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AFFINE STORAGE AND INSURANCE RISK MODELS

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ABSTRACT. The aim of this paper is to analyze a general class of storage processes, in which the rate at which the storage level increases or decreases is assumed to be an affine function of the current storage level, and in addition both upward and downward jumps are allowed. To do so, we first focus on a related insurance risk model, for which we determine the ruin probability at an exponentially distributed epoch jointly with the corresponding undershoot and overshoot, given that the capital level at time 0 is exponentially distributed as well. The obtained results for this insurance risk model can be translated in terms of two types of storage models, in one of those two cases by exploiting a duality relation. Many well-studied models are shown to be special cases of our insurance risk and storage models. We conclude by showing how the exponentiality assumptions can be greatly relaxed.

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1. Introduction

In conventional storage models [4, Ch. XIV] non-negative amounts of, say, fluid arrive at a reservoir that is emptied according to a certain release rule, where there is truncation at 0 to prevent the storage level from becoming negative. It is typically assumed that the reservoir's input process is a compound Poisson process with non-negative jumps. In addition the release rate, representing the rate at which the content of the reservoir is drained, is usually taken either independent of the current storage level (giving rise to classical workload models; see e.g. [18]) or taken proportional to the current storage level (so that the storage level decreases exponentially between jumps).

The main objective of the present paper is to generalize the above setup in various ways. Most notably, we consider storage systems with *jumps in both directions*, and in addition we assume an *affine* release rate (i.e., a constant plus a term that is proportional to the current storage level), thus considerably extending the results in the existing literature. A second major contribution concerns a direct link with a class of insurance risk models in which, next to the linear arrival of premiums and non-negative claims arriving according to a Poisson process (as covered by the classical Cramér-Lundberg model [4, Section XIII.5]), we include the interest and incidental 'capital injections' that the insurance firm receives. In our paper this class of insurance risk models will be presented first, as we use it to obtain results for two related storage models.

Both in the operations research literature and the insurance risk literature there is a vast body of work on related models. A more detailed comparison with our results will be given later in this paper, but we now include a brief historic account without attempting to provide an exhaustive overview. (i) The study of storage models goes back to the 1950s, with the seminal contributions [20, 24, 29] focusing on constant release rules. An early paper on level-dependent release rules in [21]; see also e.g. [15]. The overview in [4, Ch. XIV] provides the main results, including an exploration of the relations between storage systems and other classes of well-studied models from the applied probability literature. (ii) The insurance risk literature, starting from the works by Cramér [17] and Lundberg [27, 28] in the first part of the previous century, focuses on models that quantify an insurer's vulnerability to ruin. The basic model considers a setting in which claims with independent and identically distributed sizes arrive according to a Poisson process, whereas premiums arrive at a constant rate. The main objective is to compute (or approximate) the ruin probability. Many generalizations and ramifications have been studied, such as time-dependent ruin [5, Ch. V] and the claim process being of Lévy type rather than compound Poisson [5, Ch. X and XI]. We in particular mention the highly general setup studied in [2], which also includes as a special case the model in which interest is received on the available surplus. (iii) Duality relations between storage models and insurance risk counterparts are discussed in e.g. [5, Section III.2].

We proceed by discussing the paper's main contributions. Throughout this work we analyze two types of storage models, but we do so by first considering a related insurance risk model and then translating the obtained results in terms of the storage models. At a more detailed level, our contributions are the following:

- In the insurance risk model under consideration claims arrive linearly in time, but we add the realistic feature that at any point in time the insurance firm is allowed to invest its capital surplus (at a given interest rate). In addition to the claims (leading to negative jumps in the firm's surplus level process), there are also upward jumps, which can for instance be thought of as capital injections. In the resulting model, between two jumps the rate of growth of the surplus level is an affine function of this level.

Transforming with respect to the initial surplus level (which essentially corresponds to the initial surplus being exponentially distributed), we find the probability of ruin before an independently sampled exponentially distributed time, jointly with the corresponding undershoot and overshoot. While setting up a differential equation for the object of interest is to some extent classical, solving it is rather subtle. In particular the identification of a number of unknown constants requires a delicate reasoning.

- The first storage model we consider is a reflected version of the above insurance risk model. Observe that in the insurance risk model, as soon as the surplus level drops below zero, the insurance firm goes bankrupt. In the associated storage version, however, the surplus level is kept non-negative. We show how our results for the insurance risk model can be directly used to analyze the constructed storage model.
- We then shift our attention to a dual storage model, in which the release rate of fluid is affine. Establishing a duality relation between this storage model and the insurance risk model, we derive expressions describing the storage process' transient and stationary distribution. At this point, we include a detailed comparison with existing results. We conclude that our results significantly generalize the results obtained so far; various models that been intensively studied in the operations research and insurance risk literature, appear as special cases of our model.
- A seeming drawback of our results concerns the exponentiality assumptions. More concretely, in the setup described above we imposed the assumption that the initial level and the time horizon be exponentially distributed; in addition, the distribution of the process' jumps in one direction was allowed to be general but in the other direction it had to be exponential. Importantly, however, we show that these exponentiality assumptions can be greatly relaxed, in the sense that all results extend to a wide class of phase-type distributions (covering mixtures and sums of exponential distributions). Interestingly, the case of Erlang distributed jumps is rather subtle, as we have to deal with issues related to the non-analyticity of certain functions.

The remainder of the paper is organized as follows. Section 2 covers our insurance risk model. The next sections consider the storage processes: Section 3 deals with the reflection of the insurance risk model, while Section 4 focuses on the dual counterpart of the insurance risk model. Then Section 5 points out how the exponentiality assumptions can be relaxed. A discussion and concluding remarks are provided in Section 6.

2. Insurance risk model

This section deals with our insurance risk process. After introducing the model and some additional notation, we analyze it by focusing on the process until it drops below zero (given this happens); this time we refer to as the 'residual busy period'.

2.1. Model.

We consider an insurance process $(X_t)_{t\geqslant 0}$, recording the insurance firm's capital surplus, with a gradual non-linear growth and jumps in both directions. It is formally defined as follows.

- The initial capital surplus level X_0 is assumed to equal $x > 0$.
- The process jumps at Poisson epochs. These jumps can have positive and negative values. The jumps in the upward direction arrive according to a Poisson process with rate $\lambda_+ \geqslant 0$, and are assumed to be i.i.d. samples from an exponential distribution with mean μ^{-1}. The downward jumps occur according to an (independent) Poisson process with rate $\lambda_- \geqslant 0$, and are i.i.d. samples from a general non-negative distribution, with Laplace-Stieltjes transform $\delta(\cdot)$ and density $d(\cdot)$.

- Between the jumps the level grows according to the differential equation
$$\mathrm{d}X_t = p\,\mathrm{d}t + rX_t\,\mathrm{d}t, \tag{1}$$
for $p, r \geqslant 0$. Suppose there are no jumps in (t_-, t_+) and $X_{t_-} = x_-$, then it is readily verified that, for $t \in (t_-, t_+)$,
$$X_t = x_-\,\mathrm{e}^{r(t-t_-)} + \frac{p}{r}\big(\mathrm{e}^{r(t-t_-)} - 1\big)$$
(where the obvious limit is taken when $r = 0$).

Remark 1. In the insurance context, p can be interpreted as the premium rate (where we follow the usual assumption that premiums arrive at a constant rate), whereas r represents the interest rate. The independent and identically distribited claims arrive according to the Poisson process with rate λ_-, and in this model there are in addition exponentially distributed 'capital injections' arriving according to a Poisson process with rate λ_+. In classical insurance models (those of the Cramér-Lundberg type, that is) it is assumed that $\lambda_+ = r = 0$. $\Diamond$

The main objective of this section is to analyze the insurance risk process $(X_t)_{t\geqslant 0}$ until it hits 0 for the first time (see in particular Section 2.3). Then we use these findings to derive distributional properties pertaining to two associated storage processes; the results can be found in Sections 3 and 4.

2.2. **Additional notation.** In our study, a key quantity is the *residual busy period*, i.e., the time it takes before the insurance process drops below 0, as a function of the initial capital surplus level:
$$\tau_x := \inf\{s > 0 : X_s \leqslant 0 \,|\, X_0 = x\};$$
in case of no ruin (i.e., $X_s > 0$ for all $s > 0$), we set $\tau_x := \infty$. Interestingly, without much additional effort various related quantities can be analyzed as well:

- $X_{\tau_x^-}$ is the 'undershoot', i.e., the capital surplus level just before the residual busy period ends.
- The corresponding 'overshoot' is X_{τ_x}. This quantity is particularly relevant in the insurance setting, as it represents the 'loss' of the insurance firm (in case of ruin); see e.g. [26].

A graphical representation of a sample path of $(X_t)_{t\geqslant 0}$ is given in Figure 1.

Ideally, we would like to identify the probability that τ_x is smaller than some $t > 0$ for a given initial capital surplus level x (preferably jointly with determining the distributions of the undershoot and overshoot). This, however, turns out to be prohibitively difficult. It explains why we follow a transform-based procedure instead. More specifically, we transform with respect to x and t, which concretely means that we consider the above setting, but with an exponentially distributed initial capital surplus level, and at an exponentially distributed time. Evidently, such transforms uniquely characterize the quantity of interest (for given initial surplus level x and time t, that is); Laplace inversion could be relied on to numerically evaluate the quantity of interest; see e.g. [1, 23] for powerful approaches.

We now introduce our key quantities more formally. Let T_ν be exponentially distributed with mean ν^{-1}, for some $\nu > 0$, independent of the insurance risk process $(X_t)_{t\geqslant 0}$. Then we define, for $\vartheta > 0$, and given (non-negative) values of $\alpha, \beta, \gamma, \nu$, the transform
$$\varphi(\vartheta \,|\, \alpha, \beta, \gamma, \nu) \equiv \varphi(\vartheta) := \int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\,\mathbb{E}\left(\mathrm{e}^{-\alpha\tau_x + \beta X_{\tau_x} - \gamma X_{\tau_x^-}} 1_{\{\tau_x < T_\nu\}} \,|\, X_0 = x\right)\mathrm{d}x.$$

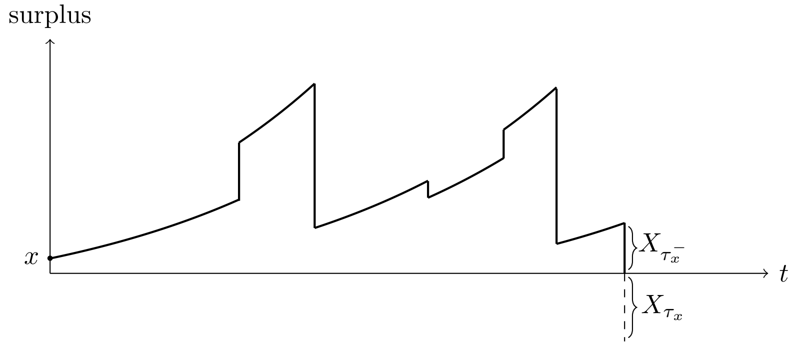

Figure 1. Sample path of $(X_t))_{t\geqslant 0}$ until τ_x.

This quantity can be interpreted as

$$\mathbb{E}\left(\mathrm{e}^{-\alpha\tau_x+\beta X_{\tau_x}-\gamma X_{\tau_x^-}}1_{\{\tau_x<T_\nu\}}\,|\,X_0=T_\vartheta\right),$$

with T_ϑ exponentially distributed with mean ϑ^{-1}, for some $\vartheta>0$, independent of both the insurance risk process $(X_t)_{t\geqslant 0}$ and T_ν. In Section 2.3 we present a procedure to evaluate the transform $\varphi(\vartheta)$.

Remark 2. Notice that three exponentiality assumptions have been made: the upward jumps are exponential with mean μ^{-1}, the initial level is exponential with mean ϑ^{-1}, and the time horizon considered is exponential with mean ν^{-1}. As it turns out, however, there are ways to substantially relax these exponentiality assumptions. We extensively discuss this theme later in the paper; see Section 5. ♢

2.3. **Analysis of residual busy period.** In this section we focus, for $x>0$, on analyzing the following performance metric pertaining to the residual busy period in the insurance risk model:

$$\varrho(x\,|\,\alpha,\beta,\gamma,\nu)\equiv\varrho(x):=\mathbb{E}\left(\mathrm{e}^{-\alpha\tau_x+\beta X_{\tau_x}-\gamma X_{\tau_x^-}}1_{\{\tau_x<T_\nu\}}\,|\,X_0=x\right).$$

We know that $\varphi(\infty)=\varrho(0)\in(0,1)$.

The first step in our analysis concerns the characterization of $\varrho(x)$ through an integro-differential equation. We do so relying on classical 'Markovian reasoning'. We consider the following options regarding the system's behavior in an (infinitesimally small) time interval of length h. With $\lambda:=\lambda_-+\lambda_+$, using that the surplus level behaves according to (1) between jumps,

$$\begin{aligned}\varrho(x)=\mathrm{e}^{-\alpha h}\Big(&\lambda_-h\int_0^x d(y)\varrho(x-y)\,\mathrm{d}y+\lambda_-h\int_x^\infty d(y)\mathrm{e}^{-\beta(y-x)}\,\mathrm{e}^{-\gamma x}\mathrm{d}y\\&+\lambda_+h\int_0^\infty\mu\mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y\\&+(1-\lambda h-\nu h)\,\varrho(x+ph+rxh)\Big)+o(h),\quad h\downarrow 0.\end{aligned}\tag{2}$$

To understand this relation, observe that (i) in this time interval (of length h) τ_x grows by h, and (ii) X_{τ_x} and $X_{\tau_x^-}$ can be assigned their values when the surplus level hits 0, which can only happen as a consequence of a sufficiently large negative jump (larger than x, that is). By linearizing $\mathrm{e}^{-\alpha h}$ and $\varrho(x+ph+rxh)$, the relation (2) can be rewritten as

$$\varrho(x)=\varrho(x+ph+rxh)+\lambda_-h\int_0^x d(y)\varrho(x-y)\,\mathrm{d}y+\lambda_-h\int_x^\infty d(y)\mathrm{e}^{-\beta(y-x)}\,\mathrm{e}^{-\gamma x}\mathrm{d}y$$

$$+\lambda_+ h\int_0^\infty \mu \mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y-(\alpha+\lambda+\nu)h\,\varrho(x)+o(h),\quad h\downarrow 0.$$

Bringing $\varrho(x+ph+rxh)$ to the left-hand side and dividing by h, we obtain

$$-\frac{\varrho(x+ph+rxh)-\varrho(x)}{ph+rxh}\,(p+rx)=\lambda_-\int_0^x d(y)\varrho(x-y)\,\mathrm{d}y+\lambda_-\int_x^\infty d(y)\mathrm{e}^{-\beta(y-x)}\,\mathrm{e}^{-\gamma x}\mathrm{d}y$$
$$+\lambda_+\int_0^\infty \mu\mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y$$
$$-(\alpha+\lambda+\nu)\,\varrho(x)+o(1),\quad h\downarrow 0.$$

Taking the limit we thus arrive at the following integro-differential equation.

Lemma 1. *For any $x>0$,*

$$-\varrho'(x)\,(p+rx)=\lambda_-\int_0^x d(y)\varrho(x-y)\,\mathrm{d}y+\lambda_-\int_x^\infty d(y)\mathrm{e}^{-\beta(y-x)}\,\mathrm{e}^{-\gamma x}\mathrm{d}y$$
$$+\lambda_+\int_0^\infty \mu\mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y-(\alpha+\lambda+\nu)\,\varrho(x).$$

This characterization of $\varrho(\cdot)$ is in line with [30, Eqn. (2.1)]; various versions of this result can be found in e.g. [19, 22, 31, 35].

The idea is now that we obtain a more explicit characterization of $\varphi(\vartheta)$ by considering an exponentially distributed initial capital surplus level (rather than a fixed level x). This effectively means that we transform with respect to x: we multiply the full equation in Lemma 1 by $\vartheta\mathrm{e}^{-\vartheta x}$, and integrate over $x\in(0,\infty)$, so as to obtain an equation in terms of the function $\varphi(\cdot)$. We do this term by term.

- We start with the terms on the left hand side of the equation. By integration by parts we obtain
$$-\int_0^\infty \varrho'(x)p\,\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x=p\vartheta\big(\varrho(0)-\varphi(\vartheta)\big).$$
- Along the same lines,
$$-\int_0^\infty \varrho'(x)\,rx\,\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x=r\vartheta\int_0^\infty \varrho(x)\big(\mathrm{e}^{-\vartheta x}-x\,\vartheta\mathrm{e}^{-\vartheta x}\big)\mathrm{d}x=r\vartheta\varphi'(\vartheta),$$
using the identity
$$\varphi'(\vartheta)=\frac{\varphi(\vartheta)}{\vartheta}-\int_0^\infty x\,\vartheta\mathrm{e}^{-\vartheta x}\varrho(x)\mathrm{d}x.$$
- We now move to the terms on the right-hand side of the equation. The first term leads to, by swapping the order of the integrals,
$$\lambda_-\int_0^\infty\left(\int_0^x d(y)\varrho(x-y)\,\mathrm{d}y\right)\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x$$
$$=\lambda_-\int_0^\infty \mathrm{e}^{-\vartheta y}\left(\int_y^\infty \varrho(x-y)\,\vartheta\mathrm{e}^{-\vartheta(x-y)}\mathrm{d}x\right)d(y)\mathrm{d}y=\lambda_-\delta(\vartheta)\varphi(\vartheta).$$
- The second term yields
$$\lambda_-\int_0^\infty\left(\int_x^\infty d(y)\mathrm{e}^{-\beta(y-x)}\,\mathrm{e}^{-\gamma x}\,\mathrm{d}y\right)\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x$$
$$=\lambda_-\vartheta\int_0^\infty\frac{\mathrm{e}^{-\beta y}-\mathrm{e}^{-(\vartheta+\gamma)y}}{\vartheta+\gamma-\beta}d(y)\mathrm{d}y=\lambda_-\vartheta\,\frac{\delta(\beta)-\delta(\vartheta+\gamma)}{\vartheta+\gamma-\beta}.$$
Notice that $\vartheta=\beta-\gamma$ is a removable singularity.

- Also, using the transformation $z := x + y$ and adapting the integration area accordingly, one finds

$$\lambda_+ \int_0^\infty \left(\int_0^\infty \mu e^{-\mu y} \varrho(x+y)\, dy \right) \vartheta e^{-\vartheta x}\, dx = \lambda_+ \frac{\mu}{\mu - \vartheta} \varphi(\vartheta) - \lambda_+ \frac{\vartheta}{\mu - \vartheta} \varphi(\mu). \tag{3}$$

- And finally, evidently,

$$- \int_0^\infty (\alpha + \lambda + \nu)\, \varrho(x) \vartheta e^{-\vartheta x}\, dx = -(\alpha + \lambda + \nu)\, \varphi(\vartheta).$$

With $A := -(\alpha + \nu)/r$ and

$$F(\vartheta) := \bar{F}(\vartheta) + \frac{A}{\vartheta}, \qquad \bar{F}(\vartheta) := \frac{p}{r} - \frac{\lambda_-}{r} \frac{1 - \delta(\vartheta)}{\vartheta} + \frac{\lambda_+}{r} \frac{1}{\mu - \vartheta},$$
$$G(\vartheta) := \frac{\lambda_-}{r} \frac{\delta(\beta) - \delta(\vartheta + \gamma)}{\vartheta + \gamma - \beta} - \frac{p}{r} \varrho(0) - \frac{\lambda_+}{r} \frac{1}{\mu - \vartheta} \varphi(\mu),$$

we obtain the following inhomogeneous ordinary differential equation for $\varphi(\cdot)$ by combining the above computations.

Proposition 1. *$\varphi(\cdot)$ fulfils the differential equation*

$$\varphi'(\vartheta) = F(\vartheta)\varphi(\vartheta) + G(\vartheta). \tag{4}$$

Here we tacitly assumed $r > 0$; as also discussed in Section 4.6, for $r = 0$ the differential equation (4) turns into an algebraic equation.

We now solve (4). It is an ordinary inhomogeneous differential equation, that can be solved in the usual way. Up to a multiplicative constant, the solution of the homogeneous counterpart of (4) is

$$\varphi_{\text{hom}}(\vartheta) = \exp\big(F_\star(\vartheta)\big),$$

where $F_\star(\cdot)$ denotes the primitive of $F(\cdot)$. Following the standard approach, for the solution of the inhomogeneous differential equation we try $\varphi(\vartheta) = H(\vartheta)\, \varphi_{\text{hom}}(\vartheta)$, for a function $H(\cdot)$ to be determined; this $H(\cdot)$ covers the multiplicative constant mentioned above. This leads to the differential equation

$$H'(\vartheta) = \frac{G(\vartheta)}{\varphi_{\text{hom}}(\vartheta)},$$

which is solved by

$$H(\vartheta) = \int_0^\vartheta \frac{G(\eta)}{\varphi_{\text{hom}}(\eta)} d\eta + K$$

for some constant K. We thus find

$$\varphi(\vartheta) = \left(\int_0^\vartheta G(\eta) \exp\big(-F_\star(\eta)\big) d\eta + K \right) \exp\big(F_\star(\vartheta)\big). \tag{5}$$

Assume for the moment that $p > 0$; the case $p = 0$ will be dealt with in Remark 5. As a consequence of $F_\star(\vartheta) \to \infty$ as $\vartheta \to \infty$ (bearing in mind that $F(\vartheta) \to p/r$ as $\vartheta \to \infty$), $\varphi(\infty) = \varrho(0) \in (0, 1)$ necessarily implies that

$$K = - \int_0^\infty G(\eta) \exp\big(-F_\star(\eta)\big)\, d\eta, \tag{6}$$

so that

$$\varphi(\vartheta) = - \left(\int_\vartheta^\infty G(\eta) \exp\big(-F_\star(\eta)\big)\, d\eta \right) \exp\big(F_\star(\vartheta)\big). \tag{7}$$

The only unknowns left are $\varrho(0)$ and $\varphi(\mu)$. We have to identify constraints by which these two constants can be determined. To this end, we introduce the notation $G(\vartheta) = \varrho(0)\, G_1(\vartheta) + \varphi(\mu)\, G_2(\vartheta) + G_3(\vartheta)$, where

$$G_1(\vartheta) := -\frac{p}{r}, \quad G_2(\vartheta) := -\frac{\lambda_+}{r}\frac{1}{\mu-\vartheta}, \quad G_3(\vartheta) := \frac{\lambda_-}{r}\frac{\delta(\beta)-\delta(\vartheta+\gamma)}{\vartheta+\gamma-\beta}.$$

We define $I(\vartheta)$ as $\varrho(0)\, I_1(\vartheta) + \varphi(\mu)\, I_2(\vartheta) + I_3(\vartheta)$, where, for $k = 1, 2, 3$,

$$I_k(\vartheta) := \int_\vartheta^\infty G_k(\eta) \exp\big(-F_\star(\eta)\big)\mathrm{d}\eta.$$

Then we investigate the singularities around 0 and μ; here we rely on the fact that, obviously, if for some ϑ we have that $F_\star(\vartheta) = \infty$, then by (7) and the finiteness of $\varphi(\vartheta)$ necessarily $I(\vartheta) = 0$.

- From the shape of $F(\cdot)$ it is seen that, for some $D_0 < 0$,

  $$\lim_{\vartheta\downarrow 0} \frac{F_\star(\vartheta)}{\log\vartheta} = D_0,$$

  so that $F_\star(\vartheta) \to \infty$ as $\vartheta \downarrow 0$. This means that to make sure (7) holds, we should have that $I(0) = 0$ (and hence $K = 0$). With this relation we can express $\varrho(0)$ and $\varphi(\mu)$ in one another, due to the fact that $G(\cdot)$ is linear in these two constants:

  $$\varrho(0)I_1(0) + \varphi(\mu)I_2(0) = -I_3(0). \tag{8}$$

- Likewise, for some $\bar{D}_\mu \in \mathbb{R}$ and $D_\mu < 0$,

  $$\lim_{\vartheta\uparrow\mu} \frac{F_\star(\vartheta) - \bar{D}_\mu}{\log(\mu-\vartheta)} = D_\mu,$$

  entailing that $F_\star(\vartheta) \to \infty$ as $\vartheta \uparrow \mu$. Relation (7) thus implies that $I(\mu) = 0$, leading to the relation

  $$\varrho(0)I_1(\mu) + \varphi(\mu)I_2(\mu) = -I_3(\mu). \tag{9}$$

By solving the linear equations (8) and (9), it is easily verified that

$$\varrho(0) = -\frac{I_3(0)I_2(\mu) - I_3(\mu)I_2(0)}{I_1(0)I_2(\mu) - I_1(\mu)I_2(0)}, \quad \varphi(\mu) = -\frac{I_1(0)I_3(\mu) - I_1(\mu)I_3(0)}{I_1(0)I_2(\mu) - I_1(\mu)I_2(0)}. \tag{10}$$

We arrive at the following result.

Theorem 1. *If $p > 0$, then $\varphi(\vartheta)$ equals* (7)*, with $\varrho(0)$ and $\varphi(\mu)$ given by* (10).

Remark 3. Above we tacitly assumed that the integrals featuring in (8) and (9) (which are the coefficients and right-hand side of the two-dimensional system of linear equations) are well-defined. In this remark we argue why this is the case; observe that it is not immediately evident, because of the ϑ and $\mu - \vartheta$ appearing in the denominators of the functions involved.

Noting that $F_\star(\vartheta)$ behaves as $\vartheta p/r$ for large ϑ, and recalling that, for any $f \in \mathbb{R}$, $g > 0$ and $h > 0$, $\int_g^\infty x^f \mathrm{e}^{-hx}\mathrm{d}x < \infty$, to establish finiteness of the numerator and denominator, we only have to deal with their singularities at $\vartheta = 0$ and $\vartheta = \mu$.

We first concentrate on the integrals appearing in $I(0)$. Recall that $\vartheta = \beta - \gamma$ is a removable singularity (in $G_3(\cdot)$). For $\eta \downarrow 0$, $G_k(\eta)$ is a bounded function for $k = 1, 2, 3$ (on the interval $(0, \varepsilon)$ for some small ε, that is). Observe that $F_\star(\eta) = -\chi_0 \log\eta + \bar{\chi}_0(\eta)$ for $\bar{\chi}_0(\cdot)$ bounded on $(0, \varepsilon)$ and $\chi_0 > 0$. Now it follows from

$$\int_0^\varepsilon \eta^{\chi_0}\mathrm{d}\eta < \infty$$

that, for $k = 1, 2, 3$,

$$\int_0^{\varepsilon} G_k(\eta) \exp\big(- F_\star(\eta)\big) \mathrm{d}\eta < \infty.$$

We now point out how to deal with the singularity of $G_2(\cdot)$ at $\vartheta = \mu$. For $\eta \uparrow \mu$ we write $G_2(\eta) = \xi_\mu/(\mu - \eta) + \bar{\xi}_\mu(\eta)$ for a bounded function $\bar{\xi}_\mu(\cdot)$ (i.e., on the interval $(\mu - \varepsilon, \mu)$) and some ξ_μ; also, $F_\star(\eta) = -\chi_\mu \log(\mu - \eta) + \bar{\chi}_\mu(\eta)$ for $\bar{\chi}_\mu(\cdot)$ bounded on $(0, \varepsilon)$ and $\chi_\mu > 0$. Now it follows from

$$\int_{\mu-\varepsilon}^{\mu} (\mu - \eta)^{-1+\chi_\mu} \, \mathrm{d}\eta < \infty$$

that

$$\int_{\mu-\varepsilon}^{\mu} G_2(\eta) \exp\big(- F_\star(\eta)\big) \mathrm{d}\eta < \infty.$$

The integral involving $G_2(\cdot)$ between μ and $\mu + \varepsilon$ can be dealt with analogously. Due to the boundedness of $G_1(\cdot)$ and $G_3(\cdot)$ around μ, the integrals involving $G_1(\cdot)$ and $G_3(\cdot)$ do not lead to any complications at μ. This proves the finiteness of the integrals in (8). The three integrals in (9) can be dealt with similarly. $\Diamond$

Remark 4. We can compute $\varphi(0)$ using 'L'Hôpital':

$$\varphi(0) = \lim_{\vartheta \downarrow 0} \frac{I(\vartheta)}{\mathrm{e}^{-F_\star(\vartheta)}} = - \lim_{\vartheta \downarrow 0} \frac{G(\vartheta)}{F(\vartheta)} = 0.$$

This was expected, as

$$\varphi(0) = \lim_{x \to \infty} \varrho(x) \leqslant \lim_{x \to \infty} \mathbb{P}(\tau_x < T_\nu \,|\, X_0 = x) = 0.$$

In addition, along the same lines one can verify that when taking the limit $\vartheta \to \mu$ one indeed obtains $\varphi(\mu)$. $\Diamond$

An alternative way to describe $\varphi(\cdot)$ is through a power series expansion, which can be evaluated as follows. Writing $\bar{F}(\vartheta) = \sum_{\ell=0}^{\infty} \bar{f}_\ell \vartheta^\ell$ and $G(\vartheta) = \sum_{\ell=0}^{\infty} g_\ell \vartheta^\ell$, we thus obtain the differential equation

$$\varphi'(\vartheta) = \left(\sum_{\ell=0}^{\infty} \bar{f}_\ell \vartheta^\ell + \frac{A}{\vartheta} \right) \varphi(\vartheta) + \sum_{\ell=0}^{\infty} g_\ell \vartheta^\ell.$$

Writing $c_\ell := \varphi^{(\ell)}(0)$, it can be rewritten to

$$\sum_{\ell=0}^{\infty} \frac{c_{\ell+1}}{\ell!} \vartheta^\ell = \left(\sum_{\ell=0}^{\infty} \bar{f}_\ell \vartheta^\ell + \frac{A}{\vartheta} \right) \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \vartheta^\ell + \sum_{\ell=0}^{\infty} g_\ell \vartheta^\ell.$$

Observe that this relation implies $c_0 = \varphi(0) = 0$, in line with what we saw in Remark 4. In addition, by collecting the terms corresponding to the same power in both sides of the equation, we conclude that the coefficients c_k obey the following recursion.

Proposition 2. *The power series expansion of $\varphi(\vartheta)$ is $\sum_{\ell=0}^{\infty} c_\ell \, \vartheta^\ell/\ell!$, where $c_0 = 0$ and, for $\ell \in \mathbb{N}$,*

$$c_{\ell+1} = \left(\frac{1}{\ell!} - \frac{A}{(\ell+1)!} \right)^{-1} \left(\sum_{m=0}^{\ell} \bar{f}_m c_{\ell-m} + g_\ell \right).$$

Remark 5. When $p = 0$, the argumentation has to be set up slightly differently, as we do not have $F_\star(\vartheta) \to \infty$ as $\vartheta \to \infty$, so that we cannot conclude that K equals the right-hand side of (6). At the same time, note that no constant $\varrho(0)$ plays a role in this case (as can be

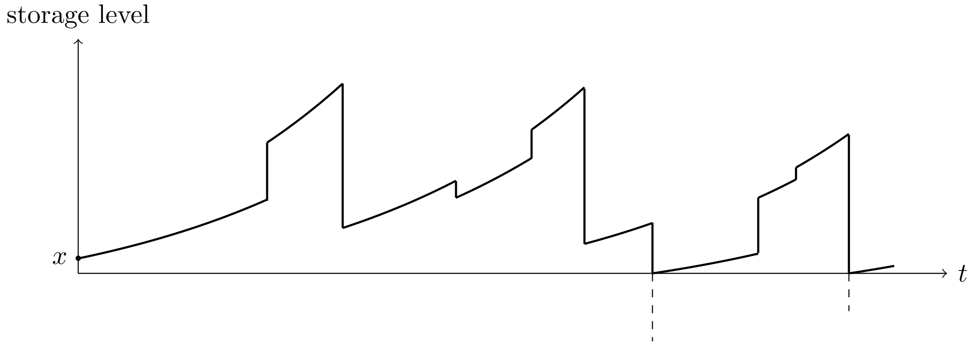

Figure 2. Sample path of $(X_t^\circ))_{t\geqslant 0}$ with $p>0$.

seen from the definition of $G(\cdot)$). The singularities at 0 and μ still exist though. Write, for $k=2,3$,

$$J_k(\vartheta) := \int_0^\vartheta G_k(\eta)\exp\big(-F_\star(\eta)\big)\mathrm{d}\eta,$$

and $J(\vartheta)=\varphi(\mu)\,J_2(\mu)+J_3(\vartheta)$. Using the same reasoning as before, and recalling (5), for any ϑ such that $F_\star(\vartheta)=\infty$, we necessarily have $J(\vartheta)+K=0$. We thus obtain the linear equations

$$\varphi(\mu)\,J_2(0)+J_3(0)+K=\varphi(\mu)\,J_2(\mu)+J_3(\mu)+K=0.$$

Using $J_2(0)=J_3(0)=0$, we thus find

$$\varphi(\mu)=-\frac{J_3(\mu)-J_3(0)}{J_2(\mu)-J_2(0)}=-\frac{J_3(\mu)}{J_2(\mu)},\quad K=J_2(\mu)\frac{J_3(\mu)-J_3(0)}{J_2(\mu)-J_2(0)}-J_3(\mu)=0.$$

As before, it can be argued that all integrals involved are well-defined. ◊

3. A first storage model

In the previous section we derived transforms pertaining to the residual busy period in our insurance risk model, given that the initial capital surplus level is exponentially distributed with mean ϑ^{-1}. In this section we consider the corresponding storage model, which is defined as the insurance model but now reflected at 0. More precisely: at moments a negative jump occurs that is larger than the current level, the storage level is set equal to 0, after which the process continues in the evident manner. The resulting process will be referred to as $(X_t^\circ)_{t\geqslant 0}$. We refer to Figure 2 for an illustration with $p>0$; if $p=0$ the picture is different in that the storage level can only become positive due to an upward jump. As it turns out, we can directly use the findings of Section 2.3 to find the transform of the storage level corresponding to $(X_t^\circ)_{t\geqslant 0}$ at a time that is exponentially distributed with mean ν^{-1}.

More concretely, the goal of this section is to evaluate, for a given $s\geqslant 0$,

$$\kappa(\vartheta):=\int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\,\mathbb{E}\left(\mathrm{e}^{-sX^\circ_{T_\nu}}\,|\,X_0^\circ=x\right)\mathrm{d}x.$$

We evaluate this quantity by using the decomposition (being an immediate consequence of the strong Markov property)

$$\kappa(\vartheta)=\int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\,\mathbb{E}\left(\mathrm{e}^{-sX^\circ_{T_\nu}}1_{\{\tau_x\geqslant T_\nu\}}\,|\,X_0^\circ=x\right)\mathrm{d}x$$
$$+\int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\,\mathbb{P}(\tau_x<T_\nu)\,\mathrm{d}x\cdot\mathbb{E}\left(\mathrm{e}^{-sX^\circ_{T_\nu}}\,|\,X_0^\circ=0\right);$$

here we distinguish between the scenarios $\tau_x \geqslant T_\nu$ and $\tau_x < T_\nu$, and use the memoryless property in the latter case. Some of the objects in this decomposition can be expressed in terms of quantities we have identified already. To this end, observe that

$$\mathbb{E}\left(\mathrm{e}^{-sX^\circ_{T_\nu}} \mid X^\circ_0 = 0\right) = \frac{\nu + \lambda_+\kappa(\mu)}{\nu+\lambda_+} 1_{\{p=0\}} + \kappa(\infty) 1_{\{p>0\}}. \tag{11}$$

It is clear that in the above expression we have to distinguish between the cases $p = 0$ and $p > 0$. In the former case the storage level can be zero for a while (until a positive jump occurs). In the latter case, however, once the storage level hits 0, it starts growing instantly.

In addition, with $\varphi(\vartheta \mid \alpha, \beta, \gamma, \nu)$ as defined in Section 2.3,

$$\pi(\vartheta) := \int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\, \mathbb{P}(\tau_x < T_\nu)\,\mathrm{d}x = \varphi(\vartheta \mid \alpha,\beta,\gamma,\nu)\big|_{\alpha=\beta=\gamma=0};$$

observe that this quantity has been identified in Theorem 1. As a consequence, we are left with evaluating

$$\varphi^\circ(\vartheta) := \int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\, \varrho^\circ(x)\mathrm{d}x, \quad \text{with} \quad \varrho^\circ(x) := \mathbb{E}\left(\mathrm{e}^{-sX_{T_\nu}} 1_{\{\tau_x \geqslant T_\nu\}} \mid X_0 = x\right).$$

Similarly to the steps followed when setting up a differential equation for $\varrho(x)$ in Section 2.3, we first derive that

$$\begin{aligned}\varrho^\circ(x) = \mathrm{e}^{-sh}\Big(\lambda_- h \int_0^x d(y)\varrho^\circ(x-y)\,\mathrm{d}y + \lambda_+ h\int_0^\infty \mu \mathrm{e}^{-\mu y}\varrho^\circ(x+y)\,\mathrm{d}y + \nu h \\ + (1-\lambda h - \nu h)\,\varrho^\circ(x+ph+rxh)\Big) + o(h),\end{aligned}$$

as $h \downarrow 0$. Bringing $\varrho^\circ(x+ph+rxh)$ to the left hand side, dividing by h and letting $h \downarrow 0$, we thus obtain the integro-differential equation

$$\begin{aligned}-(\varrho^\circ)'(x)\,(p+rx) = \lambda_- \int_0^x d(y)\varrho^\circ(x-y)\,\mathrm{d}y + \lambda_+ \int_0^\infty \mu\mathrm{e}^{-\mu y}\varrho^\circ(x+y)\,\mathrm{d}y \\ + \nu - (s+\lambda+\nu)\,\varrho^\circ(x).\end{aligned}$$

Again we convert this relation for a given initial storage level x into a relation featuring an exponentially distributed initial level. Multiplying the equation by $\vartheta\mathrm{e}^{-\vartheta x}$ and integrating over $x \in (0,\infty)$ yields, with

$$\begin{aligned}F^\circ(\vartheta) &:= \frac{p}{r} - \frac{s+\nu+\lambda_+}{r\vartheta} - \frac{\lambda_-}{r}\frac{1-\delta(\vartheta)}{\vartheta} + \frac{\lambda_+}{r\vartheta}\frac{\mu}{\mu-\vartheta},\\ G^\circ(\vartheta) &:= \nu + \frac{\lambda_-}{r}\frac{1-\delta(\vartheta)}{\vartheta} - \frac{p}{r}\varrho^\circ(0) - \frac{\lambda_+}{r}\frac{1}{\mu-\vartheta}\varphi^\circ(\mu),\end{aligned}$$

the following inhomogeneous ordinary differential equation for $\varphi^\circ(\cdot)$:

$$(\varphi^\circ)'(\vartheta) = F^\circ(\vartheta)\varphi^\circ(\vartheta) + G^\circ(\vartheta).$$

This equation can be solved in the same way as (4). We have

$$\varphi^\circ(\vartheta) = -\left(\int_\vartheta^\infty G^\circ(\eta)\exp\big(-F^\circ_\star(\eta)\big)\,\mathrm{d}\eta\right)\exp\big(F^\circ_\star(\vartheta)\big), \tag{12}$$

with $F^\circ_\star(\vartheta)$ denoting the primitive of $F^\circ(\vartheta)$. The unknowns $\varrho^\circ(0)$ and $\varphi^\circ(\mu)$ are found (following the procedure used in Section 2.3) by inserting $\vartheta = 0$ and $\vartheta = \mu$, respectively. As in Remark 3, it can be argued that the integrals involved in the resulting two-dimensional linear system are well-defined.

Now that we have found $\varphi^\circ(\cdot)$, we are in a position to determine $\kappa(\cdot)$, as follows. We distinguish between the cases $p = 0$ and $p > 0$, in light of Equation (11). For $p = 0$, collecting the above intermediate results, we arrive at

$$\kappa(\vartheta) = \varphi^\circ(\vartheta) + \pi(\vartheta)\frac{\nu + \lambda_+\kappa(\mu)}{\nu + \lambda_+}.$$

This expression still contains the unknown $\kappa(\mu)$, but this quantity can be found in an evident manner by plugging in $\vartheta = \mu$:

$$\kappa(\mu) = \frac{(\nu + \lambda_+)\varphi^\circ(\mu) + \nu\pi(\mu)}{\nu + \lambda_+ - \lambda_+\pi(\mu)}.$$

For $p > 0$ a similar procedure can be followed. We express $\kappa(\vartheta)$ in terms of the unknown constant $\kappa(\infty)$:

$$\kappa(\vartheta) = \varphi^\circ(\vartheta) + \pi(\vartheta)\kappa(\infty).$$

In this case we have to plug in $\vartheta = \infty$ to identify $\kappa(\infty)$. Doing so, we find that

$$\kappa(\infty) = \frac{\varphi^\circ(\infty)}{1 - \pi(\infty)} = \frac{\varrho^\circ(0)}{1 - \pi(\infty)}.$$

This eventually leads to the following result.

Theorem 2. *For $p = 0$,*

$$\kappa(\vartheta) = \varphi^\circ(\vartheta) + \pi(\vartheta)\left(\frac{\nu + \lambda_+\varphi^\circ(\mu)}{\nu + \lambda_+ - \lambda_+\pi(\mu)}\right),$$

whereas for $p > 0$,

$$\kappa(\vartheta) = \varphi^\circ(\vartheta) + \pi(\vartheta)\frac{\varrho^\circ(0)}{1 - \pi(\infty)}.$$

Remark 6. It can be checked that when picking $\nu = 0$ (i.e., we observe the system over an infinitely long time window) we obtain $\kappa(\vartheta) = 0$ for all ϑ. This effectively means that the system is not ergodic (i.e., the storage level eventually grows beyond any bound), irrespectively of the model parameters (as long as $r > 0$). This can be intuitively understood as follows. When the storage level is $x \geqslant 0$, then the drift is $p + rx + f$, with $f := \lambda_+/\mu + \lambda_-\delta'(0)$ (recall that $\delta'(0) < 0$, so that f can be negative). This means that as soon as x exceeds $x_0 := -(p + f)/r$ (which will eventually happen), the process has a positive drift. This drift actually becomes stronger positive if levels higher than x_0 are attained; this, informally, gives the process the possibility to 'escape to ∞'. The consequence of this observation is that, while we can consider the process' transient behavior, there are no meaningful results pertaining to its stationary behavior.

As an aside, we notice that there are various ways to make the process ergodic by adapting the model slightly. One option is to work with an arrival rate of downward jumps that is proportional to x, say $\bar\lambda_- x$ for some $\bar\lambda_- > 0$. Then the stability condition becomes

$$p + rx + \lambda_+/\mu + \bar\lambda_-\, x\delta'(0) < 0$$

for all $x \geqslant x_0$ (for some $x_0 \geqslant 0$), or equivalently $r < -\bar\lambda_-\delta'(0)$. This model can be analyzed in the same manner as the one studied in the present paper. $\Diamond$

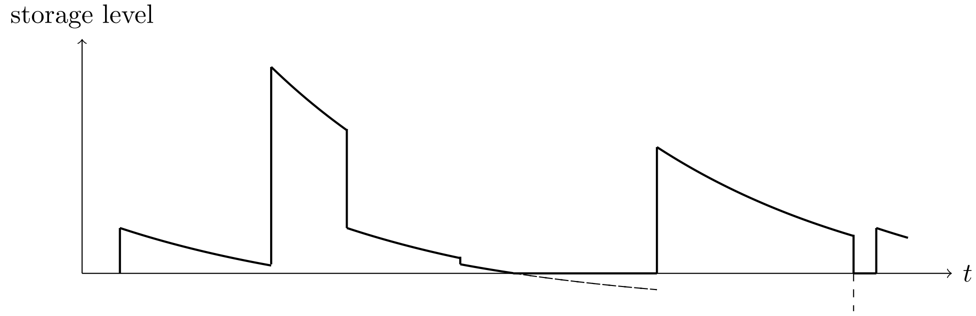

Figure 3. Sample path of $(Q_t))_{t\geqslant 0}$. The truncated downward jumps are indicated by dashed lines. Observe that there can be truncation (at 0) between jumps and at downward jumps.

4. A dual storage model

In this section we consider the storage (or workload) process $(Q_t)_{t\geqslant 0}$ with an affine release rule and jumps in both directions. As we will argue, it can be seen as a dual model of the insurance risk process $(X_t)_{t\geqslant 0}$ featuring in Section 2.3. In Section 4.1 we formally define the storage process $(Q_t)_{t\geqslant 0}$, revealing a strong similarity with $(X_t)_{t\geqslant 0}$. Then in Section 4.2 we establish a duality relation between the two processes, from which we derive expressions characterizing the transient and stationary distribution of the storage level. Section 4.3 presents an alternative compact way to identify the stationary storage-level distribution, which is used in Section 4.4 to set up a recursive procedure to derive the corresponding moments. Then Section 4.5 points out how to translate the stationary storage-level distribution into its counterpart immediately after arrivals. Finally in Section 4.6 we discuss how our model relates to the existing literature; in particular we point out how it covers various well-studied models as special cases.

4.1. **Model.** The process $(Q_t)_{t\geqslant 0}$ is defined as follows. Observe the similarity with the insurance model $(X_t)_{t\geqslant 0}$ introduced in Section 2. We refer to Figure 3 for an illustration.

- The initial workload level Q_0 is assumed to equal 0.
- The workload process jumps at Poisson epochs. These jumps can have positive and negative values, but in the latter case they are truncated such that the workload level does not become negative. The jumps in the downward direction arrive according to a Poisson process with rate $\lambda_+ \geqslant 0$, and are assumed to be i.i.d. samples from an exponential distribution with mean μ^{-1}. The upward jumps occur according to a Poisson process with rate $\lambda_- \geqslant 0$, and are i.i.d. samples from a general non-negative distribution, with Laplace-Stieltjes transform $\delta(\cdot)$ and density $d(\cdot)$; the size of a generic upward jump is denoted by the random variable D.
- Between the jumps the storage level decreases according to the differential equation

$$\mathrm{d}Q_t = -p\,\mathrm{d}t - rQ_t\,\mathrm{d}t, \tag{13}$$

  for $p, r > 0$, as long as $Q_t \geqslant 0$; if between jumps Q_t reaches 0, it remains 0 till the next upward jump.

Remark 7. It may look unnatural that the downward (upward) jumps are associated with an arrival rate λ_+ (λ_-, respectively). The reason behind our choice is that this parametrization is convenient to establish the duality with the insurance risk model of Section 2, as will become clear below. $\Diamond$

Remark 8. When $r > 0$, the steady-state distribution of $(Q_t)_{t\geqslant 0}$ always exists, because for large enough storage level the drift will always be negative. Recall that for the process $(X_t^\circ)_{t\geqslant 0}$ the opposite holds; see Remark 6. Throughout this section we assume that the steady-state distribution indeed exists. We denote by Q a random variable with the steady-state storage-level distribution. $\Diamond$

4.2. **Duality.** Comparison of $(Q_t)_{t\geqslant 0}$ with the insurance process $(X_t)_{t\geqslant 0}$ introduced in Section 2.1 reveals a close relationship: the upward jump rate becomes the downward jump rate and vice versa; the upward jump distribution becomes the downward jump distribution and vice versa; and the non-linear, level-dependent, growth between jumps of (1) becomes a mirrored non-linear, level-dependent, decrease in the $(Q_t)_{t\geqslant 0}$ process. In this subsection we establish a duality relation between the processes $(X_t)_{t\geqslant 0}$ and $(Q_t)_{t\geqslant 0}$. This is a duality of the type discussed in [5, Section III.2], there resulting in the following conclusions:

> the probability of ruin before time t (resp. ever) in an insurance risk model with initial capital x equals the probability that the storage level at time t (resp. the steady-state workload) in a dual storage model exceeds x (given it started empty at time 0).

The proof of such results typically relies on a sample-path comparison technique, cf. [5, p. 46]. The setting considered in [5, 6] is the following.

- A risk process $(R_t)_{t\geqslant 0}$ has arrivals at successive epochs $\sigma_1, \dots, \sigma_N$, with $\sigma_N \leq t$. At those epochs, positive claims $U_1, \dots, U_N$ arrive, leading to decrements of the risk process. In between jumps, the premium rate is $p(x) > 0$ when the surplus level is x.
- The storage process $(V_t)_{t\geqslant 0}$ is defined by time-reversion, reflection at zero and initial condition $V_0 = 0$. The arrival epochs are $\sigma_k^\star := t - \sigma_{N-k+1}$, for $k = 1, \dots, N$, and at those epochs upward jumps occur of size $U_k^\star = U_{N-k+1}$. In between jumps, the storage level decreases at rate $p(x)$ when the storage level is x. When 0 is reached, the storage level stays at 0 until the next upward jump.

The only two differences between the setting of [5, Section III.2] and our setting are the following: (i) in [5], more general increment/decrement functions $p(x)$ are allowed, and (ii) in [5] the risk process only has negative jumps and the storage process only has positive jumps. Closely following the arguments in [5, pp. 46–47], and in particular verifying that adding the positive jumps of the $(X_t)_{t\geqslant 0}$ process and the negative jumps of the $(Q_t)_{t\geqslant 0}$ process does not affect those arguments, we readily obtain the duality results stated in Theorem 3 below. To this end, we recall the definition of the residual busy period τ_x from Section 2, and in addition we define

$$\varrho_0(x\,|\,\nu) := \varrho(x\,|\,\alpha,\beta,\gamma,\mu)\big|_{\alpha=\beta=\gamma=0},$$

and we let $\varphi_0(\vartheta)$ be given by $\varphi(\vartheta)$ in (7), with $\alpha=\beta=\gamma=\nu=0$.

Theorem 3. *The events $\{\tau_x < t\}$ and $\{Q_t > x\}$ coincide. In particular,*

$$\varrho_0(x\,|\,\nu) = \mathbb{P}(\tau_x < T_\nu) = \mathbb{P}(Q_{T_\nu} > x), \tag{14}$$

$$\varrho_0(x\,|\,0) = \mathbb{P}(\tau_x < \infty) = \mathbb{P}(Q > x), \tag{15}$$

$$\varphi_0(\vartheta) := \int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\varrho_0(x\,|\,0)\,\mathrm{d}x = \int_0^\infty \vartheta \mathrm{e}^{-\vartheta x}\mathbb{P}(Q > x)\,\mathrm{d}x = 1 - \mathbb{E}(\mathrm{e}^{-\vartheta Q}). \tag{16}$$

4.3. **Steady-state storage-level distribution.** In this subsection we first briefly indicate how the LST $\mathbb{E}(\mathrm{e}^{-\vartheta Q})$ of the steady-state storage-level distribution could have been obtained without resorting to duality (see also [32]). We subsequently derive expressions for the moments $\mathbb{E}(Q^n)$, $n = 1, 2, \ldots$. We finally study the distribution of the storage level immediately after jump instants.

Let $q(\cdot)$ denote the density of Q, and let $\psi(\vartheta) := \int_0^\infty e^{-\vartheta x} q(x)\,dx$; observe that $\mathbb{E}(e^{-\vartheta Q}) = \psi(\vartheta) + \pi_0$, where $\pi_0 = \mathbb{P}(Q = 0)$. We shall derive an expression for $\psi(\vartheta)$ by formulating a so-called level crossings identity [14]. Using the principle that, in equilibrium, each storage level is equally often crossed from above (the left-hand side) as from below (the right-hand side), we find

$$(rx + p)q(x) + \lambda_+ \int_x^\infty q(y)e^{-\mu(y-x)}dy = \lambda_- \int_{0^+}^x q(y)\mathbb{P}(D > x - y)\,dy + \lambda_-\pi_0\,\mathbb{P}(D > x). \tag{17}$$

The first term in the left-hand side represents the downcrossing rate during service, and the second term the downcrossing rate via a jump. The two terms in the right-hand side correspond to the upcrossing rate from some positive level below x and from level 0 (i.e., from an empty system). We now multiply both sides of equation (17) with $e^{-\vartheta x}$ and then integrate over $x > 0$, so as to obtain

$$-r\psi'(\vartheta) + p\psi(\vartheta) - \lambda_+ \frac{\psi(\vartheta) - \psi(\mu)}{\vartheta - \mu} = \lambda_-\psi(\vartheta)\frac{1 - \delta(\vartheta)}{\vartheta} + \lambda_-\pi_0\frac{1 - \delta(\vartheta)}{\vartheta}. \tag{18}$$

Taking $x = 0$ in (17) yields

$$\lambda_+\psi(\mu) + pq(0) = \lambda_-\pi_0. \tag{19}$$

Observe that $\psi(\mu)$ is in fact the probability that a jump downwards will result in an empty system, i.e., a system with no storage. Rewriting Equation (18), we arrive at the differential equation

$$\psi'(\vartheta) = -\frac{1}{r}\left(\frac{\lambda_+}{\vartheta - \mu} + \frac{\lambda_-(1 - \delta(\vartheta))}{\vartheta} - p\right)\psi(\vartheta) - \frac{\lambda_-\pi_0}{r}\frac{1 - \delta(\vartheta)}{\vartheta} + \frac{\lambda_+}{r}\frac{\psi(\mu)}{\vartheta - \mu}. \tag{20}$$

Remark 9. After applying (several times) integration by parts in (17), we can rewrite this integral equation into the integro-differential equation of Lemma 1, taking $\alpha = \beta = \gamma = \nu = 0$ and observing that $-\varrho'(x) = q(x)$ (cf. (15)). Similarly, one can translate the differential equation (4) for $\varphi(\vartheta)$ into the differential equation (20) for $\psi(\vartheta)$; the only difference lies in the inhomogeneous part, and is explained by the fact that $\varphi(\vartheta)$ with $\alpha = \beta = \gamma = \nu = 0$ equals $1 - \pi_0 - \psi(\vartheta)$ (cf. (16)). Also observe that $\psi(\mu) = 1 - \pi_0 - \varphi(\mu) = \varrho(0) - \varphi(\mu)$. We conclude that $\mathbb{E}(e^{-\vartheta Q}) = \psi(\vartheta) + \pi_0$ indeed immediately follows from the results for $\varphi(\vartheta)$ in Section 2. ♢

4.4. **Stationary moments.** We now derive expressions for the moments $\mathbb{E}(Q^n)$, $n = 1, 2, \ldots$. This could be done by differentiating the steady-state storage-level LST, but we follow a convenient, direct approach based on Equation (17), leading to a recursion. The first moment $\mathbb{E}(Q)$ can be found by integrating both sides of that equation over x from $x = 0$ to ∞. Using that $\int_0^\infty \mathbb{P}(X > x)dx = \mathbb{E}(X)$ for any non-negative random variable X, we readily obtain

$$\mathbb{E}(Q) = \frac{\lambda_-\mathbb{E}(D)}{r} - \frac{p(1 - \pi_0)}{r} - \frac{\lambda_+}{r\mu}(1 - \pi_0 - \psi(\mu)); \tag{21}$$

recall that $\psi(\mu)$ has been discussed above. To identify $\mathbb{E}(Q^n)$, we need to multiply (17) with x^{n-1} and then integrate over $x \in [0, \infty)$. We thus obtain, for $n = 1, 2, \ldots$,

$$\begin{aligned} r\int_0^\infty x^n q(x)dx + p\int_0^\infty x^{n-1}q(x)dx + \lambda_+\int_0^\infty\int_x^\infty q(y)x^{n-1}e^{-\mu(y-x)}dy\,dx \qquad (22)\\ = \lambda_-\int_0^\infty\int_{0^+}^x q(y)\mathbb{P}(D > x - y)x^{n-1}dydx + \lambda_-\int_0^\infty \pi_0\mathbb{P}(D > x)x^{n-1}dx. \end{aligned}$$

We consider the five terms in this equation separately. Calling the three terms in the left-hand side $L_1(n)$, $L_2(n)$ and $L_3(n)$, we directly see

$$L_1(n) = r\,\mathbb{E}(Q^n), \qquad L_2(n) = p\,\mathbb{E}(Q^{n-1}). \tag{23}$$

We now express $L_3(n)$ in terms of $L_3(n-1)$ for $n = 2, 3, \ldots$:

$$\begin{aligned}\frac{1}{\lambda_+}L_3(n) &= \int_0^\infty \int_x^\infty q(y)x^{n-1}\mathrm{e}^{-\mu(y-x)}\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \left[\int_0^y x^{n-1}\mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y \\ &= \int_0^\infty \left[\frac{1}{\mu}x^{n-1}\mathrm{e}^{\mu x}\bigg|_0^y - \frac{n-1}{\mu}\int_0^y x^{n-2}\mathrm{e}^{\mu x}\mathrm{d}x\right] q(y)\mathrm{e}^{-\mu y}\mathrm{d}y \\ &= \frac{1}{\mu}\mathbb{E}(Q^{n-1}) - \frac{n-1}{\lambda_+\mu}L_3(n-1).\end{aligned}$$

Iterating ultimately results in the following closed-form expression for $L_3(n)$: for $n = 2, 3, \ldots$,

$$L_3(n) = \lambda_+(n-1)!\left(\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{\mu^k(n-k)!}\mathbb{E}(Q^{n-k}) - \frac{(-1)^n}{\mu^n}(1-\pi_0-\psi(\mu))\right). \tag{24}$$

We now move to the two terms in the right-hand side of (22), which we call $R_1(n)$ and $R_2(n)$. We first compute $R_1(n)$:

$$\begin{aligned}\frac{R_1(n)}{\lambda_-} &= \int_0^\infty\int_{0+}^x q(y)\mathbb{P}(D > x-y)x^{n-1}\mathrm{d}y\,\mathrm{d}x = \int_0^\infty\int_{0+}^x q(y)\left[\int_{x-y}^\infty d(t)\mathrm{d}t\right]x^{n-1}\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^\infty\int_0^\infty q(y)d(t)\int_y^{y+t}x^{n-1}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y = \int_0^\infty\int_0^\infty q(y)d(t)\frac{1}{n}\big[(y+t)^n - y^n\big]\mathrm{d}t\,\mathrm{d}y \\ &= \frac{1}{n}\big[\mathbb{E}((Q+D)^n) - \mathbb{E}(Q^n)\big].\end{aligned}$$

To determine $R_2(n)$ we rely on the identity $\mathbb{E}(X^r) = \int_0^\infty rx^{r-1}\mathbb{P}(X > x)\mathrm{d}x$ (for any non-negative random variable X and $r \in \mathbb{N}$). We thus obtain that

$$R_2(n) = \frac{\pi_0\lambda_-}{n}\mathbb{E}(D^n). \tag{25}$$

Combining the expressions for $L_1(n)$, $L_2(n)$, $L_3(n)$, $R_1(n)$ and $R_2(n)$ yields the following equality (for $n = 1, 2, \ldots$):

$$\begin{aligned} r\,\mathbb{E}(Q^n) + p\,\mathbb{E}(Q^{n-1}) + \lambda_+(n-1)!\left(\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{\mu^k(n-k)!}\mathbb{E}(Q^{n-k}) - \frac{(-1)^n}{\mu^n}(1-\psi(\mu))\right) \\ = \frac{\lambda_-}{n}\left[\mathbb{E}((Q+D)^n) - \mathbb{E}(Q^n) + \pi_0\mathbb{E}(D^n)\right].\end{aligned} \tag{26}$$

Observe that with (26), $\mathbb{E}(Q^n)$ can be expressed in terms of $\mathbb{E}(Q^i)$ with $i = 0, 1, \ldots, n-1$, so that the moments of Q can be determined recursively.

4.5. **Storage level just after arrivals.** While the above analysis provides us with characterizations of the stationary distribution, it is also possible to identify the stationary distribution of the storage level W *just after arrivals*. Observing that, by the celebrated PASTA property, the amount of work just before a jump is distributed as Q, and denoting an arbitrary jump by J, we have

$$W \stackrel{\mathrm{d}}{=} \max\{Q+J, 0\}.$$

Denoting by E_μ a random variable that is exponentially distributed with rate μ,

$$\begin{aligned}\omega(s) := \mathbb{E}(\mathrm{e}^{-sW}) = \mathbb{E}(\mathrm{e}^{-s\max\{Q+J,0\}}) &= \frac{\lambda_-}{\lambda}\mathbb{E}(\mathrm{e}^{-s\max\{Q+D,0\}}) + \frac{\lambda_+}{\lambda}\mathbb{E}(\mathrm{e}^{-s\max\{Q-E_\mu,0\}}) \\ &= \frac{\lambda_-}{\lambda}\psi(s)\delta(s) + \frac{\lambda_+}{\lambda}\left(\frac{\mu}{\mu-s}\psi(s) - \frac{s}{\mu-s}\psi(\mu)\right).\end{aligned} \tag{27}$$

It is now also possible to express the moments of W in terms of the moments of Q. We only show the result for the first moment:

$$\mathbb{E}(W) = -\frac{\mathrm{d}}{\mathrm{d}s}\omega(s)\bigg|_{s=0} = \mathbb{E}(Q) + \frac{\lambda_-}{\lambda}\mathbb{E}(D) - \frac{\lambda_+}{\lambda}\frac{1-\psi(\mu)}{\mu}. \tag{28}$$

For higher moments similar calculations can be performed, but the resulting expressions become quite tedious.

4.6. **Special cases.** As stressed before the models analyzed in this paper are quite general, in that they contain many relevant models as special cases. Below we discuss a number of these special cases.

(1) $r = 0$, $\lambda_+ = 0$. Now the storage process $(Q_t)_{t\geqslant 0}$ is the classical M/G/1 workload process, whereas the insurance process $(X_t)_{t\geqslant 0}$ corresponds to the classical Cramér-Lundberg risk process. Multiplying the differential equation (4) by r, we observe that it degenerates to an ordinary linear equation. The M/G/1 workload process and the Cramér-Lundberg risk process are intensively studied objects; see e.g. [4, 18] for various results concerning the M/G/1 workload process, and e.g. [5] for analogous findings for the Cramér-Lundberg risk process.

(2) $\lambda_+ = 0$. In the insurance literature, research on this model goes back to at least [33], where the ruin probability is computed for the special case of exponential downward jumps (in terms of incomplete gamma functions). We refer to [5, Section VIII.2] for an overview of this research area, but we mention a few main contributions here. In the first place, an extension to finite-time ruin probabilities has been provided in e.g. [3]. Furthermore, [36] identifies the joint distribution of the surplus immediately before ruin and the deficit at ruin, whereas [16] identifies the expected value of a discounted penalty function at ruin (a function of the surplus immediately prior to ruin and the deficit at ruin). In [2] the setup is generalized considerably, allowing more general dynamics between jumps than those given through (1); in addition, the interclaim times as well as the claim sizes are allowed to have a rational Laplace transform. Importantly, unlike in our work, in this branch of the literature no upward jumps are allowed, while the downward jumps are not general. If it is also assumed that $p = 0$, then the storage process $(Q_t)_{t\geqslant 0}$ becomes a shotnoise process, which is covered by e.g. [10].

(3) $r = 0$. The storage process $(Q_t)_{t\geqslant 0}$ is the workload process in an M/G/1 queue with potentially also 'negative customers' (corresponding with the exponentially distributed downward jumps). That workload process was studied in [11]; it is readily verified that the workload transform $\psi(\vartheta)$ of the present paper agrees with the workload transform obtained in [11]. In the latter paper, that workload distribution is shown to equal the waiting time distribution in a G/G/1 queue with ordinary customers only; the effect of the negative customers is incorporated by appropriately stretching the interarrival times. We are not aware of papers on the $r = 0$ case (with jumps in both directions) in the insurance context.

(4) $p = 0$. This model, allowing both positive and negative jumps in the $(Q_t)_{t\geqslant 0}$ process, has hardly been studied. If the service speed is rx at workload level x, then the

decrease during an interval $A_{i+1} - A_i$ between two successive jumps, when starting at level W_i, equals $\mathrm{e}^{-r(A_{i+1}-A_i)}W_i$. Hence we can write, for $i = 1, 2, \ldots$:

$$W_{i+1} = \max\{\mathrm{e}^{-r(A_{i+1}-A_i)}W_i + B_{i+1}, 0\}, \tag{29}$$

where B_{i+1} denotes the size of the $(i+1)$-th jump. There is a similarity with the reflected autoregressive process studied in [12], in which 'Lévy thinning' is imposed on W_i.

5. Relaxation of the exponentiality assumptions

One of the seeming drawbacks of the results presented so far concerns the underlying exponentiality assumptions. In particular, in Section 2 it was assumed that the initial level, the time parameter, and the upward jumps stem from exponential distributions. In this section we point out how the exponentiality assumptions can be greatly relaxed. We focus on these aspects for the insurance risk model of Section 2, but the reasoning is also valid for the storage processes of Sections 3 and 4.

5.1. **Initial level and time horizon.** Phase-type distributions [4, Section 3.4] are known to be able to approximate any distribution on $(0, \infty)$ arbitrarily closely [4, Thm. III.4.2]; we say that the class of phase-type distributions is dense (in the sense of weak convergence) in the set of all distributions on $(0, \infty)$. Two important subclasses op phase-type distributions are the following:

- Firstly, there is the class $\mathscr{M}$ that corresponds to *mixtures* of independent exponentially distributed random variables. More concretely, for a given $k \in \mathbb{N}$, with probability p_i $(i = 1, \ldots, k)$ the random variable is sampled from an exponential distribution with non-negative mean ν_i^{-1}; here the probabilities p_i are non-negative and summing up to 1. Without loss of generality, we assume that for distributions in $\mathscr{M}$, the corresponding random variable being denoted by $M_{k,\boldsymbol{\nu}}$, all ν_i are distinct; we write $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^\top$. The class $\mathscr{M}$ is suitable for representing random variables with a squared coefficient of variation (scov), defined as the ratio of the variance to the square of the mean, larger than 1. For random variables with scov > 1, there is a distribution in $\mathscr{M}$ with the same first two moments; we specifically refer to the *balanced-means* fit in e.g. [34, p. 359], that corresponds to $k = 2$. Higher moments can be matched by using a larger value of k.
- Secondly, there is the class $\mathscr{S}$ that corresponds to *sums* of independent exponentially distributed random variables. For a given $k \in \mathbb{N}$, we assume that these exponential random variables have non-negative means $\nu_1^{-1}, \ldots, \nu_k^{-1}$; we again write $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^\top$. We throughout assume that for distributions in $\mathscr{S}$, the corresponding random variable being denoted by $S_{k,\boldsymbol{\nu}}$, all ν_i are distinct. This class of distributions is particularly useful when working with random variables with scov smaller than 1. Put differently, for random variables with scov < 1, one can identify a distribution in $\mathscr{S}$ with the same first two moments by taking $k = 2$; again, evidently, also higher moments can be fitted by choosing a larger value of k.
  We will also pay attention to the case that the ν_i are equal, in which case the underlying random variable is of Erlang type. We denote the Erlang random variable with k phases, each with mean ν^{-1}, by $\bar{S}_{k,\nu}$; the resulting class is $\bar{\mathscr{S}}$. The Erlang distribution is often used to approximate a deterministic value; indeed, $\bar{S}_{k,k\nu}$ converges to ν^{-1} as $k \to \infty$.

We now argue that for distributions in $\mathscr{M} \cup \mathscr{S}$ the corresponding density can be written as a mixture of exponentials; this is a known property, but we include the underlying reasoning provides useful insights. As can be found in e.g. [4, Proposition III.4.1], for any phase-type distribution the density can be directly evaluated from the matrix exponential pertaining to the so-called phase generator. For both $\mathscr{M}$ and $\mathscr{S}$ this phase generator is upper-triangular with $\boldsymbol{\nu}$ on the diagonal. This implies that (i) the eigenvalues can be read off from the diagonal and equal $\nu_1, \dots, \nu_k$, and (ii) the eigenvectors can be evaluated at low computational cost by a recursive procedure. We thus find that the density is a mixture of exponentials: it is of the form $\sum_{i=1}^k \omega_i \, \mathrm{e}^{-\nu_i t}$ for constants $\omega_1, \dots, \omega_k$ (which are summing up to 1, but in case of $\mathscr{S}$ not necessarily non-negative) and $t \geqslant 0$.

The remainder of this subsection describes how to convert a transform at an exponentially distributed epoch into a transform at an epoch with a distribution in $\mathscr{M}$, $\mathscr{S}$ or $\bar{\mathscr{S}}$. To this end, let $(Z_t)_{t\geqslant 0}$ be a scalar-valued stochastic process. In addition, T_ν is an independent exponentially distributed random variable with mean ν^{-1}. We suppose that we have an expression for

$$\zeta(\nu \,|\, s) := \mathbb{E}\, \mathrm{e}^{-sZ_{T_\nu}}.$$

Assume that $M_{k,\boldsymbol{\nu}}$, $S_{k,\boldsymbol{\nu}}$, and $\bar{S}_{k,\nu}$ are independent of the process $(Z_t)_{t\geqslant 0}$. It is immediately clear that

$$\mathbb{E}\, \mathrm{e}^{-sZ_{S_{k,\boldsymbol{\nu}}}} = \sum_{i=1}^k \frac{\omega_i}{\nu_i}\, \zeta(\nu_i \,|\, s);$$

evidently, the same approach can be followed for $Z_{M_{k,\boldsymbol{\nu}}}$. We conclude that distributions in $\mathscr{M}$ and $\mathscr{S}$ can be dealt with in an elementary way, and the expressions only involve $\zeta(\cdot \,|\, s)$. As we will see now, distributions in $\bar{\mathscr{S}}$ should be treated differently, with the resulting expression also featuring derivatives of $\zeta(\cdot \,|\, s)$.

We are thus left with evaluating

$$\zeta_k(\nu \,|\, s) := \mathbb{E}\, \mathrm{e}^{-sZ_{\bar{S}_{k,\nu}}} = \int_0^\infty \frac{\nu^k t^{k-1}}{(k-1)!}\, \mathrm{e}^{-\nu t} \mathbb{E}\, \mathrm{e}^{-sZ_t}\, \mathrm{d}t. \tag{30}$$

Let $\zeta^{(\ell)}(\nu \,|\, s)$ denote the ℓ-th derivative of $\zeta(\nu \,|\, s)$ with respect to ν.

Proposition 3. *For $k \in \mathbb{N}$,*

$$\zeta_k(\nu \,|\, s) = \sum_{\ell=0}^{k-1} \frac{(-\nu)^\ell}{\ell!} \zeta^{(\ell)}(\nu \,|\, s). \tag{31}$$

Proof. There are various (roughly equally short) ways to prove Proposition 3. We include an insightful proof here. We use that (30) implies that

$$\zeta_k(\nu \,|\, s) = -\frac{(-\nu)^k}{(k-1)!} \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}\nu^{k-1}} \frac{\zeta(\nu \,|\, s)}{\nu} \right).$$

Observing that, by the binomium,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}\nu^{k-1}} \frac{\zeta(\nu \,|\, s)}{\nu} = -\sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \zeta^{(\ell)}(\nu \,|\, s) \frac{(k-1-\ell)!}{(-\nu)^{k-\ell}}, \tag{32}$$

the stated follows immediately. □

By the above findings, we can translate our results in Section 2.3 for an exponentially distributed initial level to their counterparts in which the initial level stems from $\mathscr{M}$, $\mathscr{S}$ or $\bar{\mathscr{S}}$. Along the same lines we can relax the exponentiality assumption imposed on the time horizon.

Example 1. We proceed by presenting an example that illustrates how Proposition 3 can be used. We do this in the context of Theorem 1, where we want the initial level to be sampled from an Erlang distribution with parameters k and ϑ (i.e., with mean k/ϑ), rather than from the exponential distribution. Proposition 3 requires the evaluation of the derivatives $\varphi^{(\ell)}(\cdot)$. We know $\varphi(\vartheta)$, so (4) provides us with

$$\varphi^{(1)}(\vartheta) = F(\vartheta)\varphi(\vartheta) + G(\vartheta).$$

But then also

$$\begin{aligned}\varphi^{(2)}(\vartheta) &= F^{(1)}(\vartheta)\varphi(\vartheta) + F(\vartheta)\varphi^{(1)}(\vartheta) + G^{(1)}(\vartheta)\\ &= \left(F^{(1)}(\vartheta) + (F(\vartheta))^2\right)\varphi(\vartheta) + F(\vartheta)G(\vartheta) + G^{(1)}(\vartheta).\end{aligned} \tag{33}$$

One can continue along these lines, to compute $\varphi^{(\ell)}(\vartheta)$ recursively in terms of $\varphi(\vartheta)$. More precisely, it can be checked that $\varphi^{(\ell)}(\vartheta) = A_\ell(\vartheta)\,\varphi(\vartheta) + B_\ell(\vartheta)$, where $A_\ell(\cdot)$ and $B_\ell(\cdot)$ can be computed by the recursions

$$A_{\ell+1}(\vartheta) = A_\ell'(\vartheta) + A_\ell(\vartheta)\,F(\vartheta), \quad B_{\ell+1}(\vartheta) = A_\ell(\vartheta)\,G(\vartheta) + B_\ell'(\vartheta),$$

initialized by $A_1(\vartheta) = F(\vartheta)$ and $B_1(\vartheta) = G(\vartheta)$. Proposition 3 thus yields the following counterpart of $\varphi(\vartheta)$, now with an Erlang initial storage level:

$$\sum_{\ell=0}^{k-1}\frac{(-\vartheta)^\ell}{\ell!}\varphi^{(\ell)}(\vartheta) = \sum_{\ell=0}^{k-1}\frac{(-\vartheta)^\ell}{\ell!}\left(A_\ell(\vartheta)\,\varphi(\vartheta) + B_\ell(\vartheta)\right).$$

Inserting $\vartheta = k/z$ for k large leads to an approximation of the transform conditional on the initial storage level X_0 being z. $\Diamond$

5.2. **Upward jumps.** This subsection aims at relaxing the assumption of the exponentially distributed upward jumps. We start by noticing that if the upward jumps are in $\mathscr{M}$ or $\mathscr{S}$, by the precise same argumentation as the one we have used in the previous subsection, the corresponding density can be written as $\sum_{i=1}^k g_i\,\mathrm{e}^{-\mu_i x}$ for some $k \in \mathbb{N}$, constants $g_1, \dots, g_k$ (which are summing up to 1) and $x \geqslant 0$. It is not difficult to adapt the analysis of Section 2.3 to upward jumps in $\mathscr{M}$ or $\mathscr{S}$. It is readily checked that we have to replace the right-hand side of (3) by

$$\sum_{i=1}^k \lambda_+\frac{g_i}{\mu_i - \vartheta}\varphi(\vartheta) - \sum_{i=1}^k \lambda_+\frac{g_i}{\mu_i - \vartheta}\frac{\vartheta}{\mu_i}\varphi(\mu_i).$$

Mimicking the analysis conducted in Section 2.3, it turns out that the functions $F(\cdot)$ and $G(\cdot)$ should be slightly adapted. Most notably, (i) these functions now have poles at $\mu_1, \dots, \mu_k$, and (ii) the function $G(\cdot)$ contains the unknowns $\varphi(\mu_1), \dots, \varphi(\mu_k)$. The $k+1$ unknowns $\varphi(\mu_1), \dots, \varphi(\mu_k)$ and $\varrho(0)$ can be identified as in the case of exponentially distributed upward jumps, i.e., by equating $I(0)$ as well as $I(\mu_i)$ to 0 (for all $i = 1, \dots, k$).

Having addressed the case that the upward jumps are in $\mathscr{S}$ or $\mathscr{M}$, in the remainder of this subsection we concentrate on upward jumps in $\bar{\mathscr{S}}$. With the upward jumps being distributed as $\bar{S}_{k,\mu}$, it turns out that we have to replace the left-hand side of (3) by

$$\lambda_+\int_0^\infty\left(\int_0^\infty\frac{\mu^k y^{k-1}}{(k-1)!}\mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y\right)\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x. \tag{34}$$

Applying Lemma 2 (see Appendix A), we can evaluate (34) in terms of $\varphi(\cdot)$ and its derivatives; it turns out to simplify to

$$\lambda_+\left(\frac{\mu}{\mu-\vartheta}\right)^k\varphi(\vartheta) - \lambda_+\sum_{\ell=0}^{k-1}\frac{(-\mu)^\ell}{\ell!}\varphi^{(\ell)}(\mu)\left(\left(\frac{\mu}{\mu-\vartheta}\right)^{k-\ell} - 1\right).$$

Observe that this expression involves $\varphi^{(0)}(\mu) = \varphi(\mu)$ up to $\varphi^{(k-1)}(\mu)$, and that at $\vartheta = \mu$ there are poles of multiplicity up to k. As before, we end up with the differential equation $\varphi'(\vartheta) = F(\vartheta)\varphi(\vartheta) + G(\vartheta)$, but now with

$$F(\vartheta) = \bar{F}(\vartheta) + F_0(\vartheta) + F_\mu(\vartheta), \quad G(\vartheta) = \bar{G}(\vartheta) + G_\mu(\vartheta), \tag{35}$$

where

$$\bar{F}(\vartheta) := \frac{p}{r} - \frac{\lambda_-}{r}\frac{1-\delta(\vartheta)}{\vartheta}, \quad F_0(\vartheta) := -\frac{\alpha+\nu}{r\vartheta}, \quad F_\mu(\vartheta) := \frac{\lambda_+}{r\vartheta}\left(\left(\frac{\mu}{\mu-\vartheta}\right)^k - 1\right),$$

and, using the notation $b_\ell := \varphi^{(\ell)}(\mu)$,

$$\bar{G}(\vartheta) := -\frac{p}{r}\varrho(0) + \frac{\lambda_-}{r}\frac{\delta(\beta)-\delta(\vartheta+\gamma)}{\vartheta+\gamma-\beta}, \quad G_\mu(\vartheta) := \frac{\lambda_+}{r\vartheta}\sum_{\ell=0}^{k-1}\frac{(-\mu)^\ell}{\ell!}b_\ell\left(1-\left(\frac{\mu}{\mu-\vartheta}\right)^{k-\ell}\right).$$

As in Section 2.3, again denoting by $F_\star(\cdot)$ the primitive of $F(\cdot)$, the differential equation $\varphi'(\vartheta) = F(\vartheta)\varphi(\vartheta) + G(\vartheta)$ is solved by

$$\varphi(\vartheta) = -\left(\int_\vartheta^\infty G(\eta)\exp\big(-F_\star(\eta)\big)\,\mathrm{d}\eta\right)\exp\big(F_\star(\vartheta)\big)$$

(if $p > 0$). In Appendix B it is pointed out how the unknowns $\varrho(0)$ and $\boldsymbol{b} = (b_0, \ldots, b_{k-1})^\top$ can be identified. This is a rather delicate procedure, due to issues related to the non-analyticity of certain functions; see Remark 11.

6. Discussion and concluding remarks

This paper has studied a general class of storage processes, in which the rate at which the storage level increases or decreases is an affine function of the current level. There are additional upward and downward jumps arriving according to a Poisson process. These storage processes have been analyzed by first considering a related insurance risk model (that is of independent interest) and then translating the resulting expressions.

A few possible extensions are the following. In the first place, one could pursue more general dynamics between the jumps. In our storage model, we have assumed an affine release rule, but we could aim at generalizing this to polynomials of a higher degree; cf. for instance [2]. A second extension could concern a bivariate version of our insurance risk model, in which two insurance firms experience simultaneous claim arrivals, and its dual, a storage or queueing system with simultaneous jumps. For the case of zero interest, cq. constant release rule, explicit expressions have been found; see [7, 25] for the storage/queueing setting and [9] for the insurance risk case. It would be interesting to explore whether they can also be analyzed for $r > 0$. In addition one could attempt to relax the assumption of independence between interarrival times and jump sizes; cf. [8]. A third line of research could relate to scaling limits and asymptotic results; see for instance the diffusion scaling studied in [13].

Appendix A

Lemma 2. *For $\vartheta, \mu > 0$ and $k = 1, 2, \ldots$,*

$$\int_0^\infty\left(\int_0^\infty \frac{\mu^k y^{k-1}}{(k-1)!}\mathrm{e}^{-\mu y}\varrho(x+y)\,\mathrm{d}y\right)\vartheta\mathrm{e}^{-\vartheta x}\,\mathrm{d}x \tag{36}$$

$$= \left(\frac{\mu}{\mu-\vartheta}\right)^k\varphi(\vartheta) - \sum_{\ell=0}^{k-1}\frac{(-\mu)^\ell}{\ell!}\varphi^{(\ell)}(\mu)\left(\left(\frac{\mu}{\mu-\vartheta}\right)^{k-\ell} - 1\right). \tag{37}$$

Proof: Applying the transformation $z := x + y$, the expression (36) equals

$$\int_0^\infty \left(\int_0^z \frac{\mu^k y^{k-1}}{(k-1)!} \mathrm{e}^{-(\mu-\vartheta)y} \mathrm{d}y \right) \vartheta \mathrm{e}^{-\vartheta z} \varrho(z)\,\mathrm{d}z.$$

Relying on standard identities for the Gamma function, the inner integral can be rewritten as

$$\left(\frac{\mu}{\mu-\vartheta}\right)^k \left(1 - \sum_{n=0}^{k-1} \frac{(\mu-\vartheta)^n z^n}{n!} \mathrm{e}^{-(\mu-\vartheta)z}\right).$$

This implies that (36) equals

$$\left(\frac{\mu}{\mu-\vartheta}\right)^k \int_0^\infty \left(\vartheta \mathrm{e}^{-\vartheta z} - \vartheta \sum_{n=0}^{k-1} \frac{(\mu-\vartheta)^n z^n}{n!} \mathrm{e}^{-\mu z}\right) \varrho(z)\,\mathrm{d}z.$$

In this expression we recognize derivatives of $\varphi(\cdot)$ evaluated in μ. It turns out to be convenient to work with $\breve\varphi(\vartheta) := \varphi(\vartheta)/\vartheta$, and to later translate the findings back in terms of $\varphi(\cdot)$ itself. In terms of $\breve\varphi(\vartheta)$ the last expression equals

$$\left(\frac{\mu}{\mu-\vartheta}\right)^k \left(\varphi(\vartheta) - \vartheta \sum_{n=0}^{k-1} \frac{(\vartheta-\mu)^n}{n!} \breve\varphi^{(n)}(\mu)\right).$$

Observe that, as an elementary application of the binomium,

$$\breve\varphi^{(n)}(\mu) = \sum_{\ell=0}^{n} \binom{n}{\ell} \varphi^{(\ell)}(\mu) \cdot (-1)^{n-\ell} \frac{(n-\ell)!}{\mu^{n-\ell+1}} = n! \sum_{\ell=0}^{n} \frac{\varphi^{(\ell)}(\mu)}{\ell!} \cdot \frac{(-1)^{n-\ell}}{\mu^{n-\ell+1}}.$$

Upon combining the above, we conclude that (36) equals

$$\left(\frac{\mu}{\mu-\vartheta}\right)^k \varphi(\vartheta) - \vartheta \sum_{n=0}^{k-1} \left(\frac{1}{\mu-\vartheta}\right)^{k-n} \sum_{\ell=0}^{n} (-1)^\ell \frac{\varphi^{(\ell)}(\mu)}{\ell!} \cdot \mu^{k+\ell-n-1}. \tag{38}$$

The last term we want to write as a linear combination of $\varphi^{(0)}(\mu)$ up to $\varphi^{(k-1)}(\mu)$; by swapping the order of the two summations, (38) turns out to simplify to

$$\left(\frac{\mu}{\mu-\vartheta}\right)^k \varphi(\vartheta) - \vartheta \sum_{\ell=0}^{k-1} \frac{(-\mu)^\ell}{\ell!} \varphi^{(\ell)}(\mu) \frac{\mu^{k-1}}{(\mu-\vartheta)^k} \sum_{n=\ell}^{k-1} \left(\frac{\mu-\vartheta}{\mu}\right)^n,$$

which can be verified to equal (37). □

Appendix B

This appendix focuses on the identification of the unknowns $\varrho(0)$ and $\boldsymbol{b}$ (corresponding to upward jumps in $\mathscr{S}$). Observe that the function $G(\cdot)$ in (35) is linear in the unknowns $\varrho(0)$ and $\boldsymbol{b} = (b_0, \ldots, b_{k-1})^\top$, with $b_\ell = \varphi^\ell(\mu)$.

So as to find equations that determine the unknowns, two equations derive from the poles at 0 and μ, as before. More precisely, due to $F_\star(0) = F_\star(\mu) = \infty$, we obtain the two equations that are linear in the unknowns $\varrho(0)$ and $\boldsymbol{b}$:

$$\int_0^\infty G(\eta) \exp\big(-F_\star(\eta)\big)\mathrm{d}\eta = \int_\mu^\infty G(\eta) \exp\big(-F_\star(\eta)\big)\mathrm{d}\eta = 0. \tag{39}$$

We are left with the task of finding sufficiently many additional equations.

Remark 10. By means of a sanity check, but also for later reference, we proceed by studying the singularities present in the right-hand side of the relation $\varphi'(\mu) = \lim_{\vartheta\to\mu}(F(\vartheta)\,\varphi(\vartheta) + G(\vartheta))$. In this right-hand side, there are poles of the type $(\mu-\vartheta)^{\ell-k}$, for $\ell \in \{0,\dots,k-1\}$. Evidently, one should have that the sum of all coefficients in front of terms of order $(\mu-\vartheta)^{\ell-k}$ equals 0 for all $\ell \in \{0,\dots,k-1\}$. Expanding $\varphi(\cdot)$ through a Taylor series, i.e.,

$$\varphi(\vartheta) = \sum_{\ell=0}^{k-1} \frac{(\vartheta-\mu)^\ell}{\ell!} b_\ell + O((\mu-\vartheta)^k),$$

and collecting all terms corresponding to $(\mu-\vartheta)^{\ell-k}$, this is directly verified. More specifically, the coefficient corresponding to $F(\cdot)$ is $\kappa_1 := \lambda_+\mu^{k-1}/r$, the relevant coefficient in the expansion of $\varphi(\cdot)$ is $\kappa_2 := b_\ell\,(-1)^\ell/\ell!$, whereas the coefficient corresponding to $G(\cdot)$ is $\kappa_3 := -\lambda_+ b_\ell\,(-\mu)^\ell\,\mu^{k-\ell}/(r\mu\,\ell!)$. It is directly seen that indeed $\kappa_1\kappa_2+\kappa_3 = 0$, as desired. $\Diamond$

As we are searching for derivatives of $\varphi(\cdot)$ at μ, a first thought would be to obtain additional equations to identify $\boldsymbol{b}$ from the relation $\varphi'(\mu) = \lim_{\vartheta\to\mu}(F(\vartheta)\,\varphi(\vartheta) + G(\vartheta))$ and the corresponding higher derivatives (where we could use Example 1). Unfortunately, this procedure does not lead to a tractable recursion, which can be seen as follows. The above relation for $\varphi'(\mu)$ can be written as, when considering $\vartheta \uparrow \mu$,

$$b_1 = \varphi'(\mu) = \lim_{\vartheta\to\mu}\left(\left(\bar F(\mu) + F_0(\mu) + \frac{\lambda_+}{r\vartheta}\left(\left(\frac{\mu}{\mu-\vartheta}\right)^k - 1\right)\right)\left(\sum_{\ell=0}^{k}\frac{(\vartheta-\mu)^\ell}{\ell!}\,b_\ell\right) + \right.$$
$$\left. \bar G(\mu) + \frac{\lambda_+}{r\vartheta}\sum_{\ell=0}^{k-1}\frac{(-\mu)^\ell}{\ell!} b_\ell \left(1 - \left(\frac{\mu}{\mu-\vartheta}\right)^{k-\ell}\right)\right).$$

First observe that in the right-hand side all terms that correspond to poles vanish (as they should); cf. Remark 10. When taking the limit in the right-hand side, we thus obtain

$$b_1 = \left(\bar F(\mu) + F_0(\mu) - \frac{\lambda_+}{r\mu}\right) b_0 + \frac{\lambda_+\mu^{k-1}}{r}\frac{(-1)^k}{k!} b_k + \bar G(\mu) - \frac{\lambda_+}{r}\sum_{\ell=0}^{k-1}\frac{(-\mu)^{\ell-1}}{\ell!} b_\ell$$
$$= \left(\bar F(\mu) + F_0(\mu) - \frac{\lambda_+}{r\mu}\right) b_0 + \bar G(\mu) - \frac{\lambda_+}{r}\sum_{\ell=0}^{k}\frac{(-\mu)^{\ell-1}}{\ell!} b_\ell.$$

This means that b_1 depends on b_0 up to b_k. Continuing this procedure shows that the computation of $b_0,\dots,b_{k-1}$ requires $b_k, b_{k+1},\dots$, which cannot be done in an evident recursive way (except for the case of exponential upward jumps, i.e., $k=1$). This explains why we pursue an alternative procedure, relying on the constants $c_0, c_1, \dots$, where $c_\ell := \varphi^{(\ell)}(0)$.

Importantly, the derivatives at 0, i.e., the constants $c_0, c_1, \dots$, *can* be solved recursively. The main observation is that the singularity at 0 is more benign than that at μ: the function $F(\cdot)$ essentially behaves as $1/\vartheta$ around 0 (whereas it behaves as $1/(\vartheta-\mu)^k$ around μ). Writing $A := -(\alpha+\nu)/r$ (as before),

$$\bar F(\vartheta) + F_\mu(\vartheta) = \sum_{\ell=0}^{\infty} \bar f_\ell\,\vartheta^\ell, \quad G(\vartheta) = \sum_{\ell=0}^{\infty} g_\ell\,\vartheta^\ell, \tag{40}$$

it can be checked that the recursive scheme of Proposition 2 applies to generate the constants c_ℓ (but now with the $\bar f_\ell$ and g_ℓ defined by (40)). These can be used to identify the unknowns, as can be seen as follows. Observe that the coefficients g_k are linear in $\varrho(0)$ and $\boldsymbol{b}$, and hence also the c_ℓ. It means that we have thus obtained the linear equations

$$b_\ell = \varphi^{(\ell)}(\mu) = \lim_{\vartheta\to\mu}\frac{\mathrm{d}^\ell}{\mathrm{d}\vartheta^\ell}\sum_{j=0}^{\infty}\frac{\vartheta^j}{j!}\varphi^{(j)}(0) = \sum_{j=0}^{\infty}\frac{\mu^j}{j!}\varphi^{(j+\ell)}(0) = \sum_{j=0}^{\infty}\frac{\mu^j}{j!}c_{j+\ell}(\varrho(0),\boldsymbol{b}),$$

from which $\boldsymbol{b}$ can be solved.

Remark 11. In this remark we discuss why the case of upward jumps in $\bar{\mathscr{S}}$ with $k \in \{2, 3, \ldots\}$ is crucially harder than the case of exponentially distributed upward jumps. One insightful observation, which we already made above, is that only for $k = 1$ the derivatives of $\varphi(\cdot)$ in μ can be computed recursively. There is another phenomenon worth mentioning, though.

In the evaluation of the solution an important role is played by $\mathrm{e}^{-F_\star(\eta)}$ for η around μ. For $k = 1$ and η approaching μ from below, this function effectively behaves as

$$\exp\big(-\log(\mu - \eta)\big) = \frac{1}{\mu - \eta}$$

(while for η approaching μ from above a similar property applies). For $k = 2, 3, \ldots$, however, $\mathrm{e}^{-F_\star(\eta)}$ behaves, as η approaches μ from below, as

$$\aleph(\eta) := \exp\left(-\frac{1}{(\mu - \eta)^{k-1}}\right).$$

Observe that $\aleph(\cdot)$ should be handled with care, as the function is non-analytical as $\eta \uparrow \mu$ (verify that all left-derivatives are 0); this behavior is inherited by the integral $I(\cdot)$. Recalling that $\varphi(\vartheta) = -I(\vartheta)\,\mathrm{e}^{F_\star(\vartheta)}$, we conclude that evaluation of the function $\varphi(\cdot)$ around μ cannot be done in a classical 'Taylor-based' manner. Observe that this issue does not play a role at $\eta \downarrow 0$, as there $\mathrm{e}^{-F_\star(\eta)}$ essentially behaves as $1/\eta$. ♢

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