EURANDOM PREPRINT SERIES

2019-002

March 29, 2019

On Two Classes of Reflected Autoregressive Processes

O. Boxma, A. Löpker, M. Mandjes
ISSN 1389-2355

On two Classes of Reflected Autoregressive Processes

ONNO BOXMA, ANDREAS LÖPKER, MICHEL MANDJES

Abstract

We introduce two general classes of reflected autoregressive processes, INGAR$^+$ and GAR$^+$. Here, INGAR$^+$ can be seen as the counterpart of INAR(1) with general thinning and reflection being imposed to keep the process non-negative; GAR$^+$ relates to AR(1) in an analogous manner. The two processes INGAR$^+$ and GAR$^+$ are shown to be connected via a duality relation. We proceed by presenting a detailed analysis of the time-dependent and stationary behavior of the INGAR$^+$ process, and then exploit the duality relation to obtain the time-dependent and stationary behavior of the GAR$^+$ process.

Keywords. Autoregressive processes ∘ reflection ∘ generating functions ∘ time-dependent behavior ∘ stationarity

Affiliations. *Onno Boxma* is with Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, the Netherlands. His research is partly funded by NWO Gravitation project Networks, grant number 024.002.003. Email: o.j.boxma@tue.nl.
Andreas Löpker is with HTW Dresden, University of Applied Sciences, Hochschule für Technik und Wirtschaft, Friedrich-List-Platz 1, D-01069 Dresden, Germany. Email: lopker@htw-dresden.de.
Michel Mandjes is with Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, the Netherlands. He is also with Eurandom, Eindhoven University of Technology, the Netherlands, and Amsterdam Business School, Faculty of Economics and Business, University of Amsterdam, the Netherlands. His research is partly funded by NWO Gravitation project Networks, grant number 024.002.003. Email: m.r.h.mandjes@uva.nl.

1. Introduction and model description

The primary aim of this paper is to study the transient and stationary behavior of two important classes of autoregressive processes with reflection at zero. We show that these processes are connected via a duality relation, so that the analysis of one of them provides results for the other, and vice versa.

Our first starting point is the well studied INAR(1) process, which is defined by

$$A_{n+1} = a \circ A_n + J_n, \quad n \in \mathbb{N}_0,$$

with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Here $(J_n)_{n\in\mathbb{N}_0}$ are i.i.d. (independent, identically distributed) non-negative integer-valued random variables, and the thinning operation $\circ$ is, as defined in Steutel and van Harn [15], given by $a \circ X := \sum_{k=1}^{X} U_k$, where the random variables U_k are i.i.d. Bernoulli random variables with mean $a \in [0, 1]$. We refer to e.g. McKenzie [12] and Al-Osh and Alzaid [1] for seminal contributions and Weiß [16] for more background.

When the process $(A_n)_{n\in\mathbb{N}_0}$ cannot attain negative values, one could consider a reflected version of INAR(1), defined by, using the notation $x^+ := \max\{x, 0\}$,

$$A_{n+1} = (a \circ A_n + C_n - W_n)^+, \quad n \in \mathbb{N}_0, \tag{1}$$

with i.i.d. non-negative integer-valued random variables $(C_n)_{n\in\mathbb{N}_0}$, and i.i.d. geometrically distributed random variables $(W_n)_{n\in\mathbb{N}_0}$. In the sequel, we focus on extending this recursion, still restricting ourselves to cases in which all A_n are non-negative integer-valued, which enables us to model a rather general class of stochastic processes. The resulting model has the potential to be used in any setting featuring a non-negative time series with an autoregressive correlation structure, and in addition it has obvious applications in e.g. queueing and inventory theory.

The second starting point is the classical AR(1) process, which has also been extensively studied in the literature; see e.g. the textbook treatment in Brockwell et al. [6]. It is given through the recursion

$$Z_{n+1} = aZ_n + I_n, \quad n \in \mathbb{N}_0,$$

the $(I_n)_{n\in\mathbb{N}_0}$ being i.i.d. non-negative real-valued random variables, and we assume that $a \in [0, 1]$. As in the INAR(1) case, when the series $(Z_n)_{n\in\mathbb{N}_0}$ corresponds to quantities that attain non-negative values only, one could reflect this process at 0. Recently, in Boxma et al. [5] the process

$$Z_{n+1} = (aZ_n + Y_n - B_n)^+, \quad n \in \mathbb{N}_0, \tag{2}$$

was studied, with i.i.d. real-valued non-negative $(Y_n)_{n\in\mathbb{N}_0}$, i.i.d. exponentially distributed random variables $(B_n)_{n\in\mathbb{N}_0}$, and $a \in [0, 1)$. Also this process we will generalize, still restricting ourselves to cases in which all Z_n are non-negative real-valued. Again, this enables us to set up a rather general class of stochastic processes, with abundant applications across various scientific disciplines (such as engineering, economics, and the social sciences), specifically suitable if the time series under study relates to intrinsically non-negative quantities. Notice that the boundary case $a = 1$ corresponds to the waiting time process in a conventional single-server queue.

Now that we have presented a brief account of existing models, we proceed by describing in greater detail the processes that we focus on in this paper.

Description of the INGAR$^+$ process

The first process under consideration is an integer-valued generalized autoregressive process $(A_n)_{n\in\mathbb{N}_0}$, reflected at 0. Throughout this paper we refer to it as the INGAR$^+$ process, being defined as follows. The process $(A_n)_{n\in\mathbb{N}_0}$ has values in $\mathbb{N}_0$, and is given by the recursion

$$A_{n+1} = (U_n(A_n) + C_n - W_n)^+ , \quad n \in \mathbb{N}_0, \tag{3}$$

with $(C_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$ being two mutually independent sequences of i.i.d. non-negative integer-valued random variables. It is assumed that W_n has a geometric distribution with success probability $p \in (0, 1]$, meaning that $\mathbb{P}(W_n = k) = p(1 - p)^{k-1}$ for $k \in \mathbb{N}$. Moreover, for $n \in \mathbb{N}_0$,

$$U_n(m) := \sum_{k=1}^{m} U_{n,k}$$

denotes the partial sum of m i.i.d. non-negative integer-valued random variables $U_{n,k}$ (where we assume $\mathbb{P}(U_{n,k} = 0) < 1$ to avoid trivial situations). In our model the sequences $(U_{n,1})_{n\in\mathbb{N}_0}$, $(U_{n,2})_{n\in\mathbb{N}_0}, \ldots$ are assumed independent, and they are in addition independent of $(C_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$. In what follows we use the compact notations $U(\cdot)$, C and W for generic random variables with distributions equal to those of $U_n(\cdot)$, C_n and W_n, respectively.
We throughout impose the following stability condition:

$$\mathbb{E}(U) < 1 \quad \text{and} \quad \mathbb{E}(\log(1 + C)) < \infty, \tag{S1}$$

which is shown in Theorem 10 below to be a sufficient condition for ergodicity. We remark that it turned out a delicate issue to identify a stability condition that is both sufficient and necessary; a short discussion of this issue is added at the end of Section 3.2.
We mention the following special cases:

1. Let the random variables $U_{n,k}$ be i.i.d. Bernoulli random variables with $\mathbb{P}(U_{n,k} = 1) = a \in [0, 1]$ and let $p = 1$ (so that $W_n = 1$ a.s.). Then $A_{n+1} = (a \circ A_n + C_n - 1)^+$ and if we require that $C_n \geq 1$ and we write $\varepsilon_n = C_n - 1$, then we obtain

   $$A_{n+1} = a \circ A_n + \varepsilon_n, \quad n \in \mathbb{N}_0,$$

   the defining recursion of the INAR(1) process, cf. Weiß [16].

   If we still assume $C_n = \varepsilon_n + 1 \geq 1$ and instead of Bernoulli random variables allow generally distributed $U_{n,k}$, then we obtain

   $$A_{n+1} = U_n(A_n) + \varepsilon_n, \quad n \in \mathbb{N}_0. \tag{4}$$

   Such an extension of the INAR(1) process was proposed by Latour [11]. Ristić et al. [14] discuss the case that the increments corresponding to $U_n(\cdot)$ have a geometric distribution; see also Barreto-Souza [4].

2. A rich variety of highly general queueing processes can be embedded in the INGAR$^+$ process. To start with, consider the M/G/1 queue, cf. Cohen [7, Ch. II.5], and let A_n denote the number of customers waiting immediately after the beginning of the nth

service. Let C_n denote the number of customers arriving during the nth service. Then we obtain the Lindley-type recursion

$$A_{n+1} = (A_n + C_n - 1)^+ , \quad n \in \mathbb{N}_0,$$

which is the $U_{n,k} \equiv 1$ and $p = 1$ case of the INGAR$^+$ process.

To illustrate the modeling flexibility of INGAR$^+$, consider the following setup. Suppose that each customer requires a positive service time only with probability p and no service time with probability $(1 - p)$, but every customer still has to wait in line until her/his turn. Additionally suppose that at service completion each next customer finding itself first in line but not requiring work leaves the system instantly. This means that the number of customers who leave the system between the nth and $(n + 1)$st service completion equals the geometrically distributed number W_n (with parameter p), but obviously as long as $A_n + C_n - W_n$ remains non-negative. We thus obtain the recursion

$$A_{n+1} = (A_n + C_n - W_n)^+ , \quad n \in \mathbb{N}_0,$$

which is the $U_{n,k} \equiv 1$ case of the INGAR$^+$ process.

If additionally right after the beginning of a service all waiting customers decide, independently of each other, to stay (with probability a) or to leave (before being served, that is), we end up with

$$A_{n+1} = (a \circ A_n + C_n - W_n)^+ , \quad n \in \mathbb{N}_0,$$

the INGAR$^+$ case where the $U_{n,k}$ have a Bernoulli distribution. We conclude that our model covers systems with impatient customers as a special case.

Description of the GAR$^+$ process

The second process we consider in this paper is a real-valued generalized autoregressive process $(Z_n)_{n\in\mathbb{N}_0}$, with the special feature that it is reflected at 0. We call the resulting object the GAR$^+$ process; it is formally defined as follows. The process attains values in $\mathbb{R}^+ = [0, \infty)$ and is defined by the stochastic recursion

$$Z_{n+1} = (S_n(Z_n) + Y_n - B_n)^+ , \quad n \in \mathbb{N}_0. \tag{5}$$

The components featuring in this recursion are defined as follows. In the first place, $(B_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ are sequences of i.i.d. real-valued non-negative random variables, that are in addition independent of each other. It is assumed that B_n has an exponential distribution with rate $\lambda > 0$, i.e., $\mathbb{P}(B_n \le x) = 1 - e^{-\lambda x}$ for $x \ge 0$. We allow the $\lambda = \infty$ case where $B_n \equiv 0$. As before, we write B and Y for generic random variables with distributions equal to those of B_n and Y_n, respectively. The sequence of processes $((S_n(t))_{t\in\mathbb{R}^+})_{n\in\mathbb{N}_0}$ are i.i.d. Lévy subordinators, independent of $(B_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$; we write $(S(t))_{t\in\mathbb{R}^+}$ for a generic stochastic process distributed as $(S_n(t))_{t\in\mathbb{R}^+}$. We assume that the associated Laplace-Stieltjes transform is $\mathbb{E}(e^{-sS(t)}) = e^{-\psi(s)t}$, where the Laplace exponent is necessarily of the form

$$\psi(s) = as + \int_0^\infty (1 - e^{-su})\, \mathrm{d}\nu(u),$$

for some $a \geq 0$; to see that it has this structure, recall the Lévy-Itō decomposition, and observe that increasing Lévy processes lack a Brownian term and contributions due to negative jumps. We assume that the Lévy measure ν is concentrated on $\mathbb{R}^+$ with the additional integrability constraint $\int_0^\infty (1 \wedge y)\,\mathrm{d}\nu(y) < \infty$ and exclude the trivial case where $\psi(s) \equiv 0$, i.e. $S(t) \equiv 0$. In this model, we throughout impose the stability condition

$$\mathbb{E}\left(S(1)\right) < 1 \quad \text{and} \quad \mathbb{E}\left(\log(1+Y)\right) < \infty, \tag{S2}$$

where $\mathbb{E}\left(S(1)\right)$ is the average rate pertaining to $S(\cdot)$ that can be calculated via

$$\mathbb{E}\left(S(1)\right) = a + \int_0^\infty u\,\mathrm{d}\nu(u).$$

In Section 4.2 we will prove sufficiency of (S2), and in addition equivalence with (S1) as a consequence of the duality introduced in the next section.
Throughout this work we also impose the condition

$$\exists \gamma > 0: \quad \gamma \geq \psi(\gamma). \tag{C}$$

As will be made clear in the next section, (C) guarantees the existence of a meaningful transformation between the GAR$^+$ and INGAR$^+$ processes. In Remark 5 we discuss how to deal with the situation when (C) does not hold.
The GAR$^+$ process covers the following special cases:

1. If we assume that $S(t) = at$ for some $a \in [0, 1)$ and $B_n \equiv 0$ (which can be achieved by picking $\lambda = \infty$), then (5) becomes

$$Z_{n+1} = aZ_n + Y_n, \quad n \in \mathbb{N}_0.$$

   This describes a classical autoregressive process of AR(1) type; see for more background for instance Brockwell et al. [6].

2. In the case where $S(t) = t$ the recursion (5) is equivalent to the classical Lindley recursion (see e.g. Asmussen [2, p. 92]):

$$Z_{n+1} = \left(Z_n + Y_n - B_n\right)^+, \quad n \in \mathbb{N}_0.$$

   This recursion records the waiting time at customer arrivals in an M/G/1 queue, with service times Y_n and inter-arrival times B_n.

   This model was recently extended in Boxma et al. [5], where the case of $S(t) = at$ (with $a \in [0, 1)$) was studied, leading to the recursion

$$Z_{n+1} = \left(aZ_n + Y_n - B_n\right)^+, \quad n \in \mathbb{N}_0. \tag{6}$$

   Z_n could be interpreted as the workload in a queueing model just before the nth customer arrival. Such an arrival adds Y_n work, but also makes a fixed fraction $1 - a$ of the work that is already present obsolete. Importantly, our new GAR$^+$ model covers the more general case: working with the thinning $S_n(Z_n)$ rather than aZ_n, a random part of Z_n is made obsolete (instead of a deterministic part).

For any non-negative integer-valued random variable X we introduce its 'alternate probability generating function' (in short APGF, cf. McKenzie [13]) as

$$G_X(z) := \mathbb{E}((1-z)^X), \quad z \in [0,1].$$

Note that the APGF slightly differs from the commonly used probability generating function; we use it here, rather than the conventional generating function, for reasons that will become clear soon. Given a non-negative random variable X, its LST (Laplace-Stieltjes transform) is given by

$$\varphi_X(s) := \mathbb{E}(e^{-sX}), \quad s \geq 0.$$

With these definitions in place, APGFs and LSTs are conveniently related to each other, see Theorem 1 below. The joint APGF and the joint LST of two random variables X and Y are defined in a similar manner:

$$\begin{aligned} G_{X,Y}(z,w) &:= \mathbb{E}((1-z)^X(1-w)^Y), \quad z,w \in [0,1], \\ \varphi_{X,Y}(s,t) &:= \mathbb{E}(e^{-sX-tY}), \quad s,t \geq 0. \end{aligned}$$

In the sequel we write $X =_{\mathrm{d}} Y$ if the two random objects X and Y have the same distribution.

Main contributions and organization of the paper

We conclude this introduction by a brief account of the results obtained, and an overview of the paper. We start in Section 2 by establishing a useful duality relation; see Theorem 1. Another main result of this section (Theorem 3) concerns the fact that this duality relation is well adapted to all operations that we use in our definition of reflected autoregressive processes, namely addition, reflection at zero, and the random sum and subordinator operations $U(\cdot)$ and $S(\cdot)$. Based on these results, for any GAR$^+$ process we can explicitly construct its INGAR$^+$ counterpart. In Section 3 we obtain expressions for the time-dependent (Theorem 7) and stationary (Theorem 10) APGFs corresponding to the INGAR$^+$ process. In addition moments and covariances are obtained. In Section 4 we exploit the duality relation of Section 2 to obtain expressions for the time-dependent (Theorem 17) and stationary (Theorem 19) LSTs of the GAR$^+$ process, solely relying on the INGAR$^+$ results of Section 3. We also obtain various results concerning the joint LST of Z_n and Z_{n+1} and moments. Section 5 contains a discussion and suggestions for further research.

2. Transforms and duality

In this section we establish a remarkable duality between the INGAR$^+$ model and the GAR$^+$ model. With this duality we can construct for any given GAR$^+$ process an INGAR$^+$ counterpart. Later on in this paper we will use the duality as a device to translate results for the GAR$^+$ model into results for the INGAR$^+$ model.

We introduce a family $(\boldsymbol{N}_\gamma)_{\gamma>0}$ of transformations that map non-negative random variables to non-negative integer-valued random variables as follows. Given a non-negative random variable X, let $\boldsymbol{N}_\gamma(X)$ denote *any* random variable with a *mixed Poisson distribution* of the form

$$\mathbb{P}\left(\boldsymbol{N}_\gamma(X) = k \mid X = x\right) = e^{-\gamma x}\frac{(\gamma x)^k}{k!}, \qquad k = 0, 1, 2, \ldots;$$

see e.g. Grandell [8]. Consequently,

$$\mathbb{P}\left(\boldsymbol{N}_\gamma(X) = k\right) = \int_{[0,\infty)} e^{-\gamma x}\frac{(\gamma x)^k}{k!}\, \mathbb{P}\left(X \in \mathrm{d}x\right).$$

Thus a sample of $\boldsymbol{N}_\gamma(X)$ can be obtained by letting $(N(t))_{t\geq 0}$ be an independent Poisson process with rate γ and set $\boldsymbol{N}_\gamma(X) = N(X)$. Although $\boldsymbol{N}_\gamma(X)$ actually denotes a class of random variables with a common distribution, we still write, with minor abuse of notation, $Y = \boldsymbol{N}_\gamma(X)$ to indicate that Y has the same distribution as any member of $\boldsymbol{N}_\gamma(X)$. The above transformation has been used by McKenzie [13] to describe the similarity of INAR(1) and AR(1) processes.

Theorem 1 (Duality)**.** *The APGF of the transformed variable $\boldsymbol{N}_\gamma(X)$ is related to the LST of the original variable X through*

$$G_{\boldsymbol{N}_\gamma(X)}(s) = \varphi_X(\gamma s). \tag{7}$$

In particular, given the relevant expectations and/or variances exist,

$$\mathbb{E}(\boldsymbol{N}_\gamma(X)) = \gamma\, \mathbb{E}\left(X\right), \tag{8}$$

$$\mathrm{Var}\left(\boldsymbol{N}_\gamma(X)\right) = \gamma^2 \mathrm{Var}\left(X\right) + \gamma \mathbb{E}\left(X\right). \tag{9}$$

More generally, the joint APGF of the transforms (assuming two independent Poisson transformations) is given by

$$G_{\boldsymbol{N}_{\gamma_1}(X),\boldsymbol{N}_{\gamma_2}(Y)}(s,t) = \varphi_{X,Y}(\gamma_1 s, \gamma_2 t). \tag{10}$$

Proof. We prove only (10), as (7) is obviously a special case of it. This follows by observing

$$\begin{aligned}
&G_{\boldsymbol{N}_{\gamma_1}(X),\boldsymbol{N}_{\gamma_2}(Y)}(s,t) = \mathbb{E}((1-s)^{N_{\gamma_1}(X)}(1-t)^{N_{\gamma_2}(Y)}) \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\int_0^\infty\int_0^\infty e^{-\gamma_1 x}\frac{(\gamma_1 x)^k}{k!}e^{-\gamma_2 y}\frac{(\gamma_2 y)^\ell}{\ell!}(1-s)^k(1-t)^\ell\, \mathbb{P}\left(X \in \mathrm{d}x, Y \in \mathrm{d}y\right) \\
&= \int_0^\infty\int_0^\infty e^{-\gamma_1 x s - \gamma_2 y t}\, \mathbb{P}\left(X \in \mathrm{d}x, Y \in \mathrm{d}y\right) \\
&= \mathbb{E}(e^{-\gamma_1 s X - \gamma_2 t Y}) = \varphi_{X,Y}(\gamma_1 s, \gamma_2 t).
\end{aligned}$$

The claims (8) and (9) follow directly from (7), applying standard rules for deriving moments from the respective transforms. □

We need the next proposition to establish a relation between (the transforms of) the random sum $U(\cdot)$ and subordinator $S(\cdot)$ operations which were defined in Section 1. Recall that we imposed the condition (C), guaranteeing the existence of a $\gamma > 0$ such that $\gamma \geq \psi(\gamma)$.

Proposition 2. *Let ν be the Lévy measure as defined in Section 1, and let $\gamma > 0$ be such that $\gamma \geq \psi(\gamma)$. Then*

$$G_{\Theta}(s) := 1 - \frac{\psi(\gamma s)}{\gamma}$$

is the APGF of a non-negative integer-valued random variable Θ given by the probabilities

$$\theta_0 := 1 - \frac{\psi(\gamma)}{\gamma},$$
$$\theta_k := a\mathbb{1}_{\{k=1\}} + \frac{\gamma^{k-1}}{k!}\int_0^\infty e^{-\gamma u}u^k \,\mathrm{d}\nu(u), \quad k = 1, 2, 3, \dots$$

Proof. We first show that the numbers θ_k are indeed probabilities. Obviously $\theta_0 \leq 1$, and since $\gamma \geq \psi(\gamma)$ it follows that $\theta_0 \geq 0$, so θ_0 is a probability. Moreover, $\theta_k \geq 0$ for $k \in \mathbb{N}_0$ and

$$\begin{aligned}\sum_{k=0}^{\infty}\theta_k &= 1 - \frac{\psi(\gamma)}{\gamma} + a + \sum_{k=1}^{\infty}\frac{\gamma^{k-1}}{k!}\int_0^\infty e^{-\gamma u}u^k \,\mathrm{d}\nu(u) \\ &= 1 - \frac{1}{\gamma}\Big(\psi(\gamma) - a\gamma - \int_0^\infty \left(1 - e^{-\gamma u}\right)\,\mathrm{d}\nu(u)\Big) = 1.\end{aligned}$$

The APGF of the non-negative integer-valued random variable Θ is given by

$$\begin{aligned}G_{\Theta}(s) &= 1 - \frac{\psi(\gamma)}{\gamma} + (1-s)a + \sum_{k=1}^{\infty}(1-s)^k\frac{\gamma^{k-1}}{k!}\int_0^\infty e^{-\gamma u}u^k \,\mathrm{d}\nu(u) \\ &= 1 - \frac{\psi(\gamma)}{\gamma} + (1-s)a + \frac{1}{\gamma}\int_0^\infty (1 - e^{-\gamma u} - 1 + e^{-s\gamma u})\,\mathrm{d}\nu(u) = 1 - \frac{\psi(\gamma s)}{\gamma},\end{aligned}$$

thus establishing the claim. □

As the next theorem shows, the introduced transformation is well adapted to all operations we use to define our autoregressive processes, namely addition, reflection at zero, and the random sum and subordinator operations $U(\cdot)$ and $S(\cdot)$.

Theorem 3. *Let X be a non-negative random variable.*

1. *If Y is non-negative and independent of X, then*

$$\boldsymbol{N}_{\boldsymbol{\gamma}}(X + Y) =_{\mathrm{d}} \boldsymbol{N}_{\boldsymbol{\gamma}}(X) + \boldsymbol{N}_{\boldsymbol{\gamma}}(Y), \tag{11}$$

   with the two random variables on the righthand side being independent.

2. *If $\gamma \geq \psi(\gamma)$ and $U_k =_{\mathrm{d}} \Theta$ for every $k \in \mathbb{N}_0$, where Θ is as in Proposition 2, then*

$$\boldsymbol{N}_{\boldsymbol{\gamma}}(S(X)) =_{\mathrm{d}} U(\boldsymbol{N}_{\boldsymbol{\gamma}}(X)). \tag{12}$$

3. *Let B be exponential with rate $\lambda > 0$, let W be a geometric random variable with $\mathbb{P}(W = k) = p(1-p)^{k-1}$, $k \in \mathbb{N}$, $p \in (0, 1]$, and let both random variables be independent of X. Then,*

$$\boldsymbol{N}_{\boldsymbol{\lambda/p}}((X - B)^+) =_{\mathrm{d}} (\boldsymbol{N}_{\boldsymbol{\lambda/p}}(X) - W)^+. \tag{13}$$

Proof. 1. This follows from (7):

$$G_{N_\gamma(X+Y)}(s) = \varphi_{X+Y}(\gamma s) = \varphi_X(\gamma s)\varphi_Y(\gamma s) = G_{N_\gamma(X)}(s)G_{N_\gamma(Y)}(s),$$

where the second equality is due to the independence of X and Y.

2. We have by the well-known formulas for subordination (in combination with (7))

$$\begin{aligned} G_{N_\gamma(S(X))}(s) &= \varphi_{S(X)}(\gamma s) = \varphi_X(\psi(\gamma s)) = G_{N_\gamma(X)}(\psi(\gamma s)/\gamma) \\ &= G_{N_\gamma(X)}(1-(1-\psi(\gamma s)/\gamma)) = G_{U(N_\gamma(X))}(s). \end{aligned} \tag{14}$$

3. Let $\gamma := \lambda/p$. Using (53) in the appendix in the second and fourth equality, and (7) in the third equality, we obtain

$$\begin{aligned} G_{N_{\lambda/p}((X-B)^+)}(s) &= \varphi_{(X-B)^+}(\lambda s/p) = \varphi_X(\lambda) + p\frac{\varphi_X(\lambda s/p) - \varphi_X(\lambda)}{p-s} \\ &= G_{N_{\lambda/p}(X)}(p) + p\frac{G_{N_{\lambda/p}(X)}(s) - G_{N_{\lambda/p}(X)}(p)}{p-s} \\ &= G_{(N_{\lambda/p}(X)-W)^+}(s). \end{aligned}$$

□

The main question of this section is: given a GAR$^+$ process $(Z_n)_{n\in\mathbb{N}_0}$, can we explicitly construct an integer-valued counterpart, i.e., an INGAR$^+$ process $(A_n)_{n\in\mathbb{N}_0}$? (And, if yes, how?) To study this, let $S(\cdot)$, Y_n, and λ (defining the GAR$^+$ process $(Z_n)_{n\in\mathbb{N}_0}$) be given. In a naïve construction one would take, for some value of γ,

$$K_n =_{\mathrm{d}} \boldsymbol{N}_\gamma(S(Z_n)),\ \ C_n =_{\mathrm{d}} \boldsymbol{N}_\gamma(Y_n),\ \ W_n =_{\mathrm{d}} \mathrm{Geom}(\lambda/\gamma),\ \text{ and }\ A_{n+1} := (K_n + C_n - W_n)^+.$$

Note that since ψ is concave, if $\gamma' \geq \psi(\gamma')$ for some $\gamma' > 0$ (which we have, as Condition (C) is in place), then $\gamma \geq \psi(\gamma)$ for every $\gamma > \gamma'$. In other words, under Condition (C) one can always achieve $\gamma \geq \lambda$. Then indeed, by Theorem 3,

$$\begin{aligned} A_{n+1} &=_{\mathrm{d}} \left(\boldsymbol{N}_\gamma(S(Z_n)) + \boldsymbol{N}_\gamma(Y_n) - W_n\right)^+ =_{\mathrm{d}} \left(\boldsymbol{N}_\gamma(S(Z_n) + Y_n) - W_n\right)^+ \\ &=_{\mathrm{d}} \boldsymbol{N}_\gamma((S(Z_n) + Y_n - B_n)^+) =_{\mathrm{d}} \boldsymbol{N}_\gamma(Z_{n+1}). \end{aligned}$$

However, if we do not carefully select the appropriate Poisson transformations, the *joint* distribution of A_n and A_{n+1} might be different from the required INGAR$^+$-type bivariate relation

$$(A_n, A_{n+1}) =_{\mathrm{d}} (A_n, (U_n(A_n) + C_n - W_n)^+) \tag{15}$$

and hence $(A_n)_{n\in\mathbb{N}_0}$ would not necessarily qualify as an INGAR$^+$ process. In the next theorem, we point out how $(A_n)_{n\in\mathbb{N}_0}$ should be properly defined.

Theorem 4. *Let $(Z_n)_{n\in\mathbb{N}_0}$ be a GAR$^+$ process as in* (5)*. Suppose that the Conditions* (S1) *and* (S2) *hold and that Condition* (C) *is in place with $\gamma \geq \lambda$. Then there is an INGAR$^+$ process $(A_n)_{n\in\mathbb{N}_0}$ as in* (3) *such that*

1. $A_n =_{\mathrm{d}} \boldsymbol{N}_\gamma(Z_n)$*,*

2. $U_n(A_n) =_{\mathrm{d}} \boldsymbol{N}_\gamma(S_n(Z_n))$ *and the i.i.d. summands* $U_{n,k}$*,* $k \in \mathbb{N}_0$*, have the same distribution as* Θ *in Proposition* 2*,*

3. $C_n =_{\mathrm{d}} \boldsymbol{N}_\gamma(Y_n)$*,*

4. $W_n =_{\mathrm{d}} \mathrm{Geom}(\lambda/\gamma)$.

Proof. We construct A_n inductively. To this end, let $A_0 =_{\mathrm{d}} \boldsymbol{N}_\gamma(Z_0)$ and suppose that we already constructed $A_1, A_2, \ldots, A_n$ complying with 1–4. Let E_n be a random variable with the distribution of $\boldsymbol{N}_\gamma(S_n(Z_n) + Y_n)$, but chosen in a way such that jointly $(A_n, E_n) =_{\mathrm{d}} (A_n, U_n(A_n) + C_n)$, which is possible since $E_n =_{\mathrm{d}} U_n(A_n) + C_n$ by (11) and (12). Let W_n have a geometric distribution with success probability $\lambda/\gamma \leq 1$ and set $A_{n+1} = (E_n - W_n)^+$. Then (15) holds, i.e., $(A_n)_{n\in\mathbb{N}_0}$ is an INGAR$^+$ process and we also have

$$A_{n+1} =_{\mathrm{d}} \boldsymbol{N}_\gamma((S_n(Z_n) + Y_n - B_n)^+) =_{\mathrm{d}} \boldsymbol{N}_\gamma(Z_{n+1}),$$

by (13). □

Remark 5. Suppose condition (C) is not fulfilled, i.e., $\gamma < \psi(\gamma)$ for all $\gamma > 0$. Then relation (12), i.e., $\boldsymbol{N}_\gamma(S(X)) =_{\mathrm{d}} U(\boldsymbol{N}_\gamma(X))$, is no longer true if we interpret $U(X)$ as a random sum. The reason is that, in (14), $1-\psi(\gamma s)/\gamma$ no longer is the APGF of a non-negative integer-valued random variable. It is still, however, true that

$$G_{\boldsymbol{N}_\gamma(S(X))}(s) = \varphi_{S(X)}(\gamma s) = \varphi_X(\psi(\gamma s)) = G_{\boldsymbol{N}_\gamma(X)}(\psi(\gamma s)/\gamma),$$

implying that whenever $Y = \boldsymbol{N}_\gamma(X)$ for some X we can define $U(Y)$ as any random variable for which

$$G_{U(Y)}(s) = G_Y(\psi(\gamma s)/\gamma). \tag{16}$$

Recalling the above construction we see that $A_0 =_{\mathrm{d}} \boldsymbol{N}_\gamma(Z_0)$ is permissible, and hence an appropriate $U(A_0)$ can be defined. Continuing with $A_1, A_2, \ldots$ we could define a 'generalized' INGAR$^+$ process, where $U(\cdot)$ no longer describes a random sum but some abstract operation. In what follows, we throughout impose condition (C), so that the random sum interpretation applies. ◇

3. The INGAR$^+$ model

In this section we analyze the INGAR$^+$ model, with as main objective to uniquely characterize its time-dependent and stationary behavior. Recall that the INGAR$^+$ model is defined by the recursion

$$A_{n+1} := (U_n(A_n) + C_n - W_n)^+, \tag{17}$$

with $(U_n(\cdot))_{n\in\mathbb{N}_0}$, $(C_n)_{n\in\mathbb{N}_0}$, and $(W_n)_{n\in\mathbb{N}_0}$ as introduced earlier; in particular W_n has a geometric distribution with success probability p. It requires a direct verification to see that the APGF of the random sum $U_n(A_n)$ is given by $G_{A_n}(\Psi(s))$, where $\Psi(s) := 1 - G_U(s)$. The function $\Psi(\cdot)$ is increasing and concave with $\Psi(0) = 0$. We will make frequent use of the iterates

$$\Psi^{(0)}(s) = s, \quad \Psi^{(k)}(s) = \Psi(\Psi^{(k-1)}(s)), \quad k = 1, 2, \ldots.$$

The time-dependent behavior of A_n is studied in Section 3.1 and the stationary behavior in Section 3.2. Joint APGFs and moments are derived in Section 3.3.

3.1. Time-dependent analysis

A specific type of functional difference equation naturally appears in the analysis of the INGAR$^+$ and GAR$^+$ models. The following lemma gives a solution of this difference equation, for a sufficiently general setup. The proof is standard, in that it follows directly by iterating the equation (and is therefore omitted).

Lemma 6. *Suppose that, for a given initial value f_0, a sequence of functions $f = (f_n)_{n\in\mathbb{N}_0}$ is defined by*

$$f_n(s) = \pi(s) f_{n-1}(\Psi(s)) - \varrho(s) f_{n-1}(\Psi(p)) + \kappa, \quad n \geq 1, \tag{18}$$

for functions $\pi(\cdot)$ and $\varrho(\cdot)$ and a constant κ. Then

$$\begin{aligned} f_n(s) = {} & f_0(\Psi^{(n)}(s)) \prod_{i=0}^{n-1} \pi(\Psi^{(i)}(s)) \\ & + \kappa \sum_{i=0}^{n-1} \prod_{j=0}^{i-1} \pi(\Psi^{(j)}(s)) - \sum_{i=1}^{n} f_{n-i}(\Psi(p)) \varrho(\Psi^{(i-1)}(s)) \prod_{j=0}^{i-2} \pi(\Psi^{(j)}(s)), \end{aligned} \tag{19}$$

for $n \in \mathbb{N}_0$. The values of $f_j(\Psi(p))$ follow recursively by inserting $s = \Psi(p)$ into (19).

We apply the above lemma in order to obtain the APGFs $G_{A_n}(\cdot)$, $n = 1, 2, \ldots$, when $G_{A_0}(\cdot)$ is given. Define

$$\Pi_n(s) := \prod_{k=0}^{n-1} \frac{p G_C(\Psi^{(k)}(s))}{p - \Psi^{(k)}(s)}, \qquad \Gamma_n(s) := \frac{\Psi^{(n)}(s)}{p - \Psi^{(n)}(s)} \Pi_n(s), \tag{20}$$

with empty products to be defined equal to one. Whenever the infinite product $\lim_{n\to\infty} \Pi_n(s)$ converges we simply write $\Pi_\infty(s)$ for its value. The following result provides the APGFs $G_{A_n}(\cdot)$ in terms of the functions $\Pi_n(\cdot)$ and $\Gamma_n(\cdot)$ featuring in (20).

Theorem 7. *For $n = 0, 1, \ldots$ and $s \in [0, 1]$,*

$$G_{A_n}(s) = G_{A_0}(\Psi^{(n)}(s))\, \Pi_n(s) - G_C(p) \sum_{j=0}^{n-1} G_{A_{n-j-1}}(\Psi(p))\, \Gamma_j(s). \tag{21}$$

The values of $G_{A_n}(\Psi(p))$ follow recursively by inserting $s = \Psi(p)$ into (21)*; see Remark* 9.

Proof. By rearranging relation (53) in the appendix, we obtain from (17):

$$
\begin{aligned}
G_{A_{n+1}}(s) &= \frac{p}{p-s} G_{U_n(A_n)+C_n}(s) - \frac{s}{p-s} G_{U_n(A_n)+C_n}(p) \\
&= \underbrace{\frac{pG_C(s)}{p-s}}_{\pi(s)} G_{A_n}(\Psi(s)) - \underbrace{\frac{sG_C(p)}{p-s}}_{\varrho(s)} G_{A_n}(\Psi(p)).
\end{aligned} \tag{22}
$$

This function is of the type (18) with $\kappa = 0$. □

Remark 8. Since (21) is a consequence of the purely arithmetic Lemma 6, there are no issues in relation to convergence. Relation (21) is true for all $s \in [0, 1]$. Note that the first term of the difference on the right-hand side has the same singularities as the second term (which are the values s for which $p = \Psi^{(k)}(s)$ for some k). It can be verified that these singularities are removable; each singularity in the first term of the righthand side of (21) is compensated by that same singularity in the second term of the righthand side. In this respect, observe that $s = p$ is a removable singularity in (22). ◇

Remark 9. Inserting $s = \Psi(p)$ into (21) shows that

$$
G_{A_n}(\Psi(p)) = G_{A_0}(\Psi^{(n+1)}(p))\,\Pi_n(\Psi(p)) - G_C(p)\sum_{j=0}^{n-1} G_{A_{n-j-1}}(\Psi(p))\,\Gamma_j(\Psi(p)). \tag{23}
$$

With this relation the constants $G_{A_n}(\Psi(p))$ can be found recursively. ◇

3.2. Stationary analysis

Now we turn to the stationary analysis. In the analysis, an important role is played by ξ, denoting the limit as $n \to \infty$ of the probability that $U_n(A_n) + C_n - W_n$ is strictly smaller than zero, i.e.,

$$
\xi = \lim_{n\to\infty} \mathbb{P}\left(W_n > U_n(A_n) + C_n\right),
$$

whenever it exists.

Theorem 10. *If* (S1) *holds, then the INGAR$^+$ process $(A_n)_{n\in\mathbb{N}_0}$ is positive recurrent. If it is also aperiodic and irreducible, then the stationary APGF is given by*

$$
G_A(s) = \Pi_\infty(s) - \xi\,\Sigma(s), \tag{24}
$$

where $\Sigma(s) := \sum_{n=0}^{\infty} \Gamma_n(s)$, and

$$
\xi = G_C(p)\,G_A(\Psi(p)) = \frac{G_C(p)\,\Pi_\infty(\Psi(p))}{1 + G_C(p)\,\Sigma(\Psi(p))}. \tag{25}
$$

Proof. Let the process $(A_n^+)_{n\in\mathbb{N}_0}$ be defined by $A_0^+ = A_0$ and $A_{n+1}^+ = U_n(A_n^+) + C_n$. Then $(A_n^+)_{n\in\mathbb{N}_0}$ is a Galton-Watson branching process with immigration. As follows from Heathcote [9, 10], under (S1) this process is positive recurrent and since it majorizes $(A_n)_{n\in\mathbb{N}_0}$, the same follows for our INGAR$^+$ process.

To determine the APGF of the stationary distribution we use the generating functions

$$A(r,s) = \sum_{n=0}^{\infty} r^n G_{A_n}(s), \quad B(r,s) = \sum_{n=0}^{\infty} r^n \beta_n(s), \quad D(r,s) = \sum_{n=0}^{\infty} r^n \Gamma_n(s), \quad r \in (-1,1),$$

where $\beta_n(s) := G_{A_0}(\Psi^{(n)}(s))\, \Pi_n(s)$. It follows from (23) that

$$G_{A_n}(\Psi(p)) = \beta_n(\Psi(p)) - G_C(p) \sum_{j=0}^{n-1} G_{A_{n-j-1}}(\Psi(p))\, \Gamma_j(\Psi(p)),$$

and hence, after standard algebraic manipulations,

$$A(r,\Psi(p)) = \frac{B(r,\Psi(p))}{1 + G_C(p)\, rD(r,\Psi(p))}.$$

Under condition (S1) we have $\mathbb{E}(U) < 1$ and $\Psi^{(n)}(s) = O((\mathbb{E}(U))^n) \downarrow 0$ as $n \to \infty$; see Athreya and Ney [3, Theorem 11.1]. It follows that the product

$$\prod_{k=0}^{n-1} \frac{p}{p - \Psi^{(k)}(s)}$$

tends to a finite non-zero limit as $n \to \infty$. Moreover, $\prod_{k=0}^{\infty} G_C(\Psi^{(k)}(s))$ is the LST of the limit distribution of the Galton-Watson process $(A_n^+)_{n\in\mathbb{N}_0}$; see e.g. Heathcote [9]. Hence $\beta_n(s)$ tends to a finite non-zero limit $\Pi_\infty(s)$. The convergence of β_n together with $\Psi^{(n)}(s) = O((\mathbb{E}(U))^n) \downarrow 0$ implies the convergence of $\sum_{k=0}^{n} \Gamma_k(s)$ to $\Sigma(\Psi(s))$ as $n \to \infty$. Hence, using Abel's theorem, we obtain

$$G_A(\Psi(p)) = \lim_{r\uparrow 1}(1-r)A(r,\Psi(p)) = \frac{\lim_{r\uparrow 1}(1-r)B(r,\Psi(p))}{1 + G_C(p) \lim_{r\uparrow 1} rD(r,\Psi(p))} = \frac{\Pi_\infty(\Psi(p))}{1 + G_C(p)\,\Sigma(\Psi(p))}, \tag{26}$$

and

$$\begin{aligned} G_A(s) = \lim_{r\uparrow 1}(1-r)A(r,s) &= \lim_{r\uparrow 1}(1-r)\; B(r,s) - \lim_{r\uparrow 1}\;(1-r)G_C(p)D(r,s)A(r,\Psi(p)) \\ &= \Pi_\infty(s) - G_A\,(\Psi(p))\, G_C(p)\, \Sigma(s). \end{aligned}$$

It remains to prove that $\xi_n = \mathbb{P}(W_n > U_n(A_n) + C_n)$ indeed converges to $G_C(p)\, G_{A_n}(\Psi(p))$ as $n \to \infty$. According to (54) in the appendix and the recurrence relation (17) we obtain

$$\xi_n = G_{U_n(A_n)+C_n}(p) = G_C(p)\, G_{A_n}(\Psi(p)).$$

Claim (25) thus follows by sending n to ∞. □

Remark 11. Condition (S1) is clearly not necessary in the case where $U_n \equiv 1$. In this case the INGAR$^+$ process $A_{n+1} = (A_n + C_n - W_n)^+$ is a reflected random walk and, as is well known, $\mathbb{E}(C) < \mathbb{E}(W)$ ensures positive recurrence. However, it is not obvious how necessary conditions can be derived in the general INGAR$^+$/GAR$^+$ setting. To illustrate the complications one encounters in studying the $\mathbb{E}(U) = 1$ case, assume that additionally $\mathrm{Var}(U) < \infty$. One can show that in this case $\Psi^{(k)}(s) \sim 1/k$ as $k \to \infty$; see Athreya and Ney [3, Theorem 11.1]. This implies that both main terms in (21), i.e.,

$$G_{A_0}(\Psi^{(n)}(s))\,\Pi_n(s) \quad \text{and} \quad G_C(p)\sum_{j=0}^{n-1} G_{A_{n-j-1}}(\Psi(p))\,\Gamma_n(s),$$

tend to infinity as $n \to \infty$. It follows that it is not clear whether their difference tends to zero, tends to a finite non-zero limit, or does not converge at all. ◇

Remark 12. In passing, we have also shown that

$$A(r,s) = B(r,s) - G_C(p)\,A(r,\Psi(p))\,rD(r,s) = B(r,s) - G_C(p)\frac{B(r,\Psi(p))\,rD(r,s)}{1 + G_C(p)\,rD(r,\Psi(p))},$$

as revealed by the proof of Theorem 10.
In case $A_0 = \ell$, we have that $B(r,s)$ equals $B(r,s\,|\,\ell)$, given by

$$B(r,s\,|\,\ell) := \sum_{n=0}^{\infty} r^n (1 - \Psi^{(n)}(s))^{\ell}\,\Pi_n(s). \tag{27}$$

The fact that this is a power in ℓ will be exploited in the proof of Theorem 16. ◇

3.3. Moments and covariance structure

In this subsection we include various results concerning the moments and covariance structure of the INGAR$^+$ process $(A_n)_{n\in\mathbb{N}_0}$. As before, ξ_n is defined by $\mathbb{P}(W_n > U_n(A_n) + C_n)$, which we have seen to equal $G_C(p)\,G_{A_n}(\Psi(p))$.

Theorem 13. *The mean and the variance of the INGAR$^+$ process fulfil the following recursions:*

$$\mathbb{E}(A_{n+1}) = \mathbb{E}(U)\,\mathbb{E}(A_n) + \mathbb{E}(C) - \frac{1-\xi_n}{p}, \tag{28}$$

$$\mathrm{Var}(A_{n+1}) = \mathrm{Var}(A_n)\,\mathbb{E}(U)^2 + \mathbb{E}(A_n)\,\mathrm{Var}(U) + \mathrm{Var}(C) - \frac{2\xi_n}{p}(\mathbb{E}(A_n)\,\mathbb{E}(U) + \mathbb{E}(C)) + \frac{(1-\xi_n)(1-p+\xi_n)}{p^2}. \tag{29}$$

In stationarity,

$$\mathbb{E}(A) = \frac{\mathbb{E}(C) - \dfrac{1-\xi}{p}}{1 - \mathbb{E}(U)}, \tag{30}$$

$$\mathrm{Var}(A) = \frac{\mathbb{E}(A)\,\mathrm{Var}(U) + \mathrm{Var}(C) - \dfrac{2\xi}{p}(\mathbb{E}(A)\,\mathbb{E}(U) + \mathbb{E}(C)) + \dfrac{(1-\xi)(1-p+\xi)}{p^2}}{1 - (\mathbb{E}(U))^2}. \tag{31}$$

Proof. We start by multiplying (22) by $p - s$; recalling that $\xi_n = G_{U_n(A_n)+C_n}(p)$, we obtain by differentiating with respect to s

$$-G_{A_{n+1}}(s) + (p-s)G'_{A_{n+1}}(s) = pG'_C(s)G_{A_n}(\Psi(s)) + pG_C(s)G'_{A_n}(\Psi(s))\Psi'(s) - \xi_n. \quad (32)$$

Letting $s \to 0$ we obtain

$$-1 - p\mathbb{E}\,(A_{n+1}) = -p\mathbb{E}\,(C) - p\mathbb{E}\,(A_n)\,\mathbb{E}\,(U) - \xi_n,$$

from which (28) follows. Moreover, taking another derivative in (32) and letting $s \to 0$ we obtain

$$\begin{aligned} p(\mathbb{E}(A_{n+1}^2) - \mathbb{E}(A_{n+1})) + 2\mathbb{E}\,(A_{n+1}) = {} & p(\mathbb{E}(C^2) - \mathbb{E}(C)) + 2p\mathbb{E}\,(C)\,\mathbb{E}\,(A_n)\,\mathbb{E}\,(U) + \\ & p(\mathbb{E}(A_n^2) - \mathbb{E}(A_n))\mathbb{E}\,(U)^2 - p\mathbb{E}\,(A_n)\,(\mathbb{E}(U) - \mathbb{E}(U^2)), \end{aligned}$$

which leads to (29). The stationary mean (30) and variance (31) follow by letting n tend to infinity and solving $\mathbb{E}\,(A)$ and $\mathrm{Var}\,(A)$, respectively. □

Besides the mean and variance of A_n, we can use similar techniques to obtain insight into the process' correlation structure. Our next objective is to evaluate the joint APGF of A_n and A_{n+1}. It expresses this joint APGF $G_{A_n,A_{n+1}}(s,t)$ in terms of the (univariate) APGF of A_n, which is given by Theorem 7. We restrict ourselves to $t \neq p$; the result for $t = p$ follows in an elementary way by taking a limit.

Theorem 14. *For $t \neq p$,*

$$\begin{aligned} G_{A_n,A_{n+1}}(s,t) = {} & \frac{p}{p-t}\,G_C(t)\,G_{A_n}(1-(1-s)G_U(t)) \\ & - \frac{t}{p-t}\,G_C(p)\,G_{A_n}(1-(1-s)G_U(p)). \end{aligned} \quad (33)$$

Proof. By conditioning we obtain

$$\begin{aligned} G_{A_n,A_{n+1}}(s,t) &= \mathbb{E}((1-s)^{A_n}(1-t)^{(U_n(A_n)+C_n-W_n)^+}) \\ &= \mathbb{E}\Big((1-s)^{A_n}\mathbb{E}((1-t)^{(U_n(A_n)+C_n-W_n)^+}|C_n, W_n, U(A_n))\Big). \end{aligned}$$

By (53) in the appendix it follows that for every $k \in \mathbb{N}_0$, provided that $p \neq t$,

$$\mathbb{E}((1-t)^{(k-W_n)^+}) = \frac{p}{p-t}(1-t)^k - \frac{t}{p-t}(1-p)^k.$$

Hence, for $p \neq t$,

$$\begin{aligned} & \mathbb{E}\Big((1-s)^{A_n}\mathbb{E}((1-t)^{(U_n(A_n)+C_n-W_n)^+}|C_n, W_n, U(A_n))\Big) \\ &= \mathbb{E}((1-s)^{A_n}\frac{p}{p-t}(1-t)^{U_n(A_n)+C_n} - (1-s)^{A_n}\frac{t}{p-t}(1-p)^{U(A_n)+C_n}) \\ &= \frac{p}{p-t}G_C(t)\,G_A(1-(1-s)G_U(t)) - \frac{t}{p-t}G_C(p)\,G_A(1-(1-s)G_U(p)), \end{aligned}$$

where we used the fact that

$$\begin{aligned} \mathbb{E}((1-s)^A(1-t)^{U(A)}) &= \mathbb{E}((1-s)^A\mathbb{E}((1-t)^{U(A)}|A)) = \mathbb{E}((1-s)^A G_{U(A)}(t)) \\ &= \mathbb{E}((1-s)^A G_U(t)^A) = G_A(1-(1-s)\,G_U(t)). \end{aligned}$$

□

Theorem 15. *The covariance of A_n and A_{n+1} is given by*

$$\mathrm{Cov}\,(A_n, A_{n+1}) = \mathbb{E}\,(U)\,\mathrm{Var}\,(A_n) - \frac{\xi_n}{p}\mathbb{E}\,(A_n) - \frac{1}{p}G_C(p)G_U(p)G'_{A_n}(\Psi(p)). \tag{34}$$

Proof. To derive an expression for $\mathbb{E}\,(A_n\,A_{n+1})$, we first take the derivative of (33) with respect to s, to obtain

$$\begin{aligned}\frac{\partial}{\partial s}G_{A_n,A_{n+1}}(s,t) &= \frac{p}{p-t}G_C(t)G'_{A_n}(1-(1-s)G_U(t))\,G_U(t)\\ &\quad+\Big(1-\frac{p}{p-t}\Big)G_C(p)G'_{A_n}(1-(1-s)G_U(p))\,G_U(p).\end{aligned}$$

Then taking the derivative with respect to t yields

$$\begin{aligned}\frac{\partial^2}{\partial t\partial s}G_{A_n,A_{n+1}}(s,t) &= \frac{p}{(p-t)^2}G_C(t)\,G'_{A_n}(1-(1-s)G_U(t))\,G_U(t)\\ &\quad+\frac{p}{p-t}G'_C(t)\,G'_{A_n}(1-(1-s)G_U(t))\,G_U(t)\\ &\quad-(1-s)\frac{p}{p-t}G_C(t)G''_{A_n}(1-(1-s)G_U(t))G'_U(t)\,G_U(t)\\ &\quad+\frac{p}{p-t}G_C(t)\,G'_{A_n}(1-(1-s)G_U(t))\,G'_U(t)\\ &\quad-\frac{p}{(p-t)^2}\,G_C(p)\,G'_{A_n}(1-(1-s)G_U(p))\,G_U(p).\end{aligned}$$

Letting $s\downarrow 0$ and $t\downarrow 0$ we obtain

$$\mathbb{E}\,(A_n\,A_{n+1}) = \mathbb{E}\,(A_n)\Big(\mathbb{E}\,(C)-\frac{1}{p}\Big)+\mathbb{E}(A_n^2)\mathbb{E}\,(U)-\frac{G_C(p)\,G'_{A_n}(\Psi(p))\,G_U(p)}{p}. \tag{35}$$

Then

$$\begin{aligned}\mathrm{Cov}\,(A_n, A_{n+1}) = &\,\mathbb{E}\,(U)\,\mathbb{E}(A_n^2)+\mathbb{E}\,(A_n)\left(\mathbb{E}\,(C)-\frac{1}{p}\right)-\frac{1}{p}G_C(p)\,G_U(p)\,G'_{A_n}(\Psi(p))\\ &-\mathbb{E}\,(A_n)\Big(\mathbb{E}\,(A_n)\,\mathbb{E}\,(U)+\mathbb{E}\,(C)-\frac{1-\xi_n}{p}\Big),\end{aligned}$$

which can be checked to equal the righthand side of (34). □

We conclude this section by presenting the joint APGF of A_0 and A_N, where N is (shifted) geometrically distributed, i.e., $\mathbb{P}(N=n)=r^n(1-r)$ for $n\in\mathbb{N}_0$ and $r\in[0,1]$. We assume that A_0 satisfies the equilibrium distribution, characterized in Theorem 10 (which, evidently, implies that also A_N follows the stationary distribution). Combining the above results, we obtain a representation for $G_{A_0,A_N}(t,s)=\mathbb{E}((1-t)^{A_0}(1-s)^{A_N})$, as follows. First observe that

$$\begin{aligned}G_{A_0,A_N}(t,s) &= (1-r)\sum_{n=0}^{\infty}r^n\sum_{\ell=0}^{\infty}\mathbb{E}((1-t)^{A_0}(1-s)^{A_n}\mid A_0=\ell)\,\mathbb{P}\,(A_0=\ell)\\ &= (1-r)\sum_{n=0}^{\infty}r^n\sum_{\ell=0}^{\infty}(1-t)^{\ell}G_{A_n}(s\,|\,\ell)\,\mathbb{P}\,(A_0=\ell)\,,\end{aligned}$$

where $G_{A_n}(s\,|\,\ell) := \mathbb{E}((1-s)^{A_n}\,|\,A_0=\ell)$. Relying on Remark 12, and remarking that in $A(r,s)$ only $B(r,s)$ depends on the distribution of A_0,

$$\sum_{n=0}^{\infty} r^n G_{A_n}(s\,|\,\ell) = \sum_{n=0}^{\infty} r^n (1-\Psi^{(n)}(s))^{\ell}\Pi_n(s) \\ - \frac{G_C(p)\,rD(r,s)}{1+G_C(p)\,rD(r,\Psi(p))}\sum_{n=0}^{\infty} r^n (1-\Psi^{(n+1)}(p))^{\ell}\Pi_n(\Psi(p)).$$

Combining the above elements, and using that A_0 obeys the equilibrium distribution, we arrive after some algebra at the following result.

Theorem 16. *The joint APGF of A_0 and A_N (in stationarity) is given by, with $G_A(\cdot)$ as determined in Theorem 10,*

$$G_{A_0,A_N}(t,s) = (1-r)\sum_{n=0}^{\infty} r^n G_A\big(t+\Psi^{(n)}(s)-t\Psi^{(n)}(s)\big)\,\Pi_n(s) - \\ (1-r)\frac{G_C(p)\,rD(r,s)}{1+G_C(p)\,rD(r,\Psi(p))}\sum_{n=0}^{\infty} r^n G_A(t+\Psi^{(n+1)}(p)-t\Psi^{(n+1)}(p))\,\Pi_n(\Psi(p)).$$

4. The GAR$^+$ model

In this section we investigate the GAR$^+$ model as specified in Section 1:

$$Z_{n+1} = (S_n(Z_n)+Y_n-B_n)^+\,, \tag{36}$$

with the random objects $(S_n(\cdot))_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, and $(Y_n)_{n\in\mathbb{N}_0}$ as introduced earlier; in particular, we have $\mathbb{P}(B_n \le x) = 1-e^{-\lambda x}$ for $x \ge 0$. Recall that $(S(t))_{t\in\mathbb{R}^+}$ is a Lévy subordinator with Laplace exponent ψ.

This section has the same structure as the previous one, but, as it will turn out, we will greatly benefit from the the duality property that was described in Section 2, facilitating direct translation of the INGAR$^+$ results into their GAR$^+$ counterparts. The time-dependent behavior of Z_n is addressed in Section 4.1, while the stationary behavior is covered by Section 4.2; joint LSTs and moments are derived in Section 4.3. While our approach heavily relies on the duality, it is of course also possible to derive the results for GAR$^+$ from scratch, by a similar iterative approach as the one we developed to analyze the INGAR$^+$ model.

4.1. Time-dependent analysis

We choose a probability $p \in (0,1)$ such that for $\gamma := \lambda/p$ the condition $\gamma \ge \psi(\gamma)$ is fulfilled. Then by Theorem 4 there is a dual INGAR$^+$ process

$$A_{n+1} = (U_n(A_n)+C_n-W_n)^+\,,$$

when choosing

$$A_n =_{\rm d} \boldsymbol{N}_\gamma(Z_n), \qquad U_n(A_n) =_{\rm d} \boldsymbol{N}_\gamma(S(Z_n)) \quad \text{(i.e., } U_1 =_{\rm d} \Theta\text{)},$$

$$C_n =_{\mathrm{d}} \boldsymbol{N}_\gamma(Y_n), \qquad\qquad W_n =_{\mathrm{d}} \mathrm{Geom}(p).$$

This identification enables us to translate INGAR$^+$ results to their GAR$^+$ counterparts, as we will show below. Since $\varphi_{Z_n}(s) = G_{A_n}(\gamma s)$ by (7), we can express the results for generating functions in terms of Laplace transforms. Also note that $\Psi(s) = 1 - G_U(s) = \psi(\gamma s)/\gamma$ and it follows that $\psi^{(k)}(s) = \gamma\Psi^{(k)}(s/\gamma)$ and $\Psi^{(k)}(s) = \psi^{(k)}(\gamma s)/\gamma$. Define

$$\Pi_n^*(s) := \prod_{k=0}^{n-1} \frac{\lambda\varphi_Y(\psi^{(k)}(s))}{\lambda - \psi^{(k)}(s)}, \qquad \Gamma_n^*(s) := \frac{\psi^{(n)}(s)}{\lambda - \psi^{(n)}(s)}\,\Pi_n^*(s).$$

The following theorem immediately follows from the duality relations of Section 2 (see, in particular, Theorem 4) and Theorem 7.

Theorem 17. *For $n = 0, 1, \ldots$ and $s \in [0, 1]$,*

$$\varphi_{Z_n}(s) = \varphi_{Z_0}(\psi^{(n)}(s))\,\Pi_n^*(s) - \varphi_Y(\lambda)\sum_{j=0}^{n-1}\varphi_{Z_{n-j-1}}(\psi(\lambda))\,\Gamma_j^*(s). \tag{37}$$

The values of $\varphi_{Z_n}(\psi(\lambda))$ follow recursively by inserting $s = \psi(\lambda)$ into (37).

4.2. Stationary analysis

Regarding the stationary behavior we will mimic Theorem 10. First we show that the stability conditions (S1) and (S2) are equivalent here, if A_n, $U_n(A_n)$, C_n, and W_n are as defined above.

Lemma 18. (S1) *and* (S2) *are equivalent.*

Proof. It follows from Theorem 1 that $\mathbb{E}(Y) < \infty$ is equivalent to $\mathbb{E}(C) < \infty$. Moreover, using integration by parts,

$$\mathbb{E}(\log(1+Y)) = \int_0^\infty \log(1+y)\,\mathbb{P}(Y \in \mathrm{d}y) = \int_0^\infty \frac{1}{1+y}\mathbb{P}(Y > y)\,\mathrm{d}y. \tag{38}$$

Note that, for any given $\varepsilon > 0$,

$$\frac{1}{1+y} \sim \frac{1-e^{-\varepsilon y}}{y},$$

as $y \to \infty$. Hence the integral on the right-hand side of (38) is finite iff

$$\int_0^\infty \frac{1-e^{-\varepsilon y}}{y}\mathbb{P}(Y > y)\,\mathrm{d}y = \int_0^\varepsilon \frac{1-\varphi_Y(s)}{s}\,\mathrm{d}s$$

is finite for some $\varepsilon > 0$ (where the last equality is a consequence of the observation that $(1-e^{-\varepsilon y})/y = \int_0^\varepsilon e^{-sy}\,\mathrm{d}s$). This finiteness condition is, by our duality, equivalent to

$$\int_0^\varepsilon \frac{1}{s}(1 - G_C(s))\,\mathrm{d}s < \infty$$

for some $\varepsilon > 0$. But, due to Heathcote [10], this condition is equivalent to $\mathbb{E}(\log C) < \infty$. $\square$

The following theorem immediately follows from the duality relations of Section 2 (see, in particular, Corollary 4) and Theorem 10.

Theorem 19. *If* (S2) *holds then the GAR$^+$ process* $(Z_n)_{n\in\mathbb{N}_0}$ *is positive recurrent. If it is also aperiodic and irreducible then the limit stationary LST is given by*

$$\varphi_Z(s) = \Pi^*_\infty(s) - \eta\Sigma^*(s), \tag{39}$$

where $\Sigma^*(s) := \sum_{n=0}^\infty \Gamma^*_n(s)$ *and*

$$\eta = \mathbb{P}(Z=0) = \varphi_Y(\lambda)\varphi_Z(\psi(\lambda)) = \frac{\varphi_Y(\lambda)\Pi^*_\infty(\psi(\lambda))}{1+\varphi_Y(\lambda)\Sigma^*(\psi(\lambda))}. \tag{40}$$

4.3. Moments and covariance structure

In this subsection we focus on deriving explicit formulas for moments and joint LSTs.

Theorem 20. *The mean and the variance of the GAR$^+$ process fulfil the following recursions:*

$$\begin{aligned}
\mathbb{E}(Z_{n+1}) &= \mathbb{E}(S(1))\,\mathbb{E}(Z_n) + \mathbb{E}(Y) - \frac{1-\eta_n}{\lambda}, &(41)\\
\operatorname{Var}(Z_{n+1}) &= \operatorname{Var}(Z_n)\,\mathbb{E}(S(1))^2 + \mathbb{E}(Z_n)\operatorname{Var}(S(1)) + \operatorname{Var}(Y)\\
&\qquad - \frac{2\eta_n}{\lambda}\left(\mathbb{E}(Z_n)\,\mathbb{E}(S(1)) + \mathbb{E}(Y)\right) + \frac{(1-\eta_n)(1+\eta_n)}{\lambda^2}, &(42)
\end{aligned}$$

where $\eta_n = \mathbb{P}(Z_{n+1}=0) = \mathbb{P}(B_n > S(Z_n)+Y_n) = \varphi_Y(\lambda)\varphi_{Z_n}(\psi(\lambda))$. *In stationarity,*

$$\begin{aligned}
\mathbb{E}(Z) &= \frac{\mathbb{E}(Y) - \dfrac{1-\eta}{\lambda}}{1-\mathbb{E}(S(1))}, &(43)\\
\operatorname{Var}(Z) &= \frac{\mathbb{E}(Z)\operatorname{Var}(S(1)) + \operatorname{Var}(Y) - \dfrac{2\eta}{\lambda}\left(\mathbb{E}(Z)\,\mathbb{E}(S(1)) + \mathbb{E}(Y)\right) + \dfrac{(1-\eta)(1+\eta)}{\lambda^2}}{1-\mathbb{E}(S(1))^2}. &(44)
\end{aligned}$$

Proof. Just translate Theorem 13 via the duality. Note that

$$\begin{aligned}
&\mathbb{E}(A_n) = \gamma\mathbb{E}(Z_n), \quad \mathbb{E}(C) = \gamma\mathbb{E}(Y),\\
&\operatorname{Var}(A_n) = \gamma^2\operatorname{Var}(Z_n) + \gamma\mathbb{E}(Z_n), \quad \operatorname{Var}(C_n) = \gamma^2\operatorname{Var}(Y_n) + \gamma\mathbb{E}(Y_n),\\
&\mathbb{E}(U) = \mathbb{E}(S(1)), \quad \operatorname{Var}(U) = \mathbb{E}(S(1)) + \gamma\operatorname{Var}(S(1)) - \mathbb{E}(S(1))^2.
\end{aligned}$$

Moreover, since $\lambda = \gamma p$ and $\psi(\gamma p)/\gamma = \Psi(p)$, we have

$$\eta_n = \varphi_{Y_n}(\lambda)\varphi_{Z_n}(\psi(\lambda)) = G_{C_n}(p)G_{A_n}(\psi(\gamma p)/\gamma) = \xi_n. \qquad \square$$

We continue with discussing various results concerning the correlation structure of the $(Z_n)_{n\in\mathbb{N}_0}$ process. We start by evaluating the joint LST of Z_n and Z_{n+1}. It expresses this joint LST $\varphi_{Z_n,Z_{n+1}}(s,t)$ in terms of the (univariate) LST of Z_n, which is characterized through Theorem 10. We only cover the case $t \neq \lambda$; if $t = \lambda$ the result follows by L'Hôpital's rule.

Theorem 21. *For $t \neq \lambda$,*

$$\varphi_{Z_n,Z_{n+1}}(s,t) = \frac{\lambda}{\lambda - t}\varphi_Y(t)\varphi_{Z_n}(s+\psi(t)) - \frac{t}{\lambda - t}\varphi_Y(\lambda)\varphi_{Z_n}(s+\psi(\lambda)). \tag{45}$$

Proof. By conditioning we obtain

$$G_{Z_n,Z_{n+1}}(s,t) = \mathbb{E}(e^{-sZ_n}e^{-t(S_n(Z_n)+Y_n-B_n)^+}) = \mathbb{E}\Big(e^{-sZ_n}\mathbb{E}(e^{-t(S_n(Z_n)+Y_n-B_n)^+} \mid S_n, B_n, S(Z_n))\Big).$$

By (55), for every $z \in \mathbb{R}$,

$$\varphi_{(z-B)^+}(t) = \frac{\lambda e^{-tz} - te^{-\lambda z}}{\lambda - t}e^{-tz}, \qquad \lambda \neq t,$$

so that

$$\begin{aligned}\mathbb{E}\Big(e^{-sZ_n}\mathbb{E}(e^{-t(S_n(Z_n)+Y_n-B_n)^+} \mid S_n, B_n, S(Z_n))\Big) &= \mathbb{E}\Big(\frac{\lambda e^{-sZ_n-t(S_n(Z_n)+Y_n)} - te^{-sZ_n-\lambda(S_n(Z_n)+Y_n)}}{\lambda - t}\Big)\\ &= \frac{\lambda}{\lambda - t}\varphi_Y(t)\,\mathbb{E}\Big(e^{-sZ_n-tS_n(Z_n)}\Big) - \frac{t}{\lambda - t}\varphi_Y(\lambda)\,\mathbb{E}\left(e^{-sZ_n-\lambda(S_n(Z_n))}\right).\end{aligned} \tag{46}$$

By the definition of $\psi(\cdot)$,

$$\mathbb{E}(e^{-sS(X)}) = \int e^{-\psi(s)x}\,\mathbb{P}\,(X \in \mathrm{d}x) = \varphi_X(\psi(s)),$$

and

$$\mathbb{E}\Big(e^{-tZ_n-sS_n(Z_n)}\Big) = \mathbb{E}\Big(\mathbb{E}(e^{-tZ_n-sS_n(Z_n)}\big|Z_n)\Big) = \mathbb{E}\Big(e^{-sZ_n-\psi(t)Z_n}\Big) = \varphi_{Z_n}(s+\psi(t)).$$

Now some elementary algebra shows that (46) equals (45), as desired. □

Theorem 22. *The covariance of Z_n and Z_{n+1} is given by*

$$\mathrm{Cov}\,(Z_n, Z_{n+1}) = \mathbb{E}\,(S(1))\,\mathrm{Var}\,(Z_n) - \frac{\eta_n}{\lambda}\mathbb{E}\,(Z_n) - \frac{\varphi_Y(\lambda)}{\lambda}\varphi'_{Z_n}(\psi(\lambda)). \tag{47}$$

Proof. Note that the supposed straightforward approach via the duality and using the relation (10), which leads to $\mathrm{Cov}\,(A_n, A_{n+1}) = \gamma^2\,\mathrm{Cov}\left(\boldsymbol{N}_\gamma(A_n), \boldsymbol{N}_\gamma(A_{n+1})\right)$, would yield a wrong result since a simple transformation $(A_n, A_{n+1}) \mapsto (\boldsymbol{N}_\gamma(A_n), \boldsymbol{N}_\gamma(A_{n+1}))$ does not preserve the dependence structure of the INGAR$^+$ process. Instead we used Theorem 21 and direct computations, analogous to those underlying Theorem 15. □

The joint LST of Z_0 and Z_N, with N being geometrically distributed and the process being in equilibrium at time 0, can be computed as well. As this amounts to paralleling the approach underlying Theorem 16, we omit this result.

5. Conclusion and suggestions for further research

In this paper we have introduced and analyzed two general classes of reflected autoregressive processes, INGAR^+ and GAR^+. In our approach, a crucial role is played by a powerful duality relation that connects both classes of processes. We have shown that, despite the models' general nature, a detailed analysis of the time-dependent and stationary behavior is possible. We started by analyzing the INGAR^+ process, and subsequently we have used the duality relation to obtain the analogous results for the GAR^+ process.

Various options for follow-up research arise. In this study the focus was primarily on transforms and moments, but one may wonder whether, in asymptotic regimes, the (time-dependent or stationary) distribution function can be explicitly given. The results in [5] suggest potential scaling limits when approaching the stability limit (i.e., $\mathbb{E}(U) \uparrow 1$ and $\mathbb{E}(S(1)) \uparrow 1$ for the INGAR^+ and GAR^+ model, respectively). In addition, one could try to derive the system's tail behavior from the corresponding transforms; e.g. in the regime with heavy-tailed jumps in the upward direction, Tauberian techniques could be applied. We also aim to investigate some generalizations of the INGAR^+ and GAR^+ processes, allowing distributions for W_n and B_n that are more general than just the geometric and exponential distributions, respectively.

A. Appendix: The APGF and the LST

This appendix covers a set of technical results regarding the APGF of a non-negative integer-valued r.v. X and the LST of a non-negative r.v. Y,

$$G_X(s) = \mathbb{E}((1-s)^X), \quad \varphi_Y(s) = \mathbb{E}(e^{-sY}).$$

Most of the results are standard, but we have included them for completeness and easy reference.

Provided the first two moments exist, as $s \to 0$, the expansions

$$G_X(s) = 1 - \mathbb{E}(X)\, s + \frac{\mathbb{E}(X^2) - \mathbb{E}(X)}{2} s^2 + o(s^2), \tag{48}$$

$$\varphi_Y(s) = 1 - \mathbb{E}(Y)\, s + \frac{\mathbb{E}(Y^2)}{2} s^2 + o(s^2), \tag{49}$$

are valid. It follows that

$$\mathbb{E}(X) = -G_X'(0), \quad \mathbb{E}(Y) = -\varphi_Y'(0), \tag{50}$$

$$\mathbb{E}(X^2) = G_X''(0) - G_X'(0), \quad \mathbb{E}(Y^2) = \varphi_Y''(0), \tag{51}$$

$$\text{Var}(X) = G_X''(0) - G_X'(0) - G_X'(0)^2, \quad \text{Var}(Y) = \varphi_Y''(0) - \varphi_Y'(0)^2. \tag{52}$$

Probabilities can be recovered from the APGF if the limit as $s \to 1$ is considered:

$$\mathbb{P}(X = k) = \frac{(-1)^k}{k!} G_X^{(k)}(1-).$$

The following results are used several times in the paper; hence we have collected them in this appendix. Their proof is omitted, as these results follow after straightforward calculations.

Lemma 23. *(i) If X is a non-negative integer-valued random variable and W is an independent geometric random variable with success probability $p \in (0, 1]$ then*

$$G_{(X-W)^+}(s) = G_X(p) + p\frac{G_X(s) - G_X(p)}{p - s} = \frac{p}{p-s}G_X(s) - \frac{s}{p-s}G_X(p), \tag{53}$$

for $s \neq p$, and $G_{(X-W)^+}(p) = G_X(p) - pG'_X(p)$. Moreover,

$$\mathbb{P}(W > X) = G_X(p), \quad \mathbb{P}(W \geq X) = \frac{1}{1-p}G_X(p) - \frac{p}{1-p}G_X(1). \tag{54}$$

(ii) If X is a non-negative random variable and if B has an exponential distribution with parameter $\lambda > 0$, independent of X, then

$$\varphi_{(X-B)^+}(s) = \varphi_X(\lambda) + \lambda\frac{\varphi_X(s) - \varphi_X(\lambda)}{\lambda - s} = \frac{\lambda}{\lambda - s}\varphi_X(s) - \frac{s}{\lambda - s}\varphi_X(\lambda), \tag{55}$$

for $\lambda \neq s$ and $\varphi_{(X-B)^+}(\lambda) = \varphi_X(\lambda) - \lambda\varphi'_X(\lambda)$. Moreover,

$$\mathbb{P}(B > X) = \varphi_X(\lambda). \tag{56}$$

References

[1] M. Al-Osh and A. Alzaid. First-order integer-valued autoregressive (INAR(1)) process. *Journal of Time Series Analysis*, 8:261–275, 1987.

[2] S. Asmussen. *Applied Probability and Queues*. Springer, 2008.

[3] K. B. Athreya and P. E. Ney. *Branching Processes*. Springer, 1972.

[4] W. Barreto-Souza. Zero-modified geometric INAR(1) process for modelling count time series with deflation or inflation of zeros. *Journal of Time Series Analysis*, 36(6):839–852, 2015.

[5] O. Boxma, M. Mandjes, and J. Reed. On a class of reflected AR(1) processes. *Journal of Applied Probability*, 53(3):818–832, 2016.

[6] P. J. Brockwell, R. A. Davis, and M. V. Calder. *Introduction to Time Series and Forecasting*. Springer, 2002.

[7] J. W. Cohen. *The Single Server Queue, 2nd ed.* North-Holland, 1982.

[8] J. Grandell. *Mixed Poisson Processes*. CRC Press, 1997.

[9] C. Heathcote. A branching process allowing immigration. *Journal of the Royal Statistical Society, Series B (Methodological)*, 27:138–143, 1965.

[10] C. Heathcote. Corrections and comments on the paper 'A branching process allowing immigration'. *Journal of the Royal Statistical Society, Series B (Methodological)*, 28: 213–217, 1966.

[11] A. Latour. Existence and stochastic structure of a non-negative integer-valued autoregressive process. *Journal of Time Series Analysis*, 19(4):439–455, 1998.

[12] E. McKenzie. Some simple models for discrete variate time series. *Journal of the American Water Resources Association*, 21:645–650, 1985.

[13] E. McKenzie. Autoregressive moving-average processes with negative-binomial and geometric marginal distributions. *Advances in Applied Probability*, 18:679–705, 1986.

[14] M. M. Ristić, H. S. Bakouch, and A. S. Nastić. A new geometric first-order integer-valued autoregressive (INGAR(1)) process. *Journal of Statistical Planning and Inference*, 139(7):2218–2226, 2009.

[15] F. Steutel and K. van Harn. Discrete analogues of self-decomposability and stability. *Annals of Probability*, 7:893–899, 1979.

[16] C. H. Weiß. *An Introduction to Discrete-Valued Time Series*. John Wiley & Sons, 2017.