CHAPTER I

BASIC THEORY

1. The Topic of this Chapter. The mathematical methods used in this book are largely based on the various solutions of a general recurrence relation. These solutions have some interest of their own and can be used in solving many problems in the theory of probability and stochastic processes. In this chapter we shall develop the basic theory for finding these solutions and in the following chapters we shall deal with its applications in fluctuation theory.

To describe it briefly, the basic theory is concerned with various solutions of the problem of finding a sequence of functions $\Gamma_n(s)$ $(n=1,2,\ldots)$ defined for $\mathrm{Re}(s) = 0$ by a recurrence relation

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{1}$$

where $\gamma(s)$ and $\Gamma_0(s)$ are elements of a commutative Banach algebra $\underset{\sim}{R}$, and $\underset{\sim}{T}$ is a projection. We shall define $\underset{\sim}{R}$ in such a way that on the one hand $\underset{\sim}{R}$ is large enough to contain all the important functions arising in fluctuation theory and on the other hand $\underset{\sim}{R}$ is small enough to allow an explicit representation of the transformation $\underset{\sim}{T}$, which is suitable for calculations.

First we shall give explicit expressions for $\Gamma_n(s)$ $(n=1,2,\ldots)$ in the cases where $\Gamma_0(s) \equiv 1$ and where $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$.

Second, we shall give closed expressions for the generating function

$$U(s, \rho) = \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n \tag{2}$$

in the cases where $\Gamma_0(s) \equiv 1$ and where $T\{\Gamma_0(s)\} \equiv \Gamma_0(s)$.

Third, we shall show how the generating fucntion $U(s, \rho)$ can be obtained by using the method of factorization.

Afterwards, we shall show that the above results can also be obtained in a simpler way if we restrict ourself to the case where $\gamma(s)$ and $\Gamma_0(s)$ belong to a suitably chosen subspace of R .

Finally, we shall obtain analogous results for the case where $\gamma(s)$ and $\Gamma_0(s)$ belong to a space A which is isomorphic to a subspace of R , and T is replaced by a corresponding transformation Π .

The method developed in this chapter is completely elementary and self-contained. The only auxiliary theorem which we use is Cauchy's integral formula.

The mentioned problems have been solved in a particular case by F. Pollaczek [26], [27] . In his studies F. Pollaczek considered a smaller class of functions than R . For this smaller class he gave an explicit representation of T and found the generating function $U(s, \rho)$ as the solution of a singular integral equation. Pollaczek's method has the advantage that it yields $U(s, \rho)$ in a closed form, but it has also the disadvantage that some restrictions should be imposed on the functions $\gamma(s)$ and $\Gamma_0(s)$. Our method can be considered as an extension of

Pollaczek's method to the general case. The general method presented in this chapter does not require to impose any unnecessary restrictions on the functions considered.

In solving the mentioned problems we can use also algebraic methods (G. Baxter [6], [7], [8], J. G. Wendel [46], [47], J. F. C. Kingman [19], [20], G.-C. Rota [31]), combinatorial methods (E. S. Andersen [1], [2], F. Spitzer [35], W. Feller [13], the author [38]) and analytic methods (I. J. Good [14], J. H. B. Kemperman [18], A. A. Borovkov [11]). The algebraic methods are mostly descriptive, and even in the particular case of $\Gamma_0(s) \equiv 1$, the solution does not appear in a closed form. In general, combinatorial methods do not provide the solution in a closed form either, but fortunately, in some particular cases we can obtain explicit results. (See the author [38].). The most useful analytic method is the method of factorization which yields simple solutions in many cases; however, this method has been applied only in particular cases in the past. The method of factorization has been introduced by N. Wiener and E. Hopf [49] for solving integral equations. (See also F. Smithies [33], H. Widom [48], N. I. Muskhelishvili [22] and M. G. Krein [21].)

The results presented in this chapter have been developed by the author [39], [40], [41], [42], [43].

2. <u>A Space $\underset{\sim}{R}$</u>. Denote by $\underset{\sim}{R}$ the space of all those functions $\Phi(s)$ defined for $\mathrm{Re}(s) = 0$ on the complex plane, which can be represented in the form

$$\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\} \tag{1}$$

where ζ is a complex (or real) random variable with $\underset{\sim}{E}\{|\zeta|\} < \infty$, and η is a real random variable. The function $\Phi(s)$ is uniquely determined by the joint distribution of ζ and η. However, there are infinitely many possible distributions which yield the same $\Phi(s)$. It follows from (1) that $|\Phi(s)| \leqq \underset{\sim}{E}\{|\zeta|\}$ for $\mathrm{Re}(s) = 0$. It can easily be seen that $\Phi(s)$ is a continuous function of s for $\mathrm{Re}(s) = 0$.

Let us define the norm of $\Phi(s)$ by

$$||\Phi|| = \inf_{\zeta} \underset{\sim}{E}\{|\zeta|\} \tag{2}$$

where the infimum is taken for all admissible ζ, that is, for all those ζ for which (1) holds. Obviously $|\Phi(s)| \leqq ||\Phi||$ for $\mathrm{Re}(s) = 0$.

We have $||\Phi|| \geqq 0$, and $||\Phi|| = 0$ if and only if $\Phi(s) \equiv 0$. If α is a complex (or real) number and $\Phi(s) \in \underset{\sim}{R}$, then $\alpha\Phi(s) \in \underset{\sim}{R}$ and $||\alpha\Phi|| = |\alpha| \; ||\Phi||$. Furthermore, if $\Phi_1(s) \in \underset{\sim}{R}$ and $\Phi_2(s) \in \underset{\sim}{R}$, then $\Phi_1(s) + \Phi_2(s) \in \underset{\sim}{R}$ and $||\Phi_1 + \Phi_2|| \leqq ||\Phi_1|| + ||\Phi_2||$. This last statement can be proved as follows:

For any $\varepsilon > 0$ let $\Phi_1(s) = \underset{\sim}{E}\{\zeta_1 e^{-s\eta_1}\}$ where $\underset{\sim}{E}\{|\zeta_1|\} \leqq ||\Phi_1|| + \varepsilon$ and let $\Phi_2(s) = \underset{\sim}{E}\{\zeta_2 e^{-s\eta_2}\}$ where $\underset{\sim}{E}\{|\zeta_2|\} \leqq ||\Phi_2|| + \varepsilon$. Let ν be a random variable which is independent of (ζ_1, η_1) and (ζ_2, η_2) and for which $\underset{\sim}{P}\{\nu = 1\} = \underset{\sim}{P}\{\nu = 2\} = \frac{1}{2}$. Let us define $\zeta = 2\zeta_\nu$ and $\eta = \eta_\nu$. Then

(3) $\underset{\sim}{E}\{\zeta e^{-s\eta}\} = \Phi_1(s)+\Phi_2(s)$ and $\underset{\sim}{E}\{|\zeta|\} = \underset{\sim}{E}\{|\zeta_1|\}+ \underset{\sim}{E}\{|\zeta_2|\} < \infty$.

Thus $\Phi_1(s)+\Phi_2(s) \in \underset{\sim}{R}$, and $\|\Phi_1+ \Phi_2\| \leqq \|\Phi_1\| + \|\Phi_2\| + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves the statement. Accordingly, $\underset{\sim}{R}$ is a **normed linear** space. In what follows we shall not make use of the completeness of $\underset{\sim}{R}$. However, we can prove that $\underset{\sim}{R}$ is complete, and hence it follows that $\underset{\sim}{R}$ is a Banach space. (See Problem **13.1.**)

Next we observe that if $\Phi_1(s) \in \underset{\sim}{R}$ and $\Phi_2(s) \in \underset{\sim}{R}$, then $\Phi_1(s)\Phi_2(s) \in \underset{\sim}{R}$ and $\|\Phi_1\Phi_2\| \leqq \|\Phi_1\| \|\Phi_2\|$. To prove this let us define $\Phi_1(s)$ and $\Phi_2(s)$ in exactly the same way as above. However, let us assume now that (ζ_1, η_1) and (ζ_2, η_2) are independent and define $\zeta = \zeta_1\zeta_2$ and $\eta = \eta_1+ \eta_2$. Then

(4) $\underset{\sim}{E}\{\zeta e^{-s\eta}\} = \Phi_1(s)\Phi_2(s)$ and $\underset{\sim}{E}\{|\zeta|\} = \underset{\sim}{E}\{|\zeta_1|\}\underset{\sim}{E}\{|\zeta_2|\} < \infty$.

Thus $\Phi_1(s)\Phi_2(s) \in \underset{\sim}{R}$ and $\|\Phi_1\Phi_2\| \leqq (\|\Phi_1\| +\varepsilon)(\|\Phi_2\| +\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, this proves the statement.

Accordingly, $\underset{\sim}{R}$ can be characterized as a commutative Banach algebra.

3. A Linear Transformation T. Let us define a transformation T in the following way. If $\Phi(s) \in R$ and $\Phi(s)$ is given by (2.1), then let

$$T\{\Phi(s)\} = \Phi^+(s) = E\{\zeta e^{-s\eta^+}\} \tag{1}$$

for $\mathrm{Re}(s) = 0$ where $\eta^+ = \max(0, \eta)$. It can easily be seen that the function $\Phi^+(s)$ is independent of the particular representation (2.1). It depends solely on $\Phi(s)$. If $\Phi(s) \in R$, then obviously $\Phi^+(s) \in R$.

If α is a complex (or real) number and $\Phi(s) \in R$, then $T\{\alpha\Phi(s)\} = \alpha T\{\Phi(s)\}$. If $\Phi_1(s) \in R$ and $\Phi_2(s) \in R$, then $T\{\Phi_1(s)+\Phi_2(s)\} =$ ⁄ This follows immediately from the representation (2.3). Obviously $\|T\| = 1$. ($\|T\| = \sup\{\|T\Phi\| : \Phi \in R \text{ and } \|\Phi\| \leq 1\}$.) Accordingly, T is a bounded linear transformation. Since $T^2 = T$, therefore T is a projection.

Lemma 1. If $\Phi_1(s) \in R$ and $\Phi_2(s) \in R$, then

$$T\{\Phi_1(s)\Phi_2(s)\} = T\{\Phi_1(s)T\Phi_2(s)\} + T\{\Phi_2(s)T\Phi_1(s)\} - (T\Phi_1(s))(T\Phi_2(s)) \,. \tag{2}$$

Proof. For any real x and y we have the identity

$$e^{-s[x+y]^+} = e^{-s[x+y^+]^+} + e^{-s[x^++y]^+} - e^{-s(x^++y^+)} \tag{3}$$

where we used the notation $[x]^+ = x^+ = \max(0, x)$.

Let us suppose that $\Phi_1(s) = E\{\zeta_1 e^{-s\eta_1}\}$ and $\Phi_2(s) = E\{\zeta_2 e^{-s\eta_2}\}$

⁄ $T\{\Phi_1(s)\} + T\{\Phi_2(s)\}$.

where (ζ_1, η_1) and (ζ_2, η_2) are independent. If we put $x = \eta_1$ and $y = \eta_2$ in (3), multiply it by $\zeta_1\zeta_2$ and form its expectation, then we obtain (2).

We note that (2) is equivalent to the following relation. If $\Psi_1(s) = \Phi_1(s) - \underset{\sim}{T}\{\Phi_1(s)\}$ and $\Psi_2(s) = \Phi_2(s) - \underset{\sim}{T}\{\Phi_2(s)\}$, then

(4)
$$\underset{\sim}{T}\{\psi_1(s)\psi_2(s)\} = 0 ,$$

which can easily be seen to be true.

We mention two particular cases of (2), which will frequently be used in this book. If $\underset{\sim}{T}\{\Phi_1(s)\} = \Phi_1(s)$ and $\underset{\sim}{T}\{\Phi_2(s)\} = \Phi_2(s)$, then $\underset{\sim}{T}\{\Phi_1(s)\Phi_2(s)\} = \Phi_1(s)\Phi_2(s)$. If $\underset{\sim}{T}\{\Phi_1(s)\} = c_1$ and $\underset{\sim}{T}\{\Phi_2(s)\} = c_2$, where c_1 and c_2 are complex (or real) constants, then $\underset{\sim}{T}\{\Phi_1(s)\Phi_2(s)\} = c_1c_2$. These statements can easily be proved directly.

In what follows we shall make some general observations concerning $\Phi^+(s)$ and $\Phi(s) - \Phi^+(s)$. If $\Phi(s) \varepsilon \underset{\sim}{R}$, then $\Phi(s)$ can be represented in the form (2.1) and

(5)
$$\Phi^+(s) = \underset{\sim}{E}\{\zeta e^{-s\eta^+}\}$$

for $Re(s) = 0$. If we extend the definition of $\Phi^+(s)$ for $Re(s) \geqq 0$ by (5), then $\Phi^+(s)$ becomes regular in the domain $Re(s) > 0$ and continuous for $Re(s) \geqq 0$. Furthermore, $|\Phi^+(s)| \leqq \|\Phi\|$ for $Re(s) \geqq 0$. If $\Phi(s) \varepsilon \underset{\sim}{R}$, then $\Phi(s)$ can be represented in the form (2.1) and

(6)
$$\Phi(s)-\Phi^+(s) = \underset{\sim}{E}\{\zeta e^{s[-\eta]^+}\} - \underset{\sim}{E}\{\zeta\}$$

for $Re(s) = 0$. This follows from the following identity

(7) $$e^{-sx} - e^{-sx^+} = e^{s[-x]^+} - 1$$

which holds for any real x. If we put $x = \eta$ in (7), multiply it by ζ and form its expectation, then we obtain (6). If we extend the definition of $\Phi(s) - \Phi^+(s)$ for $\text{Re}(s) \leqq 0$ by (6), then $\Phi(s) - \Phi^+(s)$ becomes regular in the domain $\text{Re}(s) < 0$ and continuous for $\text{Re}(s) \leqq 0$. Obviously $|\Phi(s) - \Phi^+(s)| \leqq 2||\Phi||$ for $\text{Re}(s) \leqq 0$.

We note that if $\underset{\sim}{T}\{\Phi(s)\} = \Phi(s)$, then $\Phi(s) = \Phi^+(s) = \underset{\sim}{E}\{\zeta e^{-s\eta^+}\}$, that is, $\Phi(s)$ can be represented as $\underset{\sim}{E}\{\zeta e^{-s\eta}\}$ where η is a nonnegative random variable. If $\underset{\sim}{T}\{\Phi(s)\} = 0$, then $\Phi^+(s) = 0$ and $\Phi(0) = \Phi^+(0) = 0$ and by (6) we have $\Phi(s) = \underset{\sim}{E}\{\zeta e^{s[-\eta]^+}\}$, that is, $\Phi(s)$ can be represented as $\underset{\sim}{E}\{\zeta e^{-s\eta}\}$ where η is a nonpositive random variable.

The last remark implies, for example, that (4) is true. For, if $\underset{\sim}{T}\{\Psi_1(s)\} = 0$ and $\underset{\sim}{T}\{\Psi_2(s)\} = 0$, then we may assume that $\Psi_1(s) = \underset{\sim}{E}\{\zeta_1 e^{-s\eta_1}\}$ and $\Psi_2(s) = \underset{\sim}{E}\{\zeta_2 e^{-s\eta_2}\}$ where η_1 and η_2 are nonpositive random variables. If (ζ_1, η_1) and (ζ_2, η_2) are chosen to be independent, then it follows immediately that $\underset{\sim}{T}\{\Psi_1(s)\Psi_2(s)\} = \underset{\sim}{E}\{\zeta_1\zeta_2\} = \Psi_1(0)\Psi_2(0) = 0$. This proves Lemma 1 once again.

We shall also need the following auxiliary theorem.

<u>Lemma 2</u>. <u>Let</u> $\Phi_n(s) \in \underset{\sim}{R}$ <u>for</u> $n=0,1,2,\ldots$ <u>and let</u> a_n $(n=0,1,2,\ldots)$ <u>be complex (or real) numbers</u>. <u>If</u>

(8) $$\sum_{n=0}^{\infty} |a_n| \, ||\Phi_n|| < \infty ,$$

<u>then</u>

$$\Psi(s) = \sum_{n=0}^{\infty} a_n \Phi_n(s) \;\varepsilon\; \mathbf{R}\,, \tag{9}$$

$$\|\Psi\| \leqq \sum_{n=0}^{\infty} |a_n|\, \|\Phi_n\|\,, \tag{10}$$

<u>and</u>

$$\mathbf{T}\{\Psi(s)\} = \sum_{n=0}^{\infty} a_n \mathbf{T}\{\Phi_n(s)\}\,. \tag{11}$$

<u>Proof</u>. If we refer to the facts that $\mathbf{R}$ is complete and $\mathbf{T}$ is continuous, then Lemma 2 follows immediately. However, we are not making use of the completeness of $\mathbf{R}$ and therefore a separate proof is required.

For $n=0,1,2,\ldots$ let $\Phi_n(s) = \mathbf{E}\{\zeta_n e^{-s\eta_n}\}$ where $\mathbf{E}\{|\zeta_n|\} \leqq \omega \|\Phi_n\|$. Let ν be a discrete random variable which is independent of the sequence (ζ_n, η_n) $(n=0,1,2,\ldots)$ and which takes on nonnegative integral values with some probabilities $\mathbf{P}\{\nu = n\} = p_n > 0$ for $n = 0,1,2,\ldots$. For example, we may choose $p_n = 1/(n+1)(n+2)$ for $n = 0,1,2,\ldots$. Define $\zeta = a_\nu \zeta_\nu / p_\nu$ and $\eta = \eta_\nu$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\} = \sum_{n=0}^{\infty} \mathbf{P}\{\nu = n\} \frac{a_n}{p_n} \mathbf{E}\{\zeta_n e^{-s\eta_n}\} = \sum_{n=0}^{\infty} a_n \Phi_n(s) \tag{12}$$

and

$$\mathbf{E}\{|\zeta|\} = \sum_{n=0}^{\infty} \mathbf{P}\{\nu = n\} \frac{|a_n|}{p_n} \mathbf{E}\{|\zeta_n|\} \leqq \omega \sum_{n=0}^{\infty} |a_n|\, \|\Phi_n\| < \infty\,. \tag{13}$$

Accordingly, $\Psi(s) = \mathbf{E}\{\zeta e^{-s\eta}\}$ and $\Psi(s) \;\varepsilon\; \mathbf{R}$. The inequality (13) implies that (10) holds. Now we have

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and ω is an arbitrary positive number greater than 1 .

$$(14)\qquad \underset{\sim}{T}\{\Psi(s)\} = \underset{\sim}{E}\{\zeta e^{-s\eta^{+}}\} = \sum_{n=0}^{\infty} P\{\nu=n\}\frac{a_n}{p_n}\underset{\sim}{E}\{\zeta_n e^{-s\eta_n^{+}}\} = \sum_{n=0}^{\infty} a_n \underset{\sim}{T}\{\Phi_n(s)\}$$

which is in agreement with (11). This completes the proof of Lemma 2.

In particular, it follows from Lemma 2 that if $\Phi(s) \in \underset{\sim}{R}$, then $e^{\rho\Phi(s)} \in \underset{\sim}{R}$ for any ρ and

$$(15)\qquad \underset{\sim}{T}\{e^{\rho\Phi(s)}\} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\underset{\sim}{T}\{[\Phi(s)]^n\} ,$$

furthermore $[1-\rho\Phi(s)]^{-1} \in \underset{\sim}{R}$ and $\log[1-\rho\Phi(s)] \in \underset{\sim}{R}$, whenever $|\rho|\,\|\Phi\| < 1$ and

$$(16)\qquad \underset{\sim}{T}\{[1-\rho\Phi(s)]^{-1}\} = \sum_{n=1}^{\infty} \rho^n\, \underset{\sim}{T}\{[\Phi(s)]^n\}$$

and

$$(17)\qquad \underset{\sim}{T}\{\log[1-\rho\Phi(s)]\} = -\sum_{n=1}^{\infty} \frac{\rho^n}{n}\underset{\sim}{T}\{[\Phi(s)]^n\}$$

for $|\rho|\,\|\Phi\| < 1$. The function $\log[1-\rho\Phi(s)]$ is defined by

$$(18)\qquad \log[1-\rho\Phi(s)] = -\sum_{n=1}^{\infty}\frac{\rho^n}{n}[\Phi(s)]^n$$

for $|\rho\,\Phi(s)| < 1$.

4. A Recurrence Relation. Many problems in the theory of probability and stochastic processes can be reduced to the problem of finding a sequence of functions $\Gamma_n(s)$ $(n=1,2,\ldots)$ defined for $\mathrm{Re}(s) \geq 0$ by the recurrence relation

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{1}$$

where $n=1,2,\ldots$, $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \in \underset{\sim}{R}$ and $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$. Obviously $\Gamma_n(s) \in \underset{\sim}{R}$ for all $n=1,2,\ldots$, and $\Gamma_n(s)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geq 0$.

Theorem 1. *Let us suppose that* $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \in \underset{\sim}{R}$ *and* $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$. *Define* $\Gamma_n(s)$ *for* $n=1,2,\ldots$ *by the following recurrence relation*

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} . \tag{2}$$

If $|\rho|\,\|\gamma\| < 1$, *then*

$$\sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}}\underset{\sim}{T}\{\Gamma_0(s)e^{-\log[1-\rho\gamma(s)]+\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}}\} \tag{3}$$

for $\mathrm{Re}(s) \geq 0$. *If, in particular,* $\Gamma_0(s) \equiv 1$, *then* (3) *reduces to*

$$\sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}} = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{T}\{[\gamma(s)]^n\}} \tag{4}$$

where $|\rho|\,\|\gamma\| < 1$.

Proof. Let us denote the right hand side of (3) by $U(s,\rho)$. Obviously, $U(s,\rho) \in \underset{\sim}{R}$ and $\underset{\sim}{T}\{U(s,\rho)\} = U(s,\rho)$. Now we shall show that

$U(s,\rho)$ satisfies the following equation

(5) $$U(s,\rho) - \rho \underset{\sim}{T}\{\gamma(s)U(s,\rho)\} = \Gamma_0(s) .$$

This can be proved as follows. Let

(6) $$h(s,\rho) = e^{\log[1-\rho\gamma(s)] - \underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}}$$

for $\operatorname{Re}(s) = 0$, and $|\rho|\,\|\gamma\| < 1$. Evidently $h(s,\rho) \in \underset{\sim}{R}$, $1/h(s,\rho) \in \underset{\sim}{R}$ and $\Gamma_0(s)/h(s,\rho) \in \underset{\sim}{R}$. We can see immediately that

(7) $$\underset{\sim}{T}\{h(s,\rho)\} = 1$$

and

(8) $$\underset{\sim}{T}\left\{\frac{\Gamma_0(s)}{h(s,\rho)} - \underset{\sim}{T}\frac{\Gamma_0(s)}{h(s,\rho)}\right\} = 0 .$$

By Lemma 3.1 it follows from (7) and (8) that

(9) $$\underset{\sim}{T}\left\{h(s,\rho)\left[\frac{\Gamma_0(s)}{h(s,\rho)} - \underset{\sim}{T}\frac{\Gamma_0(s)}{h(s,\rho)}\right]\right\} = 0 ,$$

that is,

(10) $$\underset{\sim}{T}\{[1-\rho\gamma(s)]U(s,\rho)\} = \Gamma_0(s)$$

whence (5) follows.

Let us expand $U(s,\rho)$ in a power series as follows

(11) $$U(s,\rho) = \sum_{n=0}^{\infty} U_n(s)\rho^n .$$

This series is convergent if $|\rho|\,\|\gamma\| < 1$ and evidently $U_n(s) \in \underset{\sim}{R}$ for $n=0,1,2,\ldots$. If we put (11) into (5) and form the coefficient of ρ^n, then we obtain that $U_0(s) = \Gamma_0(s)$ and

$$U_n(s) = \underset{\sim}{T}\{\gamma(s)U_{n-1}(s)\} \tag{12}$$

for $n=1,2,\ldots$. Accordingly, the sequence $\{U_n(s)\}$ satisfies the same recurrence relation, and the same initial condition as the sequence $\{\Gamma_n(s)\}$. Thus $U_n(s) = \Gamma_n(s)$ for $n=0,1,2,\ldots$ which was to be proved.

In the particular case of $\Gamma_0(s) \equiv 1$ the proof of (4) is much simpler. If now $U(s,\rho)$ denotes the right-hand side of (4), then it follows immediately that

$$\underset{\sim}{T}\{[1-\rho\gamma(s)]U(s,\rho)\} = 1 \tag{13}$$

and therefore (5) holds with $\Gamma_0(s) \equiv 1$. The remainder of the proof follows as in the general case.

The usefulness of formulas (3) and (4) depends on the applicability of the transformation $\underset{\sim}{T}$. In the following two sections we shall give a method for finding $\underset{\sim}{T}\{\Phi(s)\}$ for $\Phi(s) \in \underset{\sim}{R}$, and, in particular, for finding $\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}$ if $\gamma(s) \in \underset{\sim}{R}$ and $|\rho|\,\|\gamma\| < 1$. First, however, we shall give some alternative proofs for (3) and (4).

<u>Theorem 2</u>. <u>If</u> $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \equiv 1$ <u>and</u>

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{14}$$

<u>for</u> $n=1,2,\ldots$, <u>then</u>

$$(15) \qquad \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \exp\{ \sum_{k=1}^{\infty} \frac{\rho^k}{k} \gamma_k^+(s)\}$$

<u>for</u> $\mathrm{Re}(s) \geqq 0$ <u>and</u> $|\rho| \, \|\gamma\| < 1$ <u>where</u> $\gamma_k(s) = [\gamma(s)]^k$ <u>and</u>

$$(16) \qquad \gamma_k^+(s) = \underset{\sim}{T}\{[\gamma(s)]^k\} \quad .$$

<u>for</u> $k=1,2,\ldots$.

<u>Proof</u>. Starting from $\Gamma_0(s)$ we can obtain $\Gamma_n(s)$ for every $n=1,2,\ldots$ by the recurrence formula (14). We observe, however, that $\Gamma_n(s)$ $(n=1,2,\ldots)$ can also be obtained by the following recurrence relation

$$(17) \qquad \Gamma_n(s) = \frac{1}{n} \sum_{k=1}^{n} \gamma_k^+(s)\Gamma_{n-k}(s)$$

which holds if $\mathrm{Re}(s) \geqq 0$ and $n=1,2,\ldots$.

We shall prove by mathematical induction that (17) holds for $n=1,2,\ldots$. If $n=1$, then (17) reduces to $\Gamma_1(s) = \gamma_1^+(s)$ which is obviously true. Let us assume that (17) is true for $1,2,\ldots,n$. We shall prove that it is true for $n+1$ too. Hence it follows that (17) is true for every n $(n=1,2,\ldots)$. If (17) holds for n $(n=1,2,\ldots)$, then by (14) it follows that

$$(18) \qquad \Gamma_{n+1}(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_n(s)\} = \frac{1}{n} \sum_{k=1}^{n} \underset{\sim}{T}\{\gamma(s)\gamma_k^+(s)\Gamma_{n-k}(s)\}$$

for $\mathrm{Re}(s) \geqq 0$. If we apply Lemma 3.1 to $\Phi_1(s) = \gamma(s)\Gamma_{n-k}(s)$ and

$\Phi_2(s) = \gamma_k(s)$, then we obtain that

(19) $$\underset{\sim}{T}\{\gamma(s)\overset{+}{\gamma}_k(s)\Gamma_{n-k}(s)\} = \underset{\sim}{T}\{\gamma_{k+1}(s)\Gamma_{n-k}(s)\} + $$
$$+ \overset{+}{\gamma}_k(s)\Gamma_{n-k+1}(s) - \underset{\sim}{T}\{\gamma_k(s)\Gamma_{n-k+1}(s)\}$$

for $k=1,2,\ldots,n$.

If we put (19) into (18), then we obtain that

(20) $$\Gamma_{n+1}(s) = \frac{1}{n}\sum_{k=1}^{n+1} \overset{+}{\gamma}_k(s)\Gamma_{n-k+1}(s) - \frac{1}{n}\Gamma_{n+1}(s) ,$$

that is,

(21) $$\Gamma_{n+1}(s) = \frac{1}{n+1}\sum_{k=1}^{n+1} \overset{+}{\gamma}_k(s)\Gamma_{n-k+1}(s)$$

for $\mathrm{Re}(s) \geqq 0$. Accordingly, (17) is true if n is replaced by $n+1$. Thus we can conclude that (17) is true for every $n=1,2,\ldots$.

If we introduce the generating function

(22) $$U(s,\rho) = \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho|\,\|\gamma\| < 1$, then by (17) we obtain that

(23) $$\frac{\partial U(s,\rho)}{\partial \rho} = U(s,\rho)\sum_{k=1}^{\infty} \overset{+}{\gamma}_k(s)\rho^{k-1} .$$

Since $U(s,0) = 1$, it follows that

$$\log U(s,\rho) = \sum_{k=1}^{\infty} \frac{\rho^k}{k} \gamma_k^+(s) \tag{24}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| \, \|\gamma\| < 1$. This completes the proof of the theorem. Obviously (4) and (15) are equivalent.

We can express $\Gamma_n(s)$ explicitly by $\gamma_1^+(s), \gamma_2^+(s), \ldots, \gamma_n^+(s)$ if we introduce the following polynomials. For $n = 1,2,3,\ldots$ let us define the polynomials

$$Q_n(x_1, x_2, \ldots, x_n) = \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!k_2!\ldots k_n!} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \ldots \left(\frac{x_n}{n}\right)^{k_n} \tag{25}$$

where $k_1, k_2, \ldots, k_n$ are nonnegative integers, and let $Q_0 \equiv 1$.

<u>Theorem 3</u>. <u>If</u> $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \equiv 1$ <u>and</u>

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{26}$$

<u>for</u> $n = 1,2,\ldots,$ <u>then</u>

$$\Gamma_n(s) = Q_n(\gamma_1^+(s), \gamma_2^+(s), \ldots, \gamma_n^+(s)) \tag{27}$$

<u>for</u> $\mathrm{Re}(s) \geqq 0$ <u>and</u> $n = 1,2,\ldots$ <u>where</u> $\gamma_k(s) = [\gamma(s)]^k$ <u>and</u> $\gamma_k^+(s) = \underset{\sim}{T}\{\gamma_k(s)\}$.

<u>Proof</u>. If $x_1, x_2, \ldots, x_n, \ldots$ are complex (or real) numbers for which $|x_n| \leqq a^n$ $(n = 1,2,\ldots)$ where a is a positive real number and $|\rho| a < 1$, then we have

$$(28)\qquad 1+\sum_{n=1}^{\infty} Q_n(x_1,x_2,\ldots,x_n)\rho^n = \exp\{\sum_{k=1}^{\infty}\frac{\rho^k}{k}x_k\} .$$

The proof of (28) is immediate. If we form the coefficient of ρ^n in the power series expansion of the right-hand side of (28), then we obtain $Q_n(x_1,x_2,\ldots,x_n)$ for $n=1,2,\ldots$. If we choose $a = \|\gamma\|$, then the relation (28) shows that Theorem 2 and Theorem 3 are equivalent.

In what follows, however, we shall give a direct proof for Theorem 3.

First, we note that if $|y| \leqq a$, if we multiply (28) by

$$(29)\qquad 1-\rho y = \exp\{-\sum_{k=1}^{\infty}\frac{\rho^k}{k}y^k\} ,$$

and if we form the coefficient of ρ^n, then we obtain the following identity

$$(30)\qquad Q_n(x_1,x_2,\ldots,x_n) - yQ_{n-1}(x_1,x_2,\ldots,x_{n-1}) = Q_n(x_1-y,\ x_2-y^2,\ldots,\ x_n-y^n)$$

for $n=1,2,\ldots$. Here $Q_0 \equiv 1$.

Now let us suppose that $\Gamma_n(s)$ for $n=1,2,\ldots$ is given by (27). Since the right-hand side of (27) is a polynomial of $\gamma_1^+(s), \gamma_2^+(s),\ldots,\gamma_n^+(s)$ and $\underset{\sim}{T}\{\gamma_j^+(s)\} = \gamma_j^+(s)$ for $j=1,2,\ldots,n$, it follows that

$$(31)\qquad \underset{\sim}{T}\{\Gamma_n(s)\} = \Gamma_n(s)$$

for $n=1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$.

On the other hand, by (30) we can write that

$$(32)\qquad \Gamma_n(s)-\gamma(s)\Gamma_{n-1}(s) = Q_n(\gamma_1^+(s)-\gamma_1(s),\ \gamma_2^+(s)-\gamma_2(s),\ldots,\ \gamma_n^+(s)-\gamma_n(s))$$

for $n=1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$. Since the right-hand side of (32) is a polynomial of $\gamma_1^+(s)-\gamma_1(s),\ \gamma_2^+(s)-\gamma_2(s),\ldots,\ \gamma_n^+(s)-\gamma_n(s)$ and $\underset{\sim}{T}\{\gamma_j^+(s)-\gamma_j(s)\} = 0$ for $j=1,2,\ldots,n$, it follows that

$$(33)\qquad \underset{\sim}{T}\{\Gamma_n(s) - \gamma(s)\Gamma_{n-1}(s)\} = 0$$

for $n=1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$. By (31) and (33) we obtain that

$$(34)\qquad \Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\}$$

for $n=1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$ where $\Gamma_0(s) \equiv 1$. This is in agreement with (26) and therefore (27) is indeed correct.

Now we shall give an alternative proof for (3).

Theorem 4. If $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \in \underset{\sim}{R}$, $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$ and

$$(35)\qquad \Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\}$$

for $n=1,2,\ldots$, then we have

$$(36)\qquad \Gamma_n(s) = \sum_{k=0}^{n} Q_{n-k}(s)\underset{\sim}{T}\{\Gamma_0(s)Q_k^*(s)\}$$

for $\mathrm{Re}(s) \geqq 0$ and $n=0,1,2,\ldots$ where

$$(37)\qquad Q_k(s) = Q_k(\gamma_1^+(s),\ \gamma_2^+(s),\ldots,\ \gamma_k^+(s))$$

for $k=1,2,\ldots,n$ and $Q_0(s) \equiv Q_0 \equiv 1$, and

$$(38)\qquad Q_k^*(s) = Q_k(\gamma_1(s)-\gamma_1^+(s),\gamma_2(s)-\gamma_2^+(s),\ldots,\gamma_k(s)-\gamma_k^+(s))$$

for $k=1,2,\ldots,n$, *and* $Q_0^*(s) \equiv Q_0 \equiv 1$. *The polynomial* $Q_k(x_1, x_2,\ldots, x_k)$ *for* $k=1,2,\ldots$ *is defined by* (25).

Proof. Suppose that $\Gamma_n(s)$ is given by (36) for $n=0,1,2,\ldots$. For $n=0$ formula (36) reduces to $\Gamma_0(s) = \Gamma_0(s)$. We shall prove that (35) holds for $n=1,2,\ldots$. Thus it follows that (36) is indeed the correct formula.

By (36)

$$(39)\qquad \mathbf{T}\{\gamma(s)\Gamma_n(s)\} = \sum_{k=0}^{n} \mathbf{T}\{\gamma(s)Q_{n-k}(s)\mathbf{T}\{\Gamma_0(s)Q_k^*(s)\}\} \ .$$

If we apply Lemma 3.1 to the functions $\Phi_1(s) = \gamma(s)Q_{n-k}(s)$ and $\Phi_2(s) = \Gamma_0(s)Q_k^*(s)$, where $k=0,1,\ldots,n$, then we obtain that

$$(40)\qquad \mathbf{T}\{\gamma(s)Q_{n-k}(s)\mathbf{T}\{\Gamma_0(s)Q_k^*(s)\}\} = \mathbf{T}\{\gamma(s)Q_{n-k}(s)\Gamma_0(s)Q_k^*(s)\} + Q_{n-k+1}(s)\mathbf{T}\{\Gamma_0(s)Q_k^*(s)\} - \mathbf{T}\{Q_{n-k+1}(s)\Gamma_0(s)Q_k^*(s)\} \ .$$

If we put (40) into (39) and take into consideration that

$$(41)\qquad \sum_{k=0}^{n} Q_k^*(s)[Q_{n-k+1}(s) - \gamma(s)Q_{n-k}(s)] + Q_{n+1}^*(s) = 0$$

for $n=1,2,\ldots$, then we obtain that

$$(42)\qquad \mathbf{T}\{\gamma(s)\Gamma_n(s)\} = \sum_{k=0}^{n} Q_{n-k+1}(s)\mathbf{T}\{\Gamma_0(s)Q_k^*(s)\} + \mathbf{T}\{\Gamma_0(s)Q_{n+1}^*(s)\}$$

for $n=0,1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$. By (36) the right-hand side of (42) can be written as $\Gamma_{n+1}(s)$. This proves that (35) holds for $n=1,2,\ldots$. It remains to show that (41) is true. If we multiply the left-hand side of (41) by ρ^n where $|\rho|\,\|\gamma\| < 1$ and add for $n=1,2,\ldots,$ then we obtain

$$(43) \qquad \exp\left\{\sum_{k=1}^{\infty} \frac{\rho^k}{k}[\gamma_k(s)-\gamma_k^+(s)] + \sum_{k=1}^{\infty} \frac{\rho^k}{k}[\gamma_k^+(s)-\gamma_k(s)]\right\} - 1 = 0$$

whence (41) follows.

If $\Gamma_0(s) \equiv 1$, then (36) reduces to $\Gamma_n(s) = Q_n(s)$ $(n=0,1,2,\ldots)$ which is in agreement with (27).

If we multiply (36) by ρ^n and add for $n = 0,1,2,\ldots,$ then we obtain (3) for $|\rho|\,\|\gamma\| < 1$.

5. A Representation of $\underset{\sim}{T}$. If we know $\Phi(s) \in \underset{\sim}{R}$ for $\mathrm{Re}(s) = 0$, then $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$ is uniquely determined by $\Phi(s)$ for $\mathrm{Re}(s) \geqq 0$. The function $\Phi^+(s)$ is regular in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geqq 0$. We can obtain $\Phi^+(s)$ explicitly by the following theorem.

Theorem 1. If $\Phi(s) \in \underset{\sim}{R}$, then for $\mathrm{Re}(s) > 0$ we have

$$\Phi^+(s) = \frac{1}{2}\,\Phi(0) + \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\Phi(z)}{z(s-z)}\,dz \tag{1}$$

where L_ε $(\varepsilon > 0)$ the path of integration consists of the imaginary axis from $z = -i\infty$ to $z = -i\varepsilon$ and again from $z = i\varepsilon$ to $z = i\infty$.

Proof. Let C_ε^+ $(\varepsilon > 0)$ be the path which consists of the imaginary axis from $z = -i\infty$ to $z = -i\varepsilon$, the semicircle $\{z : z = \varepsilon e^{i\alpha}, -\frac{\pi}{2} \leqq \alpha \leqq \frac{\pi}{2}\}$ and again the imaginary axis from $z = i\varepsilon$ to $z = i\infty$. Let C_ε^- $(\varepsilon > 0)$ be the path which consists of the imaginary axis from $z = -i\infty$ to $z = -i\varepsilon$, the semicircle $\{z : z = -\varepsilon e^{i\alpha}, -\frac{\pi}{2} \leqq \alpha \leqq \frac{\pi}{2}\}$, and again the imaginary axis from $z = i\varepsilon$ to $z = i\infty$. Let $C_\varepsilon^+(R)$ $(0 < \varepsilon < R)$ be the path taken in the negative (clockwise) sense and containing C_ε^+ from $z = -iR$ to $z = iR$ and the semicircle $\{z : z = Re^{-i\alpha}, -\frac{\pi}{2} \leqq \alpha \leqq \frac{\pi}{2}\}$. Let $C_\varepsilon^-(R)$ $(0 < \varepsilon < R)$ be the path taken in the positive (counter-clockwise) sense and containing C_ε^- from $z = -iR$ to $z = iR$ and the semicircle $\{z : z = -Re^{-i\alpha}, -\frac{\pi}{2} \leqq \alpha \leqq \frac{\pi}{2}\}$.

Since $\Phi^+(z)$ is regular inside $C_\varepsilon^+(R)$ and continuous on the boundary, it follows by Cauchy's integral formula (see e.g. W. F. Osgood [23] p. 112) that

$$\frac{s}{2\pi i} \int_{C_\varepsilon^+(R)} \frac{\Phi^+(z)}{z(s-z)}\, dz = \Phi^+(s) \tag{2}$$

for $0 < \varepsilon < \mathrm{Re}(s)$ and $|s| < R$. If we let $R \to \infty$ in (2), then we obtain that

$$\frac{s}{2\pi i} \int_{C_\varepsilon^+} \frac{\Phi^+(z)}{z(s-z)}\, dz = \Phi^+(s) \tag{3}$$

for $0 < \varepsilon < \mathrm{Re}(s)$. If $\varepsilon \to 0$, then in (3) the integral taken along the semicircle of radius ε tends to $\Phi^+(0)/2 = \Phi(0)/2$ and thus by (3)

$$\lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\Phi^+(z)}{z(s-z)}\, dz + \frac{1}{2}\Phi(0) = \Phi^+(s) \tag{4}$$

for $\mathrm{Re}(s) > 0$.

If we extend the definition of $\Phi(s) - \Phi^+(s)$ for $\mathrm{Re}(s) \leqq 0$ by (3.6), then $\Phi(s) - \Phi^+(s)$ becomes regular in the domain $\mathrm{Re}(s) < 0$, continuous for $\mathrm{Re}(s) \leqq 0$ and $|\Phi(s) - \Phi^+(s)| \leqq 2\|\Phi\|$ for $\mathrm{Re}(s) \leqq 0$. Then by Cauchy's integral theorem (see e.g. W. F. Osgood [23] p. 105) it follows that

$$\frac{s}{2\pi i} \int_{C_\varepsilon^-(R)} \frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\, dz = 0 \tag{5}$$

for $\mathrm{Re}(s) > 0$. If we let $R \to \infty$ in (5), then we obtain that

$$(6) \qquad \frac{s}{2\pi i} \int_{C_\varepsilon^-} \frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\, dz = 0$$

for $\text{Re}(s) > 0$. If $\varepsilon \to 0$, then in (6) the integral taken along the semicircle of radius ε tends to $[\Phi^+(0) - \Phi(0)]/2 = 0$, and thus by (6)

$$(7) \qquad \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\Phi(z)-\Phi^+(z)}{z(s-z)}\, dz = 0$$

for $\text{Re}(s) > 0$.

If we add (4) and (7), then we obtain (1) for $\text{Re}(s) > 0$ which was to be proved. For $\text{Re}(s) = 0$ the function $\Phi^+(s)$ can be obtained by continuity or by an integral representation similar to (1).

We note that if $\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$ exists for some $s = \varepsilon > 0$, that is, if $\underset{\sim}{E}\{|\zeta e^{-\varepsilon\eta}|\} < \infty$, then

$$(8) \qquad \Phi^+(s) = \frac{s}{2\pi i} \int_{C_\varepsilon^+} \frac{\Phi(z)}{z(s-z)}\, dz$$

for $\text{Re}(s) > \varepsilon > 0$. For in this case (6) remains valid if C_ε^- is replaced by C_ε^+ , and hence (8) follows by (3).

If $\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$ exists for some $s = -\varepsilon < 0$, that is, if $\underset{\sim}{E}\{|\zeta e^{\varepsilon\eta}|\} < \infty$, then we have

$$(9) \qquad \Phi^+(s) = \Phi(0) + \frac{s}{2\pi i} \int_{C_\varepsilon^-} \frac{\Phi(z)}{z(s-z)}\, dz$$

for $\text{Re}(s) \geqq 0$. For in this case if we replace C_ε^+ by C_ε^- in (3), then the right-hand side becomes $\Phi^+(s) - \Phi^+(0)$. If we add (6) to this equation, then we obtain (9).

6. <u>The Method of Factorization.</u> If $\gamma(s) \in \underset{\sim}{R}$ and $|\rho|\,\|\gamma\| < 1$, then $\log[1-\rho\gamma(s)] \in \underset{\sim}{R}$ and we can determine $\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}$ by Theorem 5.1. We can use also the expansion

(1) $$\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\} = -\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{T}\{[\gamma(s)]^n\}$$

which is convenient if $\underset{\sim}{T}\{[\gamma(s)]^n\}$ for $n = 1,2,\ldots$ can easily be obtained. In what follows we shall mention another method, namely, the method of factorization.

Let $\rho(s) \in \underset{\sim}{R}$, $|\rho|\,\|\gamma\| < 1$ and suppose that

(2) $$1 - \rho\gamma(s) = \Gamma^{+}(s,\rho)\Gamma^{-}(s,\rho)$$

for $\mathrm{Re}(s) = 0$ where $\Gamma^{+}(s,\rho)$ satisfies the requirements:

A_1 : $\Gamma^{+}(s,\rho)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$,

A_2 : $\Gamma^{+}(s,\rho)$ is continuous and free from zeros in $\mathrm{Re}(s) \geqq 0$,

A_3 : $\lim_{|s|\to\infty} [\log\Gamma^{+}(s,\rho)]/s = 0$ whenever $\mathrm{Re}(s) \geqq 0$,

and $\Gamma^{-}(s,\rho)$ satisfies the following requirements:

B_1 : $\Gamma^{-}(s,\rho)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$,

B_2 : $\Gamma^{-}(s,\rho)$ is continuous and free from zeros in $\mathrm{Re}(s) \leqq 0$,

B_3 : $\lim_{|s|\to\infty} [\log\Gamma^{-}(s,\rho)]/s = 0$ whenever $\mathrm{Re}(s) \leqq 0$.

Such a factorization always exists. For example,

(3) $$\Gamma^{+}(s,\rho) = e^{\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}}$$

for $\mathrm{Re}(s) \geqq 0$ and

(4) $$\Gamma^{-}(s,\rho) = e^{\log[1-\rho\gamma(s)]-\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\}}$$

for $\mathrm{Re}(s) \leqq 0$ satisfy all the requirements. Actually, the above requirements determine $\Gamma^{+}(s,\rho)$ and $\Gamma^{-}(s,\rho)$ up to a multiplicative factor depending only on ρ. This is the content of the next theorem.

<u>Theorem 1.</u> <u>If</u> $\gamma(s) \in \underset{\sim}{R}$, $|\rho|\,\|\gamma\| < 1$ <u>and</u>

(5) $$1 - \rho\gamma(s) = \Gamma^{+}(s,\rho)\Gamma^{-}(s,\rho)$$

<u>for</u> $\mathrm{Re}(s) = 0$ <u>where</u> $\Gamma^{+}(s,\rho)$ <u>and</u> $\Gamma^{-}(s,\rho)$ <u>satisfy the requirements</u> A_1, A_2, A_3 <u>and</u> B_1, B_2, B_3 <u>respectively, then</u>

(6) $$\underset{\sim}{T}\{\log[1-\rho\gamma(s)]\} = \log\Gamma^{+}(s,\rho) + \log\Gamma^{-}(0,\rho)$$

<u>for</u> $\mathrm{Re}(s) \geqq 0$.

<u>Proof.</u> It is sufficient to prove (6) for $\mathrm{Re}(s) > 0$. For $\mathrm{Re}(s) = 0$ (6) follows by continuity. Let us define the paths L_ε, C_ε^{+}, C_ε^{-}, $C_\varepsilon^{+}(R)$, $C_\varepsilon^{-}(R)$ in the same ways as in the proof of Theorem 5.1. By Cauchy's integral formula we can write that

(7) $$\frac{s}{2\pi i}\int_{C_\varepsilon^{+}} \frac{\log\Gamma^{+}(z,\rho)}{z(s-z)}\,dz = \log\Gamma^{+}(s,\rho)$$

for $0 < \varepsilon < \mathrm{Re}(s)$ and by Cauchy's integral theorem we can write that

(8) $$\frac{s}{2\pi i}\int_{C_\varepsilon^{-}} \frac{\log\Gamma^{-}(z,\rho)}{z(s-z)}\,dz = 0$$

for $\mathrm{Re}(s) > 0$. We can prove (7) and (8) in a similar way as (5.3) and (5.6). First we integrate along the paths $C_\varepsilon^{+}(R)$ and $C_\varepsilon^{-}(R)$ in (7) and (8) respectively and then let $R \to \infty$. If $\varepsilon \to 0$ in (7) and (8), then we get

$$\lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\log\Gamma^+(z,\rho)}{z(s-z)}\,dz + \frac{1}{2}\log\Gamma^+(0,\rho) = \log\Gamma^+(s,\rho) \tag{9}$$

and

$$\lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\log\Gamma^-(z,\rho)}{z(s-z)}\,dz' - \frac{1}{2}\log\Gamma^-(0,\rho) = 0 \tag{10}$$

for $\mathrm{Re}(s) > 0$. If we add (9) and (10), then we obtain (6) for $\mathrm{Re}(s) > 0$. This completes the proof of the theorem.

By using Theorem 1 we can express Theorem 4.1 also in the following way.

Theorem 2. *Let us suppose that* $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \in \underset{\sim}{R}$ *and* $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$. *Define* $\Gamma_n(s)$ *for* $n=1,2,\ldots$ *by the following recurrence relation*

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\}. \tag{11}$$

If $|\rho|\,\|\gamma\| < 1$ *and*

$$1 - \rho\gamma(s) = \Gamma^+(s,\rho)\Gamma^-(s,\rho) \tag{12}$$

for $\mathrm{Re}(s) = 0$ *where* $\Gamma^+(s,\rho)$ *and* $\Gamma^-(s,\rho)$ *satisfy the requirements* A_1, A_2, A_3 *and* B_1, B_2, B_3, *then*

$$\sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \frac{1}{\Gamma^+(s,\rho)} \underset{\sim}{T}\left\{\frac{\Gamma_0(s)}{\Gamma^-(s,\rho)}\right\} \tag{13}$$

for $\mathrm{Re}(s) \geq 0$. *If, in particular,* $\Gamma_0(s) \equiv 1$, *then*

$$(14)\qquad \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \frac{1}{\Gamma^+(s,\rho)\Gamma^-(0,\rho)}$$

for $\mathrm{Re}(s) \geq 0$.

Proof. If we put (6) into (4.3) and (4.4), then we obtain (13) and (14) respectively.

We note that by (13) we obtain that

$$(15)\qquad [1-\rho\gamma(s)] \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \Gamma^-(s,\rho)\underset{\sim}{T}\left\{\frac{\Gamma_0(s)}{\Gamma^-(s,\rho)}\right\}$$

for $\mathrm{Re}(s) = 0$ and $|\rho|\ \|\gamma\| < 1$.

By (14) we obtain that if $\Gamma_0(s) \equiv 1$ then

$$(16)\qquad [1-\rho\gamma(0)] \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \frac{\Gamma^+(0,\rho)}{\Gamma^+(s,\rho)}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho|\ \|\gamma\| < 1$ and

$$(17)\qquad [1-\rho\gamma(s)] \sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \frac{\Gamma^-(s,\rho)}{\Gamma^-(0,\rho)}$$

for $\mathrm{Re}(s) = 0$ and $|\rho|\ \|\gamma\| < 1$.

In finding $\Gamma^+(s,\rho)$ and $\Gamma^-(s,\rho)$ we can usually utilize the following theorem of Rouché :

If $f(z)$ and $g(z)$ are regular in a domain D (open connected set), continuous on the closure of D and satisfy $|g(z)| < |f(z)|$ on the boundary of D , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in D .

For the proof of Rouché's theorem we refer to S. Saks and A. Zygmund [32] p. 157.

7. A Subspace $\underset{\sim}{R}_0$. There are several problems in fluctuation theory which can be solved by considering a smaller class of functions than the space $\underset{\sim}{R}$. In this section we shall define a subspace $\underset{\sim}{R}_0$ of the space $\underset{\sim}{R}$ and we shall show that if we restrict ourself to functions belonging to $\underset{\sim}{R}_0$, then the problems discussed in the previous sections can be solved in a simpler way.

Define $\underset{\sim}{R}_0$ as the class of all those functions $\gamma(s)$ defined for $\mathrm{Re}(s) = 0$ on the complex plane which can be represented in the form

$$\gamma(s) = c_1\psi_1(s) + c_2\psi_2(s) + \dots + c_n\psi_n(s) \tag{1}$$

where n is a positive integer, $c_1, c_2, \dots, c_n$ are complex (or real) numbers and $\psi_1(s), \psi_2(s), \dots, \psi_n(s)$ are Laplace-Stieltjes transforms of real random variables, that is,

$$\psi_k(s) = \underset{\sim}{E}\{e^{-s\eta_k}\} \tag{2}$$

for $\mathrm{Re}(s) = 0$ and $k=1,2,\dots,n$ where $\eta_1, \eta_2, \dots, \eta_n$ are real random variables.

If $\gamma(s) \in \underset{\sim}{R}_0$, then $\gamma(s) \in \underset{\sim}{R}$. For if $\gamma(s)$ is given by (1), then $\psi_k(s) \in \underset{\sim}{R}$ for $k=1,2,\dots,n$ and therefore $\gamma(s) \in \underset{\sim}{R}$. Accordingly, $\underset{\sim}{R}_0$ is indeed a subspace of $\underset{\sim}{R}$. We can easily see that $\underset{\sim}{R}_0$ is a linear manifold.

$\underset{\sim}{R}_0$ can also be characterized as that subspace of $\underset{\sim}{R}$ which contains all those functions $\gamma(s)$ defined for $\mathrm{Re}(s) = 0$ on the complex plane which can be represented in the form

$$\gamma(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\} \tag{3}$$

where ζ is a discrete complex random variable with a finite number of possible values and η is a real random variable. We can easily see that this definition of $\underset{\sim}{R}_0$ and the previous one are equivalent. If $\gamma(s)$ is given by (1), then let us define ν as a discrete random variable which is independent of $\eta_1, \eta_2, \ldots, \eta_n$ and for which $\underset{\sim}{P}\{\nu = k\} = 1/n$ for $k=1,2,\ldots,n$. If $\zeta = nc_\nu$ and $\eta = \eta_\nu$, then (1) can be expressed in the form of (3). The converse implication is evident.

If $\gamma(s) \varepsilon \underset{\sim}{R}_0$ and $\gamma(s)$ is given by (1), then let us define the norm of $\gamma(s)$ by

$$\|\gamma\| = \inf_{\gamma}\{|c_1|+|c_2|+..+|c_n|\} \tag{4}$$

where the infimum is taken for all admissible representations of $\gamma(s)$ in the form (1). This definition of $\|\gamma\|$ is in agreement with that of Section 2.

We have $\|\gamma\| \geqq 0$, and $\|\gamma\| = 0$ if and only if $\gamma(s) \equiv 0$. If α is a complex (or real) number and $\gamma(s) \varepsilon \underset{\sim}{R}_0$, then $\alpha\gamma(s) \varepsilon \underset{\sim}{R}_0$ and $\|\alpha\gamma\| = |\alpha|\ \|\gamma\|$. Furthermore, if $\gamma_1(s) \varepsilon \underset{\sim}{R}_0$ and $\gamma_2(s) \varepsilon \underset{\sim}{R}_0$, then $\gamma_1(s) + \gamma_2(s) \varepsilon \underset{\sim}{R}_0$ and $\gamma_1(s)\gamma_2(s) \varepsilon \underset{\sim}{R}_0$ and $\|\gamma_1 + \gamma_2\| \leqq \|\gamma_1\| + \|\gamma_2\|$ and $\|\gamma_1\gamma_2\| \leqq \|\gamma_1\|\ \|\gamma_2\|$.

Let us define the transformation $\underset{\sim}{T}$ in the following way. If $\gamma(s) \varepsilon \underset{\sim}{R}_0$ and $\gamma(s)$ is given by (1), then let

(5) $$\underset{\sim}{T}\{\gamma(s)\} = \gamma^+(s) = c_1\psi_1^+(s) + c_2\psi_2^+(s) + \dots + c_n\psi_n^+(s)$$

for $\mathrm{Re}(s) = 0$ where

(6) $$\psi_k^+(s) = \underset{\sim}{T}\{\psi_k(s)\} = \underset{\sim}{E}\{e^{-s\eta_k^+}\}$$

and $\eta_k^+ = \max(0, \eta_k)$. It can easily be seen that the function $\gamma^+(s)$ is independent of the particular representation (1). It depends solely on $\gamma(s)$. This definition of $\underset{\sim}{T}\{\gamma(s)\}$ is in agreement with that of Section 3. If $\gamma(s) \in \underset{\sim}{R}_0$, then obviously $\gamma^+(s) \in \underset{\sim}{R}_0$.

If α is a complex (or real) number and $\gamma(s) \in \underset{\sim}{R}_0$, then $\underset{\sim}{T}\{\alpha\gamma(s)\} = \alpha\underset{\sim}{T}\{\gamma(s)\}$. If $\gamma_1(s) \in \underset{\sim}{R}_0$ and $\gamma_2(s) \in \underset{\sim}{R}_0$ then $\underset{\sim}{T}\{\gamma_1(s)+ \gamma_2(s)\} = \underset{\sim}{T}\{\gamma_1(s)\} + \underset{\sim}{T}\{\gamma_2(s)\}$ which follows immediately from the definition (5).

<u>Lemma 1.</u> <u>If</u> $\gamma_1(s) \in \underset{\sim}{R}_0$ <u>and</u> $\gamma_2(s) \in \underset{\sim}{R}_0$, <u>then we have</u>

(7) $$\underset{\sim}{T}\{\gamma_1(s)\underset{\sim}{T}\gamma_2(s)\} + \underset{\sim}{T}\{\gamma_2(s)\underset{\sim}{T}\gamma_1(s)\} = $$
$$= \underset{\sim}{T}\{\gamma_1(s)\gamma_2(s)\} + (\underset{\sim}{T}\gamma_1(s))(\underset{\sim}{T}\gamma_2(s)) \ .$$

<u>Proof.</u> We can easily see that for any two real random variables η_1 and η_2 we have

(8) $$\underset{\sim}{P}\{\max(0, \eta_1, \eta_1+ \eta_2) \leqq x\} + \underset{\sim}{P}\{\max(0, \eta_2, \eta_1+ \eta_2) \leqq x\} =$$
$$= \underset{\sim}{P}\{\max(0, \eta_1+ \eta_2) \leqq x\} + \underset{\sim}{P}\{\max(0, \eta_1)+ \max(0, \eta_2) \leqq x\}$$

for all x . If we assume that η_1 and η_2 are independent random variables for which $\underset{\sim}{E}\{e^{-s\eta_1}\} = \gamma_1(s)$ and $\underset{\sim}{E}\{e^{-s\eta_2}\} = \gamma_2(s)$ whenever $\mathrm{Re}(s) = 0$, and if we form the Laplace-Stieltjes transform of (8), then we obtain (7) in this particular case. The general case can immediately be reduced to this particular case by using the representation (1).

Finally, we note that if $\gamma(s) \in \underset{\sim}{R}$, then $\gamma^+(s)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$, continuous for $\mathrm{Re}(s) \geq 0$ and $|\gamma(s)| \leq \|\gamma\|$ for $\mathrm{Re}(s) \geq 0$.

Now let us consider the recurrence relation studied in Section 4 in the particular case when $\gamma(s) \in \underset{\sim}{R}_0$ and $\Gamma_0(s) \in \underset{\sim}{R}_0$ and $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$. If we define $\Gamma_n(s)$ for $n=1,2,\ldots$ by the recurrence relation

(9) $$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} ,$$

then $\Gamma_n(s) \in \underset{\sim}{R}_0$ for $n=1,2,\ldots$. First, we shall consider the particular case when $\Gamma_0(s) \equiv 1$, then the general case when $\Gamma_0(s) \in \underset{\sim}{R}_0$ and $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$.

<u>Theorem 1.</u> <u>If</u> $\gamma(s) \in \underset{\sim}{R}_0$, $\Gamma_0(s) \equiv 1$, <u>and</u>

(10) $$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\}$$

<u>for</u> $n=1,2,\ldots$, <u>then</u>

(11) $$\sum_{n=0}^{\infty} \Gamma_n(s)\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} \gamma_k^+(s)\}$$

<u>for</u> $\mathrm{Re}(s) \geq 0$ <u>and</u> $|\rho|\,\|\gamma\| < 1$ <u>where</u> $\gamma_k(s) = [\gamma(s)]^k$ <u>and</u>

(12) $$\gamma_k^+(s) = \underset{\sim}{T}\{\gamma_k(s)\} = \underset{\sim}{T}\{[\gamma(s)]^k\}$$

<u>for</u> $k=1,2,\ldots$.

Proof. The proof follows along the same lines as the proof of Theorem 4.2 . First, by using Lemma 1 we can prove by mathematical induction that

$$\Gamma_n(s) = \frac{1}{n}\sum_{k=1}^{n} \gamma_k^+(s)\Gamma_{n-k}(s) \tag{13}$$

for $\operatorname{Re}(s) \geq 0$ and $n=1,2,\ldots$. If we introduce the generating function of the sequence $\{\Gamma_n(s)\}$, then we can easily obtain (11) from (13).

Theorem 2. *If* $\gamma(s) \in \underset{\sim}{R}_0$, $\Gamma_0(s) \equiv 1$ *and*

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{14}$$

for $n=1,2,\ldots,$ *then*

$$\Gamma_n(s) = Q_n(\gamma_1^+(s),\ \gamma_2^+(s),\ldots,\ \gamma_n^+(s)) \tag{15}$$

for $\operatorname{Re}(s) \geq 0$ *and* $n=1,2,\ldots$ *and* $\Gamma_0(s) \equiv Q_0 \equiv 1$. *The polynomial* $Q_n(x_1,\ x_2,\ \ldots,\ x_n)$ *is defined by* (4.21).

Proof. The proof follows exactly along the same lines as the proof of Theorem 4.3 .

Theorem 3. *If* $\gamma(s) \in \underset{\sim}{R}_0$, $\Gamma_0(s) \in \underset{\sim}{R}_0$, $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$ *and*

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\} \tag{16}$$

for $n=1,2,\ldots,$ *then we have*

$$\Gamma_n(s) = \sum_{k=0}^{n} Q_{n-k}(s)\underset{\sim}{T}\{\Gamma_0(s)Q_k^*(s)\} \tag{17}$$

for $\operatorname{Re}(s) \geq 0$ and $n = 0,1,2,\ldots$ *where*

$$(18)\qquad Q_k(s) = Q_k(\gamma_1^+(s), \gamma_2^+(s), \ldots, \gamma_k^+(s))$$

for $k = 1,2,\ldots,n$, *and* $Q_0(s) \equiv Q_0 \equiv 1$, *and*

$$(19)\qquad Q_k^*(s) = Q_k(\gamma_1(s) - \gamma_1^+(s), \gamma_2(s) - \gamma_2^+(s), \ldots, \gamma_k(s) - \gamma_k^+(s))$$

for $k = 1,2,\ldots,n$, *and* $Q_0^*(s) \equiv Q_0 \equiv 1$. *The polynomial* $Q_k(x_1, x_2, \ldots, x_k)$ *for* $k = 1,2,\ldots$ *is defined by* (4.21).

Proof. The proof follows exactly along the same lines as the proof of Theorem 4.4 .

If $\Gamma_0(s) \equiv 1$, then (17) reduces to $\Gamma_n(s) = Q_n(s)$ $(n = 0,1,2,\ldots)$ which is in agreement with (15).

If we restrict ourself to the consideration of the class R_0 only, then from (15) and (17) we cannot deduce compact formulas analogous to (4.4) and (4.3) . For if $\gamma(s) \in \underset{\sim}{R}_0$ and $|\rho|\,\|\gamma\| < 1$, then it does not follow in general that $\log[1-\rho\gamma(s)] \in \underset{\sim}{R}_0$.

8. A Space $\underset{\sim}{A}$. There are many discrete type problems in fluctuation theory whose solutions do not require the use of the whole space $\underset{\sim}{R}$ but only a particular subspace of $\underset{\sim}{R}$. This subspace contains all those functions $\Phi(s)$ defined for $\operatorname{Re}(s) = 0$ on the complex plane which can be represented in the form

$$\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\} \tag{1}$$

where ζ is a complex (or real) random variable for which $\underset{\sim}{E}\{|\zeta|\} < \infty$ and η is a discrete real random variable taking on integral values only. This subspace of $\underset{\sim}{R}$ has exactly the same properties as $\underset{\sim}{R}$ and all those results which we deduced for $\underset{\sim}{R}$, remain valid for this subspace too. However, it will be more convenient to introduce a new variable in $\Phi(s)$ and replace $\Phi(s)$ defined for $\operatorname{Re}(s) = 0$ by

$$a(s) = \underset{\sim}{E}\{\zeta s^{\eta}\} \tag{2}$$

defined for $|s| = 1$. Thus we shall replace the mentioned subspace of $\underset{\sim}{R}$ by an isomorphic space $\underset{\sim}{A}$. For the space $\underset{\sim}{A}$ we shall prove analogous theorems as we obtained for $\underset{\sim}{R}$.

Let us denote by $\underset{\sim}{A}$ the space of all those functions $a(s)$ which are defined for $|s| = 1$ on the complex plane and which can be represented in the form

$$a(s) = \sum_{k=-\infty}^{\infty} a_k s^k \tag{3}$$

where a_k $(k = 0, \pm 1, \pm 2, \ldots)$ are complex (or real) numbers satisfying the requirement

$$\sum_{k=-\infty}^{\infty} |a_k| < \infty \, . \tag{4}$$

Let us define the norm of $a(s)$ by

$$\|a\| = \sum_{k=-\infty}^{\infty} |a_k| \, . \tag{5}$$

We have $\|a\| \geqq 0$, and $\|a\| = 0$ if and only if $a(s) = 0$. If α is a complex (or real) number and $a(s) \in \underset{\sim}{A}$, then $\alpha a(s) \in \underset{\sim}{A}$ and $\|\alpha a\| = |\alpha| \, \|a\|$. Furthermore, if $a_1(s) \in \underset{\sim}{A}$ and $a_2(s) \in \underset{\sim}{A}$, then $a_1(s) + a_2(s) \in \underset{\sim}{A}$ and $\|a_1 + a_2\| \leqq \|a_1\| + \|a_2\|$. Accordingly, $\underset{\sim}{A}$ is a normed linear space. In what follows we shall not make use of the completeness of $\underset{\sim}{A}$. However, we can easily prove that $\underset{\sim}{A}$ is complete, and hence it follows that $\underset{\sim}{A}$ is a Banach space. (See Problem 13.2.)

Next we observe that if $a_1(s) \in \underset{\sim}{A}$ and $a_2(s) \in \underset{\sim}{A}$, then $a_1(s)a_2(s) \in \underset{\sim}{A}$ and $\|a_1 a_2\| \leqq \|a_1\| \, \|a_2\|$. Accordingly, $\underset{\sim}{A}$ can be characterized as a commutative Banach algebra.

Finally, we note that the space $\underset{\sim}{A}$ can be defined in the following equivalent way. The space $\underset{\sim}{A}$ contains all those functions $a(s)$ which are defined for $|s| = 1$ on the complex plane and which can be represented in the following form

$$a(s) = \underset{\sim}{E}\{\zeta s^{\eta}\} \tag{6}$$

where ζ is a complex (or real) random variable for which $\underset{\sim}{E}\{|\zeta|\} < \infty$ and η is a discrete random variable taking on integral values only. It follows from (6) that $|a(s)| \leqq \underset{\sim}{E}\{|\zeta|\}$ for $|s| = 1$.

If $a(s)$ is given by (6) for $|s| = 1$, then evidently $a(s) \varepsilon \underset{\sim}{A}$ and $\|a\| \leqq \underset{\sim}{E}\{|\zeta|\}$. Conversely, if $a(s) \varepsilon \underset{\sim}{A}$ and $a(s)$ is given by (3), then $a(s)$ can also be expressed in the form (6). To see this let η be a discrete random variable taking on integral values only with some probabilities $\underset{\sim}{P}\{\eta = k\} = p_k > 0$ for all $k = 0, \pm 1, \pm 2, \ldots$. Define $\zeta = a_k/p_k$ if $\eta = k$. In this case $a(s)$ is given by (6) for $|s| = 1$ and $\|a\| = \underset{\sim}{E}\{|\zeta|\}$.

We note that for $|s| = 1$ the function $a(s)$ is uniquely determined by the joint distribution of ζ and η. However, there are infinitely many possible distributions which yield the same $a(s)$.

By using the representation (6) we can define the norm of $a(s)$ by

$$\|a\| = \inf_{\zeta} \underset{\sim}{E}\{|\zeta|\} \tag{7}$$

where the infimum is taken for all admissible ζ, that is, for all those ζ for which (6) holds. Obviously, $|a(s)| \leqq \|a\|$ for $|s| = 1$.

9. <u>A Linear Transformation $\underset{\sim}{\Pi}$</u>. Let us define a transformation $\underset{\sim}{\Pi}$ in the following way. If $a(s) \in \underset{\sim}{A}$ and $a(s)$ is given by (8.3), then let

$$\underset{\sim}{\Pi}\{a(s)\} = a^{+}(s) \tag{1}$$

for $|s| = 1$ where

$$a^{+}(s) = \sum_{k=-\infty}^{0} a_k + \sum_{k=1}^{\infty} a_k s^k . \tag{2}$$

If $a(s)$ is given by (6), then

$$a^{+}(s) = \underset{\sim}{E}\{\zeta s^{\eta^{+}}\} \tag{3}$$

for $|s| = 1$ where $\eta^{+} = \max(0, \eta)$. It can easily be seen that $a^{+}(s)$ is independent of the particular representation of $a(s)$. It depends solely on $a(s)$.

If $a(s) \in \underset{\sim}{A}$, then obviously $a^{+}(s) \in \underset{\sim}{A}$. We observe that $a^{+}(s)$ is a regular function of s in the domain $|s| < 1$ and continuous for $|s| \leq 1$. Furthermore, $|a^{+}(s)| \leq \|a\|$ for $|s| \leq 1$. We notice that $a(s) - a^{+}(s) \in \underset{\sim}{A}$ and

$$a(s) - a^{+}(s) = \sum_{k=-\infty}^{0} a_k (s^k - 1) \tag{4}$$

is a regular function of s in the domain $|s| > 1$, and continuous for $|s| \geq 1$. Furthermore, $|a(s) - a^{+}(s)| \leq 2\|a\|$ for $|s| \geq 1$.

If α is a complex (or real) number and $a(s) \in \underset{\sim}{A}$, then $\underset{\sim}{\Pi}\{\alpha\, a(s)\} = \alpha \underset{\sim}{\Pi}\{a(s)\}$. If $a_1(s) \in \underset{\sim}{A}$ and $a_2(s) \in \underset{\sim}{A}$ then $\underset{\sim}{\Pi}\{a_1(s) + a_2(s)\} = \underset{\sim}{\Pi}\{a_1(s)\} + \underset{\sim}{\Pi}\{a_2(s)\}$. Obviously $\|\underset{\sim}{\Pi}\| = 1$. ($\|\underset{\sim}{\Pi}\| = \sup\{\|\underset{\sim}{\Pi} a\| : a \in \underset{\sim}{A}$ and $\|a\| \leq 1\}$.) Accordingly, $\underset{\sim}{\Pi}$ is a bounded linear transformation. Since

$\underset{\sim}{\Pi}^2 = \underset{\sim}{\Pi}$, therefore $\underset{\sim}{\Pi}$ is a projection.

The following remarks are obvious. Let $a_1(s) \varepsilon \underset{\sim}{A}$ and $a_2(s) \varepsilon \underset{\sim}{A}$. If $\underset{\sim}{\Pi}\{a_1(s)\} = a_1(s)$ and $\underset{\sim}{\Pi}\{a_2(s)\} = a_2(s)$, then $\underset{\sim}{\Pi}\{a_1(s)a_2(s)\} = a_1(s)a_2(s)$. If $\underset{\sim}{\Pi}\{a_1(s)\} = c_1$ and $\underset{\sim}{\Pi}\{a_2(s)\} = c_2$ where c_1 and c_2 are complex (or real) constants, then $\underset{\sim}{\Pi}\{a_1(s)a_2(s)\} = c_1c_2$.

<u>Lemma 1.</u> <u>If</u> $a_1(s) \varepsilon \underset{\sim}{A}$ <u>and</u> $a_2(s) \varepsilon \underset{\sim}{A}$, <u>then</u>

$$\underset{\sim}{\Pi}\{a_1(s)\underset{\sim}{\Pi}a_2(s)\} + \underset{\sim}{\Pi}\{a_2(s)\underset{\sim}{\Pi}a_1(s)\} = \tag{5}$$

$$= \underset{\sim}{\Pi}\{a_1(s)a_2(s)\} + (\underset{\sim}{\Pi}a_1(s))(\underset{\sim}{\Pi}a_2(s)) .$$

<u>Proof.</u> Let $\overset{*}{a}_1(s) = a_1(s) - \overset{+}{a}_1(s)$ and $\overset{*}{a}_2(s) = a_2(s) - \overset{+}{a}_2(s)$. We can express (5) in the following equivalent form

$$\underset{\sim}{\Pi}\{\overset{*}{a}_1(s)\overset{*}{a}_2(s)\} = 0 . \tag{6}$$

This is however true, because $\underset{\sim}{\Pi}\{\overset{*}{a}_1(s)\} = 0$ and $\underset{\sim}{\Pi}\{\overset{*}{a}_2(s)\} = 0$.

We shall also need the following auxiliary theorem.

<u>Lemma 2.</u> <u>Let</u> $a_n(s) \varepsilon \underset{\sim}{A}$ <u>for</u> $n = 0,1,2,\ldots$ <u>and let</u> c_n $(n = 0,1,2,\ldots)$ <u>be complex (or real) numbers.</u> <u>If</u>

$$\sum_{n=0}^{\infty} |c_n| \|a_n\| < \infty, \tag{7}$$

<u>then</u>

$$a(s) = \sum_{n=0}^{\infty} c_n a_n(s) \varepsilon \underset{\sim}{A} , \tag{8}$$

$$\|a\| \leq \sum_{n=0}^{\infty} |c_n| \, \|a_n\| \tag{9}$$

<u>and</u>

$$\Pi\{a(s)\} = \sum_{n=0}^{\infty} c_n \Pi\{a_n(s)\} . \tag{10}$$

Proof. If we would refer to the fact that A is complete, then Lemma 2 would follow immediately. However, we are not making use of the completeness of A, and therefore a separate proof is required. In proving (8), (9) and (10) we shall use the representation (8.6). Let

$$a_n(s) = E\{\zeta_n s^{\eta_n}\} \tag{11}$$

for $|s| = 1$ and $n = 0,1,2,\ldots$ where $E\{|\zeta_n|\} \leq \omega \|a_n\|$.$^{\wedge}$ Let ν be a discrete random variable which is independent of the sequence (ζ_n, η_n) ($n = 0,1,2,\ldots$) and which takes on only nonnegative integers with probabilities $P\{\nu = n\} = p_n > 0$ for $n = 0,1,2,\ldots$. Define $\zeta = c_\nu \zeta_\nu / p_\nu$ and $\eta = \eta_\nu$. Then

$$E\{\zeta s^{\eta}\} = \sum_{n=0}^{\infty} P\{\nu = n\} \frac{c_n}{p_n} E\{\zeta_n s^{\eta_n}\} = \sum_{n=0}^{\infty} c_n a_n(s) \tag{12}$$

and

$$E\{|\zeta|\} = \sum_{n=0}^{\infty} P\{\nu = n\} \frac{|c_n|}{p_n} E\{|\zeta_n|\} \leq \omega \sum_{n=0}^{\infty} |c_n| \, \|a_n\| < \infty \tag{13}$$

Accordingly, we have $a(s) = E\{\zeta s^{\eta}\}$ and $a(s) \in A$. The inequality (13) implies (9). Furthermore, we have

$$\Pi\{a(s)\} = E\{\zeta s^{\eta^+}\} = \sum_{n=0}^{\infty} P\{\nu=n\} \frac{c_n}{p_n} E\{\zeta_n s^{\eta_n^+}\} = \sum_{n=0}^{\infty} c_n \Pi\{a_n(s)\} \tag{14}$$

which is in agreement with (10). This completes the proof of Lemma 2.

In particular, it follows from Lemma 2 that if $a(s) \in A$, then $e^{\rho a(s)} \in A$ for any ρ and

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$^{\wedge}$ and ω is an arbitrary positive number greater than 1.

$$\underset{\sim}{\Pi}\{e^{\rho a(s)}\} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \underset{\sim}{\Pi}\{[a(s)]^n\} , \tag{15}$$

furthermore $[1-\rho a(s)]^{-1} \in \underset{\sim}{A}$ and $\log [1-\rho a(s)] \in \underset{\sim}{A}$ whenever $|\rho| \, \|a\| < 1$ and

$$\underset{\sim}{\Pi}\{[1-\rho a(s)]^{-1}\} = \sum_{n=1}^{\infty} \rho^n \underset{\sim}{\Pi}\{[a(s)]^n\} \tag{16}$$

and

$$\underset{\sim}{\Pi}\{\log[1-\rho a(s)]\} = - \sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{\Pi}\{[a(s)]^n\} \tag{17}$$

for $|\rho| \, \|a\| < 1$.

10. A Recurrence Relation. Many problems in the theory of probability and stochastic processes can be reduced to the problem of finding a sequence of functions $g_n(s)$ $(n = 1,2,\ldots)$ defined for $|s| = 1$ by the recurrence relation

$$g_n(s) = \underset{\sim}{\Pi}\{\gamma(s)g_{n-1}(s)\} \tag{1}$$

where $n = 1,2,\ldots,$ $\gamma(s) \varepsilon \underset{\sim}{A}$, $g_0(s) \varepsilon \underset{\sim}{A}$ and $\underset{\sim}{\Pi}\{g_0(s)\} = g_0(s)$. Obviously $g_n(s) \varepsilon \underset{\sim}{A}$ for all $n = 1,2,\ldots$ and $g_n(s)$ is a regular function of s in the domain $|s| < 1$ and continuous for $|s| \leqq 1$.

Theorem 1. Let us suppose that $\gamma(s) \varepsilon \underset{\sim}{A}$, $g_0(s) \varepsilon \underset{\sim}{A}$ *and* $\underset{\sim}{\Pi}\{g_0(s)\} = g_0(s)$. *Define* $g_n(s)$ *for* $n = 1,2,\ldots$ *by the following recurrence relation*

$$g_n(s) = \underset{\sim}{\Pi}\{\gamma(s)g_{n-1}(s)\} . \tag{2}$$

If $|\rho|\,\|\gamma\| < 1$, *then*

$$\sum_{n=0}^{\infty} g_n(s)\rho^n = e^{-\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}}\underset{\sim}{\Pi}\{g_0(s)e^{-\log[1-\rho\gamma(s)]+\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}}\} \tag{3}$$

for $|s| \leqq 1$. *If, in particular,* $g_0(s) \equiv 1$, *then* (3) *reduces to*

$$\sum_{n=0}^{\infty} g_n(s)\rho^n = e^{-\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}} \tag{4}$$

where $|\rho|\,\|\gamma\| < 1$ *and* $|s| \leqq 1$.

Proof. Let us denote the right-hand side of (3) by $U(s,\rho)$. Obviously, $U(s,\rho) \varepsilon \underset{\sim}{A}$ and $\underset{\sim}{\Pi}\{ U(s,\rho)\} = U(s,\rho)$. Now we shall show that

$U(s,\rho)$ satisfies the following equation

(5) $$U(s,\rho)-\rho\underset{\sim}{\Pi}\{\gamma(s)U(s,\rho)\} = g_0(s) .$$

This can be proved as follows. Let

(6) $$h(s,\rho) = e^{\log[1-\rho\gamma(s)]-\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}}$$

for $|s| = 1$ and $|\rho|\,\|\gamma\| < 1$. Evidently $h(s,\rho) \in \underset{\sim}{A}$, $1/h(s,\rho) \in \underset{\sim}{A}$ and $g_0(s)/h(s) \in \underset{\sim}{A}$. We can see immediately that

(7) $$\underset{\sim}{\Pi}\{h(s,\rho)\} = 1$$

and

(8) $$\underset{\sim}{\Pi}\{ \frac{g_0(s)}{h(s,\rho)} - \underset{\sim}{\Pi}\frac{g_0(s)}{h(s,\rho)}\} = 0 .$$

Now (7) and (8) imply that

(9) $$\underset{\sim}{\Pi}\{ h(s,\rho)[\frac{g_0(s)}{h(s,\rho)} - \underset{\sim}{\Pi}\, \frac{g_0(s)}{h(s,\rho)}]\} = 0 ,$$

that is,

(10) $$\underset{\sim}{\Pi}\{[1-\rho\gamma(s)]U(s,\rho)\} = g_0(s)$$

whence (5) follows.

Let us expand $U(s,\rho)$ in a power series as follows

(11) $$U(s,\rho) = \sum_{n=0}^{\infty} u_n(s)\rho^n .$$

This series is convergent if $|\rho|\,\|\gamma\| < 1$ and evidently $u_n(s) \in \underset{\sim}{A}$ for $n = 0,1,2,\ldots$. If we put (11) into (5) and form the coefficient of ρ^n , then we obtain that $u_0(s) = g_0(s)$ and

$$u_n(s) = \underset{\sim}{\Pi}\{\gamma(s)u_{n-1}(s)\} \tag{12}$$

for $n = 1,2,\ldots$. Accordingly, the sequence $\{u_n(s)\}$ satisfies the same recurrence relation and the same initial condition as the sequence $\{g_n(s)\}$. Thus $u_n(s) = g_n(s)$ for $n = 0,1,2,\ldots$ which was to be proved.

In the particular case of $g_0(s) \equiv 1$ the proof of (4) is much simpler. If now $U(s,\rho)$ denotes the right hand side of (4), then it follows immediately that

$$\underset{\sim}{\Pi}\{[1-\rho\gamma(s)]U(s,\rho)\} = 1 \tag{13}$$

and therefore (5) holds with $g_0(s) \equiv 1$. The remainder of the proof follows as in the general case.

The following theorems follow immediately from Theorem 1. Alternately, we can prove the following theorems directly by using the same methods as we used in Section 4.

<u>Theorem 2.</u> <u>If</u> $\gamma(s) \in \underset{\sim}{A}$, $g_0(s) \equiv 1$ <u>and</u>

$$g_n(s) = \underset{\sim}{\Pi}\{\gamma(s)g_{n-1}(s)\} \tag{14}$$

<u>for</u> $n = 1,2,\ldots,$ <u>then</u>

$$\sum_{n=0}^{\infty} g_n(s)\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} \gamma_k^+(s)\} \tag{15}$$

<u>for</u> $|s| \leq 1$ <u>and</u>. $|\rho|\,\|\gamma\| < 1$ <u>where</u> $\gamma_k(s) = [\gamma(s)]^k$ <u>and</u>

$$\gamma_k^+(s) = \underset{\sim}{\Pi}\{\gamma_k(s)\} = \underset{\sim}{\Pi}\{[\gamma(s)]^k\} \tag{16}$$

for $k = 1,2,\ldots$. Furthermore we can write that

$$(17) \qquad g_n(s) = Q_n(\overset{+}{\gamma}_1(s), \overset{+}{\gamma}_2(s), \ldots, \overset{+}{\gamma}_n(s))$$

for $|s| \leq 1$ and $n = 1,2,\ldots$ where the polynomial $Q_n(x_1, x_2, \ldots, x_n)$ is defined by (4.21).

Proof. We can prove this theorem in an analogous way as Theorem 4.2 and Theorem 4.3 .

Theorem 3. If $\gamma(s) \in \underset{\sim}{A}$, $g_0(s) \in \underset{\sim}{A}$, $\underset{\sim}{\Pi}\{g_0(s)\} = g_0(s)$ and

$$(18) \qquad g_n(s) = \underset{\sim}{\Pi}\{\gamma(s)g_{n-1}(s)\}$$

for $n = 1,2,\ldots,$ then we have

$$(19) \qquad g_n(s) = \sum_{k=0}^{n} q_{n-k}(s)\underset{\sim}{\Pi}\{g_0(s)q_k^*(s)\}$$

for $|s| \leq 1$ and $n = 0,1,2,\ldots$ where

$$(20) \qquad q_k(s) = Q_k(\overset{+}{\gamma}_1(s), \overset{+}{\gamma}_2(s), \ldots, \overset{+}{\gamma}_k(s))$$

for $k = 1,2,\ldots,n$ and $q_0(s) \equiv Q_0 \equiv 1$, and

$$(21) \qquad q_k^*(s) = Q_k(\gamma_1(s) - \overset{+}{\gamma}_1(s), \gamma_2(s) - \overset{+}{\gamma}_2(s), \ldots, \gamma_k(s) - \overset{+}{\gamma}_k(s))$$

for $k = 1,2,\ldots,n$ and $q_0^*(s) \equiv Q_0 \equiv 1$. The polynomial $Q_k(x_1, x_2, \ldots, x_k)$ for $k = 1,2,\ldots$ is defined by (4.21) .

Proof. The proof follows along the same lines as the proof of Theorem 4.4 .

If $g_0(s) \equiv 1$, then (19) reduces to $g_n(s) = q_n(s)$ $(n = 0,1,2,\ldots)$ which is in agreement with (17).

If we multiply (17) by ρ^n and add for $n = 0,1,2,\ldots$) then we obtain (4) or (15) for $|\rho|\,\|\gamma\| < 1$.

If we multiply (19) by ρ^n and add for $n = 0,1,2,\ldots$, then we obtain (3) for $|\rho|\,\|\gamma\| < 1$.

The usefulness of the results of this section depends on the applicability of the transformation $\underset{\sim}{\Pi}$. In the following two sections we shall give a method for finding $\underset{\sim}{\Pi}\{a(s)\}$ for $a(s) \varepsilon \underset{\sim}{A}$, and, in particular, for finding $\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}$ if $\gamma(s) \varepsilon \underset{\sim}{A}$ and $|\rho|\,\|\gamma\| <1$.

11. <u>A Representation of $\underset{\sim}{\Pi}$.</u> If we know

(1) $$a(s) = \sum_{k=-\infty}^{\infty} a_k s^k \in \underset{\sim}{A}$$

for $|s| = 1$, then we have

(2) $$a_k = \frac{1}{2\pi i} \oint_{|z|=1} \frac{a(z)}{z^{k+1}} dz$$

for $k = 0, \pm 1, \pm 2, \ldots$ and thus

(3) $$\underset{\sim}{\Pi}\{a(s)\} = a^+(s) = \sum_{k=-\infty}^{0} a_k + \sum_{k=1}^{\infty} a_k s^k$$

for $|s| \leq 1$ is uniquely determined by $a(s)$. The function $a^+(s)$ is regular in the disc $|s| < 1$ and continuous in $|s| \leq 1$. We can obtain $a^+(s)$ explicitly by the following theorem.

<u>Theorem 1.</u> <u>If</u> $a(s) \in \underset{\sim}{A}$, <u>then for</u> $|s| < 1$ <u>we have</u>

(4) $$a^+(s) = \frac{1}{2} a(1) + \lim_{\varepsilon \to 0} \frac{1-s}{2\pi i} \int_{L_\varepsilon} \frac{a(z)}{(1-z)(s-z)} dz$$

<u>where</u> $L_\varepsilon = \{z : z = e^{i\theta},\ \varepsilon < \theta < 2\pi - \varepsilon\}$ <u>for</u> $0 < \varepsilon < \pi/2$.

<u>Proof.</u> For $0 < \varepsilon < \pi/2$ let C_ε^+ and C_ε^- be closed paths of integration taken in the positive (counter-clockwise) sense and defined as follows: The path C_ε^+ varies from $z = e^{i\varepsilon}$ to $z = e^{-i\varepsilon}$ on the longer arc of the circle $|z| = 1$ and from $z = e^{-i\varepsilon}$ to $z = e^{i\varepsilon}$ on the shorter arc of the circle $|z-1| = 2 \sin \frac{\varepsilon}{2}$. The path C_ε^- varies from $z = e^{i\varepsilon}$ to $z = e^{-i\varepsilon}$ on the longer arc of the circle $|z| = 1$ and from $z = {}^{-i\varepsilon}$ to $z = e^{i\varepsilon}$ also on the longer arc of the circle $|z-1| = 2\sin \frac{\varepsilon}{2}$. Since $a^+(z)$ is regular inside C_ε^+ and continuous on the boundary, it follows

by Cauchy's integral formula (see e.g. W. F. Osgood [23] p.112) that

(5) $$\frac{1-s}{2\pi i} \int_{C_\varepsilon^+} \frac{a^+(z)}{(1-z)(s-z)}\, dz = a^+(s)$$

for $|s| < 1$ if $\varepsilon > 0$ is small enough.

Since $a(z) - a^+(z)$ is regular outside C_ε^-, continuous on the boundary and $|a(z) - a^+(z)| \leqq 2\|a\|$ for $|z| \geqq 1$, it follows by Cauchy's integral theorem (see e.g. W. F. Osgood [23]p. 105) that

(6) $$\frac{1-s}{2\pi i} \int_{C_\varepsilon^-} \frac{a(z)-a^+(z)}{(1-z)(s-z)}\, dz = 0$$

for $|s| < 1$. For the integral in (6) remains unchanged if the path C_ε^- is replaced by the circle $|z| = R$, where $R > 1 + \varepsilon$. If $R \to \infty$, then the latter integral tends to 0.

Let $\varepsilon \to 0$ in (5) and (6). Then we obtain that

(7) $$\lim_{\varepsilon \to 0} \frac{1-s}{2\pi i} \int_{L_\varepsilon} \frac{a^+(z)}{(1-z)(s-z)}\, dz + \frac{1}{2}\, a(1) = a^+(s)$$

and

(8) $$\lim_{\varepsilon \to 0} \frac{1-s}{2\pi i} \int_{L_\varepsilon} \frac{a(z)-a^+(z)}{(1-z)(s-z)}\, dz = 0$$

for $|s| < 1$. Here we used that $a^+(1) = a(1)$. If we add (7) and (8), then we obtain $a^+(s)$ for $|s| < 1$. This proves (4). Since $a^+(s)$ is continuous for $|s| \leqq 1$, (4) determines $a^+(s)$ also for $|s| = 1$ by continuity.

We note that if $a(s) \in A$ is given by (1) and

$$(9) \qquad \sum_{n=-\infty}^{\infty} |a_n|(1-\varepsilon)^n < \infty$$

for some $0 < \varepsilon < 1$, then

$$(10) \qquad a^+(s) = \frac{1-s}{2\pi i} \int_{C_\varepsilon^+} \frac{a(z)}{(1-z)(s-z)}\,dz$$

for $|s| < 1-\varepsilon$. For in this case (6) remains valid if C_ε^- is replaced by C_ε^+ and hence (10) follows by (5).

If $a(s) \in \underset{\sim}{A}$ is given by (1) and

$$(11) \qquad \sum_{n=-\infty}^{\infty} |a_n|(1+\varepsilon)^n < \infty$$

for some $\varepsilon > 0$, then we have

$$(12) \qquad a^+(s) = a(1) + \frac{1-s}{2\pi i} \int_{C_\varepsilon^-} \frac{a(z)}{(1-z)(s-z)}\,dz$$

for $|s| \leq 1$. For in this case if we replace C_ε^+ by C_ε^- in (5), then the right-hand side becomes $a^+(s) - a^+(1)$. If we add (6) to this equation, then we obtain (12).

12. <u>The Method of Factorization.</u> If $\gamma(s) \in \underset{\sim}{A}$ and $|\rho|\,\|\gamma\| < 1$, then $\log[1-\rho\gamma(s)] \in \underset{\sim}{A}$ and we can determine $\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}$ by Theorem 11.1 . We can use also the expansion

$$(1) \qquad \underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\} = -\sum_{n=1}^{\infty} \frac{\rho^n}{n}\, \underset{\sim}{\Pi}\{[\gamma(s)]^n\}$$

which is convenient if $\underset{\sim}{\Pi}\{[\gamma(s)]^n\}$ for $n = 1,2,\ldots$ can easily be obtained. In what follows we shall mention another method, namely, the method of factorization.

Let $\gamma(s) \in \underset{\sim}{A}$, $|\rho|\,\|\gamma\| < 1$ and suppose that

$$(2) \qquad 1 - \rho\gamma(s) = g^+(s,\rho)g^-(s,\rho)$$

for $|s| = 1$ where $g^+(s,\rho)$ satisfies the requirements:

(a_1) $g^+(s,\rho)$ is a regular function of s in the disc $|s| < 1$,

(a_2) $g^+(s,\rho)$ is continuous and free from zeros in $|s| \leqq 1$,

and $g^-(s,\rho)$ satisfies the following requirements:

(b_1) $g^-(s,\rho)$ is a regular function of s in the domain $|s| > 1$,

(b_2) $g^-(s,\rho)$ is continuous and fre from zeros in $|s| \geqq 1$,

(b_3) $\lim_{|s|\to\infty} [\log g^-(s,\rho)]/s = 0$.

Such a factorization always exists. For example,

$$(3) \qquad g^+(s,\rho) = e^{\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}}$$

for $|s| \leqq 1$ and

$$(4) \qquad g^-(s,\rho) = e^{\log[1-\rho\gamma(s)]-\underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\}}$$

for $|s| \geq 1$ satisfy all the requirements. Actually, the above requirements determine $g^+(s,\rho)$ and $g^-(s,\rho)$ up to a multiplicative factor depending only on ρ. This is the content of the next theorem.

<u>Theorem 1.</u> <u>If</u> $\gamma(s) \in \underset{\sim}{A}$, $|\rho|\,\|\gamma\| < 1$ <u>and</u>

$$(5) \qquad 1 - \rho\gamma(s) = g^+(s,\rho)g^-(s,\rho)$$

<u>for</u> $|s| = 1$ <u>where</u> $g^+(s,\rho)$ <u>satisfies</u> (a_1), (a_2) <u>and</u> $g^-(s,\rho)$ <u>satisfies</u> (b_1), (b_2), (b_3), <u>then</u>

$$(6) \qquad \underset{\sim}{\Pi}\{\log[1-\rho\gamma(s)]\} = \log g^+(s,\rho) + \log g^-(1,\rho)$$

<u>for</u> $|s| \leq 1$.

<u>Proof.</u> It is sufficient to prove (6) for $|s| < 1$. For $|s| = 1$ (6) follows by continuity. Let us define the paths L_ε, C_ε^+, C_ε^- in the same way as in the proof of Theorem 11.1. By Cauchy's integral formula we can write that

$$(7) \qquad \frac{1-s}{2\pi i} \int_{C_\varepsilon^+} \frac{\log g^+(z,\rho)}{(1-z)(s-z)}\, dz = \log g^+(s,\rho)$$

for $|s| < 1$ if $\varepsilon > 0$ is small enough, and by Cauchy's integral theorem we can write that

$$(8) \qquad \frac{1-s}{2\pi i} \int_{C_\varepsilon^-} \frac{\log g^-(z,\rho)}{(1-z)(s-z)}\, dz = 0$$

for $|s| < 1$. For the integral in (8) remains unchanged if instead of C_ε^- we integrate along the circle $|z| = R$ where $R > 1 + \varepsilon$. If $R \to \infty$, then the latter integral tends to 0 .

If $\varepsilon \to 0$ in (7) and (8), then we get

$$(9) \qquad \lim_{\varepsilon \to 0} \frac{1-s}{2\pi i} \int_{L_\varepsilon} \frac{\log g^+(z,\rho)}{(1-z)(s-z)}\, dz + \frac{1}{2} \log g^+(1,\rho) = \log g^+(s,\rho)$$

and

$$(10) \qquad \lim_{\varepsilon \to 0} \frac{1-s}{2\pi i} \int_{L_\varepsilon} \frac{\log g^-(z,\rho)}{(1-z)(s-z)}\, dz - \frac{1}{2} \log g^-(1,\rho) = 0$$

for $|s| < 1$. If we add (9) and (10), then we obtain (6) for $|s| < 1$. This completes the proof of the theorem.

By using Theorem 1 we can express Theorem 10.1 also in the following way.

Theorem 2. Let us suppose that $\gamma(s) \in \underset{\sim}{A}$, $g_0(s) \in \underset{\sim}{A}$, *and* $\underset{\sim}{\Pi}\{g_0(s)\} = g_0(s)$. *Define* $g_n(s)$ *for* $n = 1,2,\ldots$ *by the following recurrence formula*

$$(11) \qquad g_n(s) = \underset{\sim}{\Pi}\{\gamma(s) g_{n-1}(s)\} .$$

If $|\rho|\, \|\gamma\| < 1$ *and*

$$(12) \qquad 1-\rho\gamma(s) = g^+(s,\rho) g^-(s,\rho)$$

for $|s| = 1$ *where* $g^+(s,\rho)$ *satisfies* (a_1), (a_2) *and* $g^-(s,\rho)$ *satisfies* (b_1), (b_2), (b_3), *then*

$$(13) \qquad \sum_{n=0}^{\infty} g_n(s)\rho^n = \frac{1}{g^+(s,\rho)} \underset{\sim}{\Pi}\left\{\frac{g_0(s)}{g^-(s,\rho)}\right\}$$

for $|s| \le 1$. *If, in particular,* $g_0(s) \equiv 1$, *then*

$$\sum_{n=0}^{\infty} g_n(s)\rho^n = \frac{1}{g^+(s,\rho)g^-(1,\rho)} \tag{14}$$

for $|s| \leq 1$.

Proof. If we put (6) into (10.3) and (10.4), then we obtain (13) and (14) respectively.

By (13) we obtain that

$$[1-\rho\gamma(s)] \sum_{n=0}^{\infty} g_n(s)\rho^n = g^-(s,\rho)\underset{\sim}{\Pi}\left\{\frac{g_0(s)}{g^-(s,\rho)}\right\} \tag{15}$$

for $|s| = 1$.

By (14) we obtain that if $g_0(s) \equiv 1$ then

$$[1-\rho\gamma(1)] \sum_{n=0}^{\infty} g_n(s)\rho^n = \frac{g^+(1,\rho)}{g^+(s,\rho)} \tag{16}$$

for $|s| \leq 1$, or

$$[1-\rho\gamma(s)] \sum_{n=0}^{\infty} g_n(s)\rho^n = \frac{g^-(s,\rho)}{g^-(1,\rho)} \tag{17}$$

for $|s| = 1$.

In finding $g^+(s,\rho)$ and $g^-(s,\rho)$ we can usually utilize the following particular case of Rouché's theorem:

If $f(z)$ *and* $g(z)$ *are regular in the disc* $|z| < 1$, *continuous in* $|z| \leq 1$ *and* $|g(z)| < |f(z)|$ *if* $|z| = 1$, *then* $f(z)$ *and* $f(z)+g(z)$ *have the same number of zeros in the disc* $|z| < 1$.

13. PROBLEMS

13.1. Prove that the space $\underset{\sim}{R}$ is complete, that is, if $\Phi_n(s) \in \underset{\sim}{R}$ for $n = 1,2,\dots$ and if $\lim_{\substack{m \to \infty \\ n \to \infty}} \|\Phi_m - \Phi_n\| = 0$, then there exists a $\Phi(s) \in \underset{\sim}{R}$ such that $\lim_{n \to \infty} \|\Phi - \Phi_n\| = 0$.

13.2. Prove that the space $\underset{\sim}{A}$ is complete, that is, if $a_n(s) \in \underset{\sim}{A}$ for $n = 1,2,\dots$ and if $\lim_{\substack{m \to \infty \\ n \to \infty}} \|a_m - a_n\| = 0$, then there exists an $a(s) \in \underset{\sim}{A}$ such that $\lim_{n \to \infty} \|a - a_n\| = 0$.

13.3. Let $\Phi(s) = 1/(1-s^2)$. Find $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$.

13.4. Let $\Phi(s) = (pe^s + qe^{-s})^m$ where $p \geq 0$, $q \geq 0$ and $p+q = 1$. Prove that $\Phi(s) \in \underset{\sim}{R}$ and determine $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$.

13.5. Let $\Phi(s) = e^{s^2/2}$ for any complex s. Prove that $\Phi(s) \in \underset{\sim}{R}$ and determine $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$.

13.6. Let $\phi(s)$ be the Laplace-Stieltjes transform of a nonnegative random variable and let λ be a positive constant. Determine $\underset{\sim}{T}\{\frac{\lambda\phi(s)}{\lambda - s}\}$.

13.7. Let $\Phi(s) \in \underset{\sim}{R}$ and $\operatorname{Re}(q) > 0$. Prove that

$$\underset{\sim}{T}\{\frac{\Phi(s)}{s-q}\} = \frac{1}{s-q}[\Phi^+(s) - \frac{s}{q}\Phi^+(q)]$$

if $s \neq q$ and $\operatorname{Re}(s) \geq 0$ where $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$.

13.8. Let $\phi(s)$ be the Laplace-Stieltjes transform of a nonnegative random variable and let λ be a positive constant. Determine $\underset{\sim}{T}\{\frac{\lambda\phi(-s)}{\lambda+s}\}$.

13.9. Let $\phi(s)$ be the Laplace-Stieltjes transform of a nonnegative ranfom variable and let λ be a positive constant. Determine $\underset{\sim}{T}\{\phi(s)(\frac{\lambda}{\lambda-s})^m\}$ where m is a positive integer.

13.10. Let $\phi(s)$ be the Laplace-Stieltjes transform of a nonnegative random variable and let λ be a positive constant. Determine $\underset{\sim}{T}\{(\frac{\lambda}{\lambda+s})^m\phi(-s)\}$ where m is a positive integer.

13.11. Let $\phi(s)$ and $\gamma(s)$ be Laplace-Stieltjes transforms of non-negative random variables and suppose that $\gamma(s)$ is a rational function of s. Find $\underset{\sim}{T}\{\phi(s)\gamma(-s)\}$.

13.12. Let $\Phi(s) \in \underset{\sim}{R}$ and let $\gamma(s)$ be the Laplace-Stieltjes transform of a nonnegative random variable. Suppose that $\gamma(s)$ is a rational function of s. Find $\underset{\sim}{T}\{\Phi(s)\gamma(-s)\}$.

13.13. Let $\phi(s)$ and $\gamma(s)$ be Laplace-Stieltjes transforms of non-negative random variables and uppose that $\gamma(s)$ is a rational function of s. Find $\underset{\sim}{T}\{\gamma(s)\phi(-s)\}$.

13.14. Let ξ be a discrete random variable taking on nonnegative integers only. Denote by $g(s)$ the generating function of ξ, that is, $g(s) = \underset{\sim}{E}\{s^\xi\}$ for $|s| \leq 1$. Determine $\underset{\sim}{\Pi}\{psg(s)/(s-q)\}$ where $p > 0$, $q > 0$ and $p+q = 1$.

13.15. Let ξ be a discrete random variable taking on nonnegative integers only. Denote by $g(s)$ the generating function of ξ, that is, $g(s) = \underset{\sim}{E}\{s^{\xi}\}$ for $|s| \leq 1$. Determine $\underset{\sim}{\Pi}\{\frac{pg(1/s)}{1-qs}\}$ where $p > 0$, $q > 0$ and $p+q = 1$.

13.16. Let ξ be a discrete random variable taking on nonnegative integers. Denote by $g(s)$ the generating function of ξ, that is, $g(s) = \underset{\sim}{E}\{s^{\xi}\}$ for $|s| \leq 1$. Determine $\underset{\sim}{\Pi}\{p^m s^m g(s)/(s-q)^m\}$ where $p > 0$, $q > 0$, $p+q = 1$ and m is a positive integer.

13.17. Let ξ be a discrete random variable taking on nonnegative integers. Denote by $g(s)$ the generating function of ξ, that is, $g(s) = \underset{\sim}{E}\{s^{\xi}\}$ for $|s| \leq 1$. Determine $\underset{\sim}{\Pi}\{p^m g(1/s)/(1-qs)^m\}$ where $p > 0$, $q > 0$, $p+q = 1$ and m is a positive integer.

13.18. Let $a(s)$ and $b(s)$ be generating functions of discrete random variables taking on nonnegative integers only. Suppose that $b(s)$ is a rational function of s. Determine $\underset{\sim}{\Pi}\{a(s)b(\frac{1}{s})\}$.

13.19. Let $a(s)$ and $b(s)$ be generating functions of discrete random variables taking on nonnegative integers only. Suppose that $b(s)$ is a rational function of s. Determine $\underset{\sim}{\Pi}\{a(\frac{1}{s})b(s)\}$.

13.20. Let $\{\xi_n ; n = 0,1,2,\dots\}$ be a homogeneous Markov chain with state space $I = \{0,1,2,\dots\}$ and transition probability matrix

$$\pi = \left\| \begin{matrix} 1 - h_0 , & h_0 , & 0 , & 0 , & \dots \\ 1 - h_0 - h_1 , & h_1 , & h_0 , & 0 , & \dots \\ 1 - h_0 - h_1 - h_2 , & h_2 , & h_1 , & h_0 , & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \right\|$$

where $h_0 > 0$, $h_0 + h_1 < 1$, $\sum_{j=0}^{\infty} h_j = 1$, and $\alpha = \sum_{j=0}^{\infty} j\, h_j < \infty$. Find the distribution of ξ_n $(n = 1,2,\dots)$ and the limiting distribution of ξ_n as $n \to \infty$. (See reference [37].)

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