. Indeed, let $\widetilde{\omega}=(\widetilde{\omega}(x))_{x\in\mathbb{Z}}=(1-\omega(x))_{x\in\mathbb{Z}}$. For $u>0$, $P_0^\omega(\frac{Z_n}{n}<-u)=P_0^{\widetilde{\omega}}(\frac{Z_n}{n}>u)$ and, denoting by $\mathcal{I}^\omega$ the quenched rate function on ω, we get

$$\mathcal{I}^{\widetilde{\omega}}(u)=\mathcal{I}^\omega(-u). \tag{3.43}$$

Therefore

$$\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}<-u\right)+\mathcal{I}^\omega(-u)=\frac{1}{n}\log P_0^{\widetilde{\omega}}\left(\frac{Z_n}{n}>u\right)+\mathcal{I}^{\widetilde{\omega}}(u) \tag{3.44}$$

and

$$\begin{aligned}&\mu\left(\omega\colon\left|\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}<-u\right)+\mathcal{I}^\omega(-u)\right|>\varepsilon\right)\\&=\mu\left(\omega\colon\left|\frac{1}{n}\log P_0^{\widetilde{\omega}}\left(\frac{Z_n}{n}>u\right)+\mathcal{I}^{\widetilde{\omega}}(u)\right|>\varepsilon\right)\\&=\tilde{\mu}\left(\omega\colon\left|\frac{1}{n}\log P_0^{\omega}\left(\frac{Z_n}{n}>u\right)+\mathcal{I}^{\omega}(u)\right|>\varepsilon\right),\end{aligned} \tag{3.45}$$

where $\tilde{\mu}[\omega\in A]=\mu[\widetilde{\omega}\in A]$ satisfies the conditions of Lemma 3.4. After proving (3.31) for μ, we obtain (3.31) for $\tilde{\mu}$, which is equivalent to the proof of (3.32) for μ.

To prove (3.31), we derive upper and lower bounds for $\frac{1}{n}\log P_0^\omega(\frac{Z_n}{n}\ge u)+\mathcal{I}(u)$.

Upper bound. To bound the probabilities on the displacements we can use the hitting times. For $u>0$,

$$P_0^\omega\left(\frac{Z_n}{n}\ge u\right)\le P_0^\omega\left(Z_n\ge\lfloor un\rfloor\right)\le P_0^\omega\left(H_{\lfloor un\rfloor}\le n\right). \tag{3.46}$$

By the Markov inequality, for $\theta<0$,

$$P_0^\omega\left(H_{\lfloor un\rfloor}\le n\right)\le e^{-\theta n}E_0^\omega\left[e^{\theta H_{\lfloor un\rfloor}}\right], \tag{3.47}$$

and so

$$\frac{1}{n}\log P_0\left(\frac{Z_n}{n}\geq u\right) \leq -\theta + \frac{1}{n}\log E_0^\omega\left[e^{\theta H_{\lfloor un\rfloor}}\right]$$
$$= -\theta + \frac{1}{n}\log E_0^\omega\left[e^{\theta H_{\lfloor un\rfloor}}\right] + u\mathcal{J}^*(\theta) - u\mathcal{J}^*(\theta) \tag{3.48}$$
$$= -u\left(\theta\frac{1}{u} + O(\omega, un, \theta) - \mathcal{J}^*(\theta)\right),$$

where

$$O(\omega, un, \theta) = \mathcal{J}^*(\theta) - \frac{1}{un}\log E_0^\omega\left[e^{\theta H_{\lfloor un\rfloor}}\right]. \tag{3.49}$$

Taking $\theta < 0$ such that $\theta\frac{1}{u} - \mathcal{J}^*(\theta) = \mathcal{J}(\frac{1}{u})$ (see Fig. 7) and using that $u\mathcal{J}(\frac{1}{u}) = \mathcal{I}(u)$, we get

$$\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\geq u\right) + \mathcal{I}(u) \leq -u\, O(\omega, un, \theta). \tag{3.50}$$

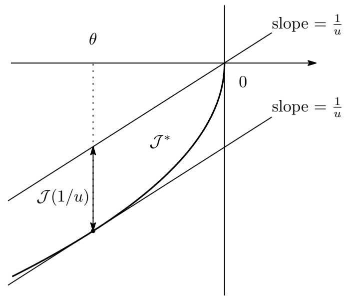

Figure 7: The function $x \mapsto \frac{1}{u}x - \mathcal{J}^*(x)$ attains its maximum at θ.

Therefore, using Proposition 3.1, we arrive at

$$\mu\left(\omega\colon \frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\geq u\right) + \mathcal{I}(u) > \varepsilon\right) \leq \mu\left(\omega\colon\ |O(\omega, un, \theta)| > \varepsilon\right) < Ce^{-cn^{1-\delta}}. \tag{3.51}$$

Lower bound. The lower bound for $P_0^\omega\left(\frac{Z_n}{n}\geq u\right)$ is more subtle. Note that, since the steps of the random walk are either $+1$ or -1, for $d > 0$, we have

$$n < H_x < n + dn \quad\Longrightarrow\quad Z_n > x - dn. \tag{3.52}$$

Therefore,

$$P_0^\omega\left(\frac{Z_n}{n}\geq u\right) \geq P_0^\omega\left(Z_n \geq \lceil un\rceil\right) \geq P_0^\omega\left(n \leq H_{\lceil un\rceil + \lfloor dn\rfloor} \leq n + \lfloor dn\rfloor\right). \tag{3.53}$$

Now, let $m = \lceil un\rceil + \lfloor dn\rfloor$. Note that

$$\frac{1}{u+d+r_n} \leq \frac{H_m}{m} \leq \frac{1+d+\tilde{r}_n}{u+d+r_n}$$

with $r_n, \tilde{r}_n \to 0$ as $n \to 0$. Let $\tilde{d}$ and $\tilde{u}$ be such that

$$\frac{1}{u+d} < \frac{1}{\tilde{u}} - \tilde{d} < \frac{1}{\tilde{u}} - \tilde{d} < \frac{1+d}{u+d}. \tag{3.54}$$

Letting $B_{\tilde d}(1/\tilde u)$ denote the ball with center $\frac{1}{\tilde u}$ and radius $\tilde d$, we have, for n large enough,

$$\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\geq u\right)\geq\frac{1}{n}\log P_0^\omega\left(\frac{H_m}{m}\in B_{\tilde d}(1/\tilde u)\right). \tag{3.55}$$

If $d\to 0$, then $|\tilde u-u|\to 0$ and $\tilde d\to 0$. Note that $E_0^\omega[e^{\zeta H_m}]<\infty$ for $\zeta<0$. We define the ζ-tilted probability measure

$$\frac{\mathrm{d}P_0^{\omega,\zeta,m}}{\mathrm{d}P_0^{\omega,m}}(y)=\frac{e^{m\zeta y}}{E_0^\omega\left[e^{\zeta H_m}\right]},\qquad P_0^{\omega,m}(\cdot)=P_0^\omega\left(\frac{H_m}{m}\in\cdot\right). \tag{3.56}$$

Recalling that $E_0^\omega\left[e^{\zeta H_m}\right]=\int e^{m\zeta y}\mathrm{d}P_0^{\omega,m}(y)$, we compute

$$\begin{aligned}\frac{1}{n}\log P_0^\omega\left(\frac{H_m}{m}\in B_{\tilde d}(1/\tilde u)\right)&=\frac{1}{n}\log\int_{B_{\tilde d}(1/\tilde u)}\mathrm{d}P_0^{\omega,m}(y)\\&=\frac{1}{n}\log\int_{B_{\tilde d}(1/\tilde u)}\frac{E_0^\omega\left[e^{\zeta H_m}\right]}{e^{m\zeta y}}\mathrm{d}P_0^{\omega,\zeta,m}(y).\end{aligned} \tag{3.57}$$

Now, since $\zeta<0$,

$$y\in B_{\tilde d}(1/\tilde u)\quad\Longrightarrow\quad e^{-m\zeta y}\geq e^{-m\zeta\left(\frac{1}{\tilde u}-\tilde d\right)}. \tag{3.58}$$

Inserting this into (3.57) and replacing $\frac{1}{n}\log E_0^\omega\left[e^{\zeta H_m}\right]$ by $-\frac{m}{n}[O(\omega,m,\zeta)-\mathcal{J}^*(\zeta)]$, yields:

$$\begin{aligned}&\frac{1}{n}\log P_0^\omega\left(\frac{H_m}{m}\in B_{\tilde d}(1/\tilde u)\right)\\&\geq\frac{1}{n}\log E_0^\omega\left[e^{\zeta H_m}\right]-\frac{m}{n}\zeta\left(\frac{1}{\tilde u}-\tilde d\right)+\frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u))\\&=-\hat u_n\left[\left(\zeta\frac{1}{\hat u_n}-\mathcal{J}^*(\zeta)\right)+O(\omega,m,\zeta)\right]-\hat d\zeta+\frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u)),\end{aligned} \tag{3.59}$$

where $\hat u_n=\frac{m}{n}$ and $\hat d$ is defined by the relation $\frac{m}{n}\left(\frac{1}{\tilde u}-\tilde d\right)=1+\hat d$. Note that

$$\hat u_n\to u,\hat d\to 0\quad\text{ as }\quad d\to 0,n\to\infty \tag{3.60}$$

Since $\zeta\frac{1}{\hat u_n}-\mathcal{J}^*(\zeta)\leq\mathcal{J}(\frac{1}{\hat u_n})$, combining (3.59) with (3.55) we obtain

$$\begin{aligned}\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\geq u\right)+\mathcal{I}(u)\geq\mathcal{I}(u)-\mathcal{I}(\hat u_n)-\hat u_n\,O(\omega,m,\zeta)-\hat d\zeta\\+\frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u)).\end{aligned} \tag{3.61}$$

Therefore, taking d small enough and n large enough so that

$$|\mathcal{I}(u)-\mathcal{I}(\hat u_n)|+|\hat d\zeta|<\tfrac{1}{2}\varepsilon, \tag{3.62}$$

we get

$$\begin{aligned}&\mu\left(\omega\colon\frac{1}{n}\log P_0^\omega\left(\frac{Z_n}{n}\geq u\right)+\mathcal{I}(u)<-\varepsilon\right)\\&\leq\mu\left(\omega\colon\ -\hat u_n\,O(\omega,m,\zeta)+\frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u))<-\tfrac{1}{2}\varepsilon\right)\\&\leq\mu\left(\omega\colon\ -\hat u_n\,O(\omega,m,\zeta)<-\tfrac{1}{4}\varepsilon\right)+\mu\left(\omega\colon\frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u))<-\tfrac{1}{4}\varepsilon\right).\end{aligned} \tag{3.63}$$

From Proposition 3.1 and the fact that $0 < \hat{u}_n \le 1$, it follows that

$$\mu\left(\omega\colon \hat{u}_n\, |O(\omega,m,\zeta)| > \tfrac{1}{4}\varepsilon\right) \le C\, e^{-cn^{1-\delta}}. \tag{3.64}$$

It therefore remains to prove that

$$\mu\left(\omega\colon \frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde{c}}(1/\tilde{u})) < -\tfrac{1}{4}\varepsilon\right) \le C\, e^{-cn^{1-\delta}}. \tag{3.65}$$

Let $E_0^{\omega,\zeta,m}[f(Y)]$ be expectation of f with respect to $P_0^{\omega,\zeta,m}(dY)$ and $E_0^{\omega,m}[f(Y)]$ be expectation of f with respect to $P_0^{\omega,m}(dY)$. Then

$$E_0^{\omega,\zeta,m}\left[e^{m\theta Y}\right] = \frac{E_0^{\omega,m}\left[e^{m(\theta+\zeta)Y}\right]}{E_0^{\omega,m}\left[e^{m\zeta Y}\right]} = \frac{E_0^{\omega}\left[e^{(\theta+\zeta)H_m}\right]}{E_0^{\omega}\left[e^{\zeta H_m}\right]}. \tag{3.66}$$

We have

$$\begin{aligned} P_0^{\omega,\zeta,m}(B_{\tilde{d}}(1/\tilde{u})^c) &= \underbrace{P_0^{\omega,\zeta,m}\left(Y > \frac{1}{\tilde{u}} + \tilde{d}\right)} + \underbrace{P_0^{\omega,\zeta,m}\left(Y < \frac{1}{\tilde{u}} - \tilde{d}\right)} \\ &= \qquad\qquad I \qquad\qquad + \qquad\qquad II. \end{aligned} \tag{3.67}$$

Bound for I. For $\theta > 0$,

$$\begin{aligned} &P_0^{\omega,\zeta,m}\left(Y > \frac{1}{\tilde{u}} + \tilde{d}\right) \le e^{-m\theta(\frac{1}{\tilde{u}}+\tilde{d})} E_0^{\omega,\zeta,m}\left[e^{\theta Y}\right] \\ &= \exp\left\{m\left[-\theta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \frac{1}{m}\left(\log E_0^{\omega}\left[e^{(\theta+\zeta)H_m}\right] - \log E_0^{\omega}\left[e^{\zeta H_m}\right]\right)\right]\right\} \\ &\le \exp\left\{m\left[-\theta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \mathcal{J}^*(\theta+\zeta) - \mathcal{J}^*(\zeta) - O(\omega,m,\theta+\zeta) + O(\omega,m,\zeta)\right]\right\}. \end{aligned} \tag{3.68}$$

By the strict convexity of $\mathcal{J}$ at $\frac{1}{\tilde{u}} > 0$, we can pick $\zeta = \mathcal{J}'(\frac{1}{\tilde{u}}) < 0$ an exposing plane, i.e., ζ is such that for any $y \neq \frac{1}{\tilde{u}}$,

$$\mathcal{J}(y) - \mathcal{J}\left(\frac{1}{\tilde{u}}\right) > \left(y - \frac{1}{\tilde{u}}\right)\zeta. \tag{3.69}$$

By the strict convexity of $\mathcal{J}$ and the fact that $\frac{1}{\tilde{u}} + \tilde{d} > \frac{1}{\tilde{u}}$, we can pick $\theta > 0$ (see Fig. 8) such that

$$(\theta+\zeta)\left(\frac{1}{\tilde{u}}+\tilde{d}\right) - \mathcal{J}\left(\frac{1}{\tilde{u}}+\tilde{d}\right) = \mathcal{J}^*(\theta+\zeta). \tag{3.70}$$

By (3.69) with $y = \frac{1}{\tilde{u}} + \tilde{d}$, we find that

$$\begin{aligned} &-\theta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \mathcal{J}^*(\theta+\zeta) - \mathcal{J}^*(\zeta) \\ &= -(\theta+\zeta)\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \mathcal{J}^*(\theta+\zeta) - \mathcal{J}^*(\zeta) + \zeta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) \\ &= -\mathcal{J}\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \zeta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) - \mathcal{J}^*(\zeta) \\ &\le -\mathcal{J}\left(\frac{1}{\tilde{u}}+\tilde{d}\right) + \zeta\left(\frac{1}{\tilde{u}}+\tilde{d}\right) - \zeta\frac{1}{\tilde{u}} + \mathcal{J}\left(\frac{1}{\tilde{u}}\right) = -d_1 < 0 \end{aligned} \tag{3.71}$$

(see Fig. 8). On the set $A_I = \{\,\omega\colon |O(\omega,m,\theta+\zeta) - O(\omega,m,\zeta)| < \frac{1}{2}|d_1|\,\}$, we have

$$P_0^{\omega,\zeta,m}\left(Y > \frac{1}{\tilde{u}} + \tilde{d}\right) < e^{-m\frac{|d_1|}{2}} \to 0. \tag{3.72}$$

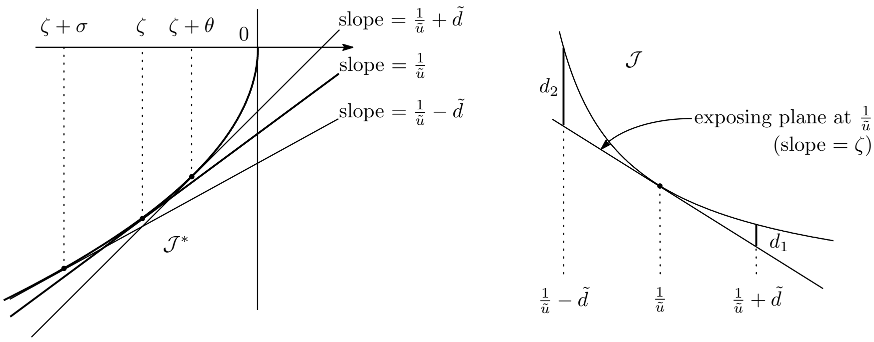

Figure 8: *Left*: The strict convexity of $\mathcal{J}^*$ shows that for a line of slope $1/\tilde u+\tilde d$ the maximum is attained at $\zeta+\theta>\zeta$, while for a line of slope $1/\tilde u-\tilde d$ the maximum is attained at $\zeta+\sigma<\zeta$ (see (3.70) and (3.74), respectively). *Right*: The exposing plane condition shows that d_1 in (3.71) and d_2 in (3.75) are strictly positive.

Bound for II. Again, for $\sigma<0$,

$$\begin{aligned}
&P_0^{\omega,\zeta,m}\left(Y<\frac{1}{\tilde u}-\tilde d\right)\le e^{-m\sigma(\frac{1}{\tilde u}-\tilde d)}\,E_0^{\omega,\zeta,m}\left[e^{\sigma Y}\right]\\
&=\exp\left\{m\left[-\sigma\left(\frac{1}{\tilde u}-\tilde d\right)+\frac{1}{m}\left(\log E_0^{\omega}\left[e^{(\sigma+\zeta)H_m}\right]-\log E_0^{\omega}\left[e^{\zeta H_m}\right]\right)\right]\right\}\\
&\le\exp\left\{m\left[-\sigma\left(\frac{1}{\tilde u}-\tilde d\right)+\mathcal{J}^*(\sigma+\zeta)-\mathcal{J}^*(\zeta)-O(\omega,m,\sigma+\zeta)+O(\omega,m,\zeta)\right]\right\}.
\end{aligned}\tag{3.73}$$

By the strict convexity of $\mathcal{J}$ and the fact that $\frac{1}{\tilde u}-\tilde d<\frac{1}{\tilde u}$, we can pick $\sigma<0$ (see Fig 8) such that

$$(\sigma+\zeta)\left(\frac{1}{\tilde u}-\tilde d\right)-\mathcal{J}\left(\frac{1}{\tilde u}-\tilde c\right)=\mathcal{J}^*(\sigma+\zeta).\tag{3.74}$$

Similarly to (3.71), using (3.69) with $y=\frac{1}{\tilde u}-\tilde d$, we obtain

$$-\sigma\left(\frac{1}{\tilde u}-\tilde d\right)+\mathcal{J}^*(\sigma+\zeta)-\mathcal{J}^*(\zeta)=-d_2<0,\tag{3.75}$$

see Fig 8. For $\omega\in A_{II}=\{\,\omega\colon\ |O(\omega,m,\sigma+\zeta)-O(\omega,m,\zeta)|<\frac12|d_2|\,\}$

$$P_0^{\omega,\zeta,m}\left(Y<\frac{1}{\tilde u}-\tilde d\right)<e^{-m\frac12|d_2|}\to 0.\tag{3.76}$$

Conclusion. For n large enough, using (3.72) and (3.76) we see that

$$\omega\in A_I\cap A_{II}\quad\Longrightarrow\quad P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u)^c)<\tfrac12\quad\Longrightarrow\quad P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u))\ge\tfrac12,\tag{3.77}$$

and therefore, for large n, we conclude that

$$\left|\frac1n\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u))\right|<\frac1n\log 2<\tfrac14\varepsilon.\tag{3.78}$$

Hence

$$\mu\left(\omega\colon \frac{1}{n}\log P_0^{\omega,\zeta,m}(B_{\tilde d}(1/\tilde u)) < -\tfrac14\varepsilon\right) \le \mu\left(A_I^c\right) + \mu\left(A_{II}^c\right). \tag{3.79}$$

To complete the proof, note that Proposition 3.1 implies

$$\mu\left(A_I^c\right) + \mu\left(A_{II}^c\right) \le C\, e^{-cn^{1-\delta}}. \tag{3.80}$$

It is worth mentioning that the constant in (3.80) does not depends on n. For fixed u we choose $\varepsilon > 0$, and (3.62) together with (3.60) gives us $d, \tilde d$ and $\tilde u$. After that, d_1, d_2 are given by the exposing plane conditions at the boundary of the ball of radius $\tilde d$ centered at $\frac{1}{\tilde u}$ (see (3.71) and (3.75)). Thus, even though the constant in (3.80) depends on d_1, d_2, the latter are functions of u and ε only, and not of n. The latter estimate, together with the bounds in (3.63) and (3.64), yield

$$\mu\left(\omega\colon \frac{1}{n}\log P_0^{\omega}\left(\frac{Z_n}{n}\ge u\right) + \mathcal{I}(u) < -\varepsilon\right) < C\, e^{-cn^{1-2\delta}}. \tag{3.81}$$

Therefore (3.31) in Lemma 3.4 follows from (3.51) and (3.81). □

4 Proofs of SLLN and LDP

4.1 Proof of Theorem 1.10

Proof. We will prove the SLLN under the annealed law. After that we get Theorem 1.10 by noting that, for any event A, if $P_0^{\mu,\tau}(A) = 1$, then, for $\mu^{\mathbb{N}}$-a.e. Ω, $P_0^{\Omega,\tau}(A) = 1$, and by taking $A = \{\lim_{n\to\infty} \frac{X_n}{n} = v_\mu\}$ we get the claim.

To prove the SLLN under the annealed law, we will use Theorem 1.12. To this aim, let $\mathbb{P}$ be the joint law of doubly indexed variables $\psi_n^{(k)}$ that are pair-wise independent in k and such that, for each k, $\psi_n^{(k)}$ has law $P_0^\mu(Z_n = \cdot)$. From (1.26) we see that $\psi_{T_k}^{(k)}$ is distributed as Y_k and $\psi_{\bar T^n}^{(\ell(n))}$ is distributed as $\bar Y^n$.

Assumptions $(A1)$ and $(A2)$ are trivially satisfied. It remains to check $(A3)$ with $L = v_\mu$, for which we use the annealed large deviation estimates for RWRE. In fact, from Proposition 1.6 we get

$$\limsup_{n\to\infty} \frac{1}{n}\log P_0^\mu\left(|Z_n/n - v_\mu| \ge \varepsilon\right) \le -\mathcal{I}(v_\mu+\varepsilon) \vee -\mathcal{I}(v_\mu-\varepsilon). \tag{4.1}$$

In the zero-speed case, since $-\mathcal{I}(v_\mu+\varepsilon) \vee -\mathcal{I}(v_\mu-\varepsilon) < 0$, the speed of decay is exponential in n and $(A3)$ holds. In the positive-speed case, $\mathcal{I}(v_\mu-\varepsilon) = 0$ and the bound in (4.1) is not useful. However, Proposition 1.7 yields the refined bound in the flat piece:

$$\limsup_{n\to\infty} \frac{1}{\log n}\log P_0^\mu\left(|Z_n/n| < v_\mu - \varepsilon\right) \le 1 - s. \tag{4.2}$$

This speed of decay is polynomial in n, and hence $(A3)$ still holds for $\delta \in (0, s-1)$. □

4.2 Proof of Theorem 1.11

Proof. We start by showing that, for $\mu^{\mathbb{N}}$-a.e. Ω, $P^{\Omega,\tau}(X_n/n \in \cdot)$ satisfies the LDP with rate n and with some convex rate function, which we denote by $\mathcal{I}_{\mu,\tau}$. This follows from Campos *et* *al.* [4, Theorem 1.2], which states that any uniformly elliptic nearest-neighbour random walk in a dynamic random environment on $\mathbb{Z}$, for which the space-time-shift operator is ergodic, satisfies a quenched LDP with rate n and with a convex rate function. In our case, the uniform ellipticity assumption is given in (1.4). For the above mentioned ergodicity we note that, in view of the i.i.d. property of μ, it suffices to establish ergodicity of the time-shift operator. The latter is true because the time-shift operator is actually strongly mixing w.r.t. to the law induced on the space of dynamic environments. In fact, translations of cylinder events over time eventually become separated by some resampling time, and therefore they are independent.

Having established the existence of a convex rate function $\mathcal{I}_{\mu,\tau}$, we can invoke Varadhan's Lemma (see e.g. [8, Chaper III]) to guarantee that

$$\exists \lim_{n\to\infty} \frac{1}{n} \log E_0^{\Omega,\tau}\left[e^{\lambda X_n}\right] = \sup_{x\in\mathbb{R}} \left[\lambda x - \mathcal{I}_{\mu,\tau}(x)\right] \qquad \forall\, \lambda \in \mathbb{R}. \tag{4.3}$$

On the other hand, we next show by means of Theorem 1.12 that

$$\lim_{n\to\infty} \frac{1}{n} \log E_0^{\Omega,\tau}\left[e^{\lambda X_n}\right] = \sup_{x\in\mathbb{R}} \left[\lambda x - \mathcal{I}(x)\right] \qquad \forall\, \lambda \in \mathbb{R}, \tag{4.4}$$

with $\mathcal{I}$ from Proposition 1.4. Hence, combining (4.3) and (4.4), and using the convexity of the rate functions involved, we obtain the identity $\mathcal{I}_{\mu,\tau} = \mathcal{I}$, which concludes the proof.

Let us finally check that indeed (4.4) holds. Note first that

$$\frac{1}{n} \log E_0^{\Omega,\tau}\left[e^{\lambda X_n}\right] = \frac{1}{n} \sum_{k=1}^{\ell(n)-1} \log E_0^{\omega_k}\left[e^{\lambda Z_{T_k}}\right] + \frac{1}{n} \log E_0^{\omega_{\ell(n)}}\left[e^{\lambda Z_{\bar{T}^n}}\right]. \tag{4.5}$$

Now, let $\mathbb{P}$ be the law induced by the doubly indexed variables $\psi_n^{(k)}(\Omega) := \log E_0^{\omega_k}\left[e^{\lambda Z_n}\right]$ under $\mu^{\mathbb{N}}$. Then

$$\psi_{T_k}^{(k)} = \log E_0^{\omega_k}\left[e^{\lambda Z_{T_k}}\right], \qquad \psi_{\bar{T}^n}^{(\ell(n))} = \log E_0^{\omega}\left[e^{\lambda Z_{\bar{T}^n}}\right]. \tag{4.6}$$

Again, Assumptions $(A1)$ and $(A2)$ of Theorem 1.12 are readily satisfied, and it remains to check (A3) with $L = \mathcal{I}^*(\lambda)$. Finally, Theorem 1.13 yields the latter assumption and gives the desired result. □

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