CHAPTER X

QUEUING, RISK AND STORAGE PROCESSES

62. Single Server Queues. The theory of queues deals with the mathematical studies of random mass service phenomena. Such phenomena appear in physics, engineering, industry, transportation, commerce, business, and several other fields. The theory of queues developed in the twentieth century with the investigation of telephone traffic problems. The pioneer work has been done by A. K. Erlang [82], [83], [84] who studied the stochastic law of the delay of calls in a telephone exchange. The mathematical theory of queues made considerable progress in the 1930's through the work of F. Pollaczek [223], [225], A. N. Kolmogorov [181], A. Ya. Khintchine [167], [168], and others. At present there is a huge literature on the theory of queues and its applications. See, for example, A. Doig [75], T. L. Saaty [262], [263] and H. O. A. Wold [346].

Many processes arising in the theory of mass service can be described by the following queuing model: In the time interval $[0, \infty)$ customers arrive at a counter at random times $\tau_0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$ and are served by one or more servers. The successive service times $\chi_0, \chi_1, \chi_2, \ldots, \chi_n, \ldots$ are random variables. The initial state is determined by the initial queue size and by the initial occupation times of the servers.

The most important problems are connected with the investigation of the stochastic behavior of the waiting time, the queue size, the busy periods and the occupation times of the servers.

In this section we shall be concerned exclusively with single server queues. One of the most important models of single server queues is the following: In the time interval $[0, \infty)$ customers arrive at a counter at times $\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$ and are served by one server in the order of arrival. The server is busy if there is at least one customer at the counter. Denote by χ_n the service time of the customer arriving at time τ_n. Denote by $\xi(0)$ the initial queue size and η_0 the initial occupation time of the server at time $t = 0$. It is assumed that the interarrival times $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots; \tau_0 = 0)$ and the service times χ_n $(n = 0,1,2,\ldots)$ are independent sequences of mutually independent and identically distributed positive random variables and they are independent of $\xi(0)$ and η_0 too.

Let

(1) $$\mathbf{P}\{\tau_n - \tau_{n-1} \leq x\} = F(x)$$

for $n = 1,2,\ldots$ and

(2) $$\mathbf{P}\{\chi_n \leq x\} = H(x)$$

for $n = 0,1,2,\ldots$.

Denote by η_n the actual waiting time of the customer arriving at time τ_n.

Denote by $\eta(t)$ the virtual waiting time at time t. The virtual waiting time at time t is defined as the time which a customer would have to wait if he arrived at time t.

Denote by $\xi(t)$ the queue size at time t, that is, the total number of customers in the system at time t.

Denote by $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ the lengths of the successive idle periods and $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ the lengths of the successive busy periods of the server. Idle periods and busy periods are successive time intervals during which there is no customer in the system or there is at least one customer in the system.

Denote by $\theta(t)$ the total idle time of the server in the time interval $(0, t)$, and $\sigma(t)$, the total occupation time of the server in the time interval $(0, t)$.

In what follows we shall deal with the problem of determining the distributions of the random variables η_n, $\eta(t)$, $\xi(t)$, θ_n, σ_n, $\theta(t)$, and $\sigma(t)$. If we want to design efficient queuing systems, then it is necessary to know these distributions.

The distribution of the waiting time. Our first aim is to determine the distribution of η_n $(n = 0,1,2,\ldots)$, the waiting time of the customer arriving at time τ_n. Obviously η_0 is the initial occupation time of the server at time $t = 0$. We can easily see that the random variables η_n $(n = 0,1,2,\ldots)$ satisfy the following recurrence relation

(3) $$\eta_{n+1} = [\eta_n + \chi_n - (\tau_{n+1} - \tau_n)]^+$$

for $n = 0,1,2,\ldots$ where $[x]^+ = \max(0, x)$.

Let us introduce the notation

$$\phi(s) = \int_0^\infty e^{-sx}\, dF(x) \tag{4}$$

and

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x) \tag{5}$$

for $\mathrm{Re}(s) \geq 0$. Furthermore, let

$$\Omega_n(s) = \underset{\sim}{E}\{e^{-s\eta_n}\} \tag{6}$$

for $\mathrm{Re}(s) \geq 0$.

The distribution function $\underset{\sim}{P}\{\eta_n \leq x\}$ is uniquely determined by $\Omega_n(s)$. The Laplace-Stieltjes transforms $\Omega_n(s)$ $(n = 1,2,\ldots)$ are determined by the following theorem. See F. Pollaczek [229].

Theorem 1. *If* $\mathrm{Re}(s) \geq 0$ *and* $|\rho| < 1$, *then*

$$\sum_{n=0}^{\infty} \Omega_n(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(-s)\psi(s)]\}}\, \underset{\sim}{T}\left\{ \frac{\Omega_0(s)e^{\underset{\sim}{T}\{\log[1-\rho\phi(-s)\psi(s))]\}}}{1-\rho\phi(-s)\psi(s)} \right\} \tag{7}$$

where $\underset{\sim}{T}$ *operates on the variable* s .

Proof. If $\underset{\sim}{R}$ denotes the space which we introduced in Section 2, then it is evident that $\Omega_0(s) \in \underset{\sim}{R}$, $\underset{\sim}{T}\{\Omega_0(s)\} = \Omega_0(s)$ and $\gamma(s) = \phi(-s)\psi(s) \in \underset{\sim}{R}$ and $\|\gamma\| = 1$. Furthermore, by (3) it follows that

$$\Omega_{n+1}(s) = \underset{\sim}{T}\{\Omega_n(s)\phi(-s)\psi(s)\} \tag{8}$$

for $n = 0,1,2,\ldots$ and $Re(s) \geq 0$. Hence (7) follows by Theorem 4.1 .

In finding (7) we can also use Theorem 6.2 .

If we introduce the notation

(9) $$\xi_n = \chi_{n-1} - (\tau_n - \tau_{n-1})$$

for $n = 1,2,\ldots$ and $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots,$ and $\zeta_0 = 0$, then by (3) we can write that

(10) $$\eta_n = \max(0, \xi_n, \xi_{n-1} + \xi_n, \ldots, \xi_2 + \ldots + \xi_n, \eta_0 + \xi_1 + \ldots + \xi_n) ,$$

for $n = 1,2,\ldots$. If in (10) we replace $\xi_n, \xi_{n-1}, \ldots, \xi_1$ by $\xi_1, \xi_2, \ldots, \xi_n$ respectively, then we obtain a new random variable

(11) $$\bar{\eta}_n = \max(\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}, \eta_0 + \zeta_n)$$

for $n = 1,2,\ldots,$ which has exactly the same distribution as η_n . Thus we can write that

(12) $$\mathbf{P}\{\eta_n \leq x\} = \mathbf{P}\{\max_{0 \leq k \leq n} \zeta_k \leq x \text{ and } \eta_0 + \zeta_n \leq x\}$$

for $n = 0,1,2,\ldots$ and all x .

The relation (12) makes it possible to find the limiting behavior of $\mathbf{P}\{\eta_n \leq x\}$ as $n \to \infty$. See <u>D. V. Lindley</u> [187] .

<u>Theorem 2.</u> <u>If</u> $\mathbf{P}\{\xi_n = 0\} < 1$, <u>then</u>

(13) $$\lim_{n \to \infty} \mathbf{P}\{\eta_n \leq x\} = W(x)$$

<u>exists and is independent of the distribution of</u> η_0 . <u>Let</u>

(14) $$M = \sum_{n=1}^{\infty} \frac{\mathbf{P}\{\zeta_n > 0\}}{n} .$$

If $\mathbf{P}\{\xi_n = 0\} < 1$ *and* $M < \infty$, *then* $W(x)$ *is a proper distribution function and*

(15) $$\Omega(s) = \int_0^{\infty} e^{-sx}\, dW(x) = e^{-\sum_{n=1}^{\infty} \frac{1}{n} [1-\mathbf{T}\{[\phi(-s)\psi(s)]^n\}]}$$

for $\mathrm{Re}(s) \geq 0$.

If $\mathbf{P}\{\xi_n = 0\} < 1$ *and* $M = \infty$, *then* $W(x) = 0$ *for all* x .

Proof. If $\mathbf{P}\{\xi_n = 0\} < 1$, then by (12) we can conclude that

(16) $$\lim_{n\to\infty} \mathbf{P}\{\eta_n \leq x\} = \lim_{n\to\infty} \mathbf{P}\{\max_{0\leq k\leq n} \zeta_k \leq x\} = \mathbf{P}\{\sup_{0\leq k<\infty} \zeta_k \leq x\}$$

for all x . This can be proved by the inequality

(17) $$\mathbf{P}\{\max_{0\leq k\leq n} \zeta_k \leq x\} - \mathbf{P}\{\eta_0 + \zeta_n > x\} \leq \mathbf{P}\{\eta_n \leq x\} \leq \mathbf{P}\{\max_{0\leq k\leq n} \zeta_k \leq x\}$$

which holds for all x and which follows from (12).

If $\mathbf{P}\{\xi_n = 0\} < 1$ and $M < \infty$, then by Theorem 43.12 we have

(18) $$\mathbf{P}\{\sup_{0\leq k<\infty} \zeta_k < \infty\} = 1 .$$

This implies that $\lim_{n\to\infty} \mathbf{P}\{\eta_0 + \zeta_n > x\} = 0$ for $x \geq 0$ in (17) . For if $x \geq 0$, then

$$P\{\eta_0 + \zeta_n > x\} \leq P\{\frac{\eta_0}{n} + \frac{\zeta_n}{n} > 0\} \tag{19}$$

and by (18) the right-hand side of (19) tends to 0 as $n \to \infty$. If we let $n \to \infty$ in (17), then we obtain (16) for $x \geq 0$. For $x < 0$ (16)′ is obvious. Thus (16) is valid and $W(x)$ is a proper distribution function. We note that $W(0) = e^{-M}$.

If $P\{\xi_n = 0\} < 1$ and $M = \infty$, then by Theorem 43.12 we have

$$P\{\sup_{0 \leq k < \infty} \zeta_k = \infty\} = 1 . \tag{20}$$

Thus by (17) $\lim_{n \to \infty} P\{\eta_n \leq x\} = 0$ for all x regardless of the distribution of η_0. This proves (16) and that $W(x) = 0$ for all x.

We note that if

$$a = \int_0^\infty x dF(x) \tag{21}$$

and

$$b = \int_0^\infty x dH(x) \tag{22}$$

are finite and $b < a$, then $M < \infty$, whereas if $b \geq a$ and $P\{\xi_n = 0\} < 1$, then $M = \infty$.

The Laplace-Stieltjes transform $\Omega(s)$ can be obtained by Theorem 43.13.

The Laplace-Stieltjes transform $\Omega(s)$ can also be obtained by the method of factorization.

Theorem 3. Let us suppose that $\mathbf{P}\{\xi_n = 0\} < 1$ and $M < \infty$ where M is defined by (14). If

$$1-\phi(-s)\psi(s) = \Phi^{+}(s)\Phi^{-}(s) \tag{23}$$

for $\mathrm{Re}(s) = 0$ where $\Phi^{+}(s)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$, continuous and free from zeros in $\mathrm{Re}(s) \geq 0$, and $\lim_{|s|\to\infty} [\log\Phi^{+}(s)]/s = 0$ whenever $\mathrm{Re}(s) \geq 0$, furthermore, $\Phi^{-}(s)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$, continuous in $\mathrm{Re}(s) \leq 0$, free from zeros in $\mathrm{Re}(s) < 0$, and $\lim_{|s|\to\infty} [\log\Phi^{-}(s)]/s = 0$ whenever $\mathrm{Re}(s) < 0$, then we have

$$\Omega(s) = \frac{\Phi^{+}(0)}{\Phi^{+}(s)} \tag{24}$$

for $\mathrm{Re}(s) \geq 0$.

Proof. The theorem follows immediately from Theorem 43.15 .

Example. Let us suppose that

$$F(x) = \begin{cases} 1-e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \tag{25}$$

Then $a = 1/\lambda$, and $\phi(s) = \lambda/(\lambda+s)$ for $\mathrm{Re}(s) > -\lambda$.

If $\lambda b < 1$, then by Theorem 3 we obtain that

$$\Omega(s) = \frac{1-\lambda b}{1-\lambda \frac{1-\psi(s)}{s}} \tag{26}$$

for $\mathrm{Re}(s) \geq 0$ where $\Omega(0) = 1$. This is the celebrated formula which was found in 1930 by F. Pollaczek [223] and in 1932 by A. Ya. Khintchine [167].

Next we shall be concerned with the distribution of the virtual waiting time $\eta(t)$. First, however, we shall consider a deterministic single-server queue and deduce a fundamental identity which makes it possible to find the distribution of $\eta(t)$ for $t \geq 0$.

Let us consider the mathematical model of a deterministic (non-random) queuing process which satisfies the following assumptions: In the time interval $[0, \infty)$ customers arrive at a counter at times $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ where $\tau_0 = 0 < \tau_1 < \ldots < \tau_n < \ldots$ and $\lim_{n \to \infty} \tau_n = \infty$. The customers are served by one server in the order of arrival. The server is busy if there is at least one customer in the system. At time $t = 0$ the server has an initial occupation time $\eta_0 \geqq 0$. The service time of the customer arriving at time τ_n is a positive quantity χ_n. Here τ_n $(n = 0,1,2,\ldots)$, χ_n $(n = 0,1,2,\ldots)$ and η_0 are numerical (non-random) quantities.

Let us define the following functions for $0 \leq t < \infty$. Let $\eta(t)$ be the virtual waiting time at time t. Let $\gamma(t)$ be the total service time of all those customers who arrive in the interval $[0, t]$ plus η_0. Let $\vartheta(t)$ be the time difference between t and the time of the first arrival in the time interval $[t, \infty)$. Denote by $\nu(t)$ the number of customers arriving in the time interval $[0, t]$.

Let us define also the following quantities for $n = 0,1,2,\ldots$. Let η_n be the waiting time of the customer arriving at time τ_n. We have $\eta_n = \eta(\tau_n - 0)$ for $n = 1,2,\ldots$ and η_0 is the initial occupation time of the server. Let $\gamma_n = \eta_0 + \chi_0 + \ldots + \chi_{n-1}$ for $n = 1,2,\ldots$ and $\gamma_0 = \eta_0$. We have

$\gamma_n = \gamma(\tau_n - 0)$ for $n = 1,2,\dots$.

We note that if $\tau_n < t < \tau_{n+1}$, then $\gamma(t) = \gamma_{n+1}$, $\vartheta(t) = \tau_{n+1} - t$ and

(27) $$\eta(t) = [\eta_n + \chi_n - (t-\tau_n)]^+ .$$

Furthermore, we have

(28) $$\eta_{n+1} = [\eta_n + \chi_n - (\tau_{n+1} - \tau_n)]^+$$

for $n = 0,1,2,\dots$.

The following theorem contains the fundamental identity which expresses a relation between the functions $\eta(t)$, $\gamma(t)$, $\vartheta(t)$, $\nu(t)$ $(0 \le t < \infty)$ and the sequences η_n, γ_n, τ_n, χ_n $(n = 0,1,2,\dots)$. See also the author [320], [321].

<u>Theorem 4.</u> <u>If</u> $\mathrm{Re}(q) > 0$, $\mathrm{Re}(v) \ge 0$, $\mathrm{Re}(s+v) \ge 0$, $\mathrm{Re}(w) \ge 0$, $\mathrm{Re}(q+v-w) \ge 0$, $q \ne w$ <u>and</u> $|\rho| \le 1$, <u>then we have</u>

(29) $$\begin{aligned} &(s+w-q)\int_0^\infty e^{-qt-s\eta(t)-v\gamma(t)-w\vartheta(t)}\,\rho^{\nu(t)}\,dt = \\ &= \sum_{n=0}^\infty [e^{-q\tau_{n+1}-s\eta_{n+1}} - e^{-q\tau_n - s\eta_n - s\chi_n - w(\tau_{n+1}-\tau_n)}]\, e^{-v\gamma_{n+1}}\rho^{n+1} \\ &- \frac{s}{q-w}\sum_{n=0}^\infty [e^{-q\tau_{n+1}-(q-w)\eta_{n+1}} - e^{-q\tau_n-(q-w)\eta_n-(q-w)\chi_n-w(\tau_{n+1}-\tau_n)}]\, e^{-v\gamma_{n+1}}\rho^{n+1} \end{aligned}$$

<u>Proof</u> We can write that

$$(30)\quad \int_0^\infty e^{-qt-s\eta(t)-v\gamma(t)-w\vartheta(t)}\rho^{\nu(t)}dt = \sum_{n=0}^{\infty} e^{-v\gamma_{n+1}}\rho^{n+1}\int_{\tau_n}^{\tau_{n+1}} e^{-qt-s\eta(t)-w\vartheta(t)}dt .$$

If we take into consideration that $\vartheta(t) = \tau_{n+1} - t$ and $\eta(t)$ is given by (27) for $\tau_n < t < \tau_{n+1}$, then by (54.17) we obtain that

$$(31)\quad (s+w-q)\int_{\tau_n}^{\tau_{n+1}} e^{-qt-s\eta(t)-w\vartheta(t)}dt = [e^{-q\tau_{n+1}-s\eta_{n+1}} - e^{-q\tau_n-s\eta_n-s\chi_n-w(\tau_{n+1}-\tau_n)}] - \frac{s}{q-w}[e^{-q\tau_{n+1}-(q-w)\eta_{n+1}} - e^{-q\tau_n-(q-w)\eta_n-(q-w)\chi_n-w(\tau_{n+1}-\tau_n)}] .$$

If we put (31) into (30), then we get (29) which was to be proved.

If we suppose that $\{\tau_n\}$, $\{\chi_n\}$ and η_0 are random variables and $\underset{\sim}{P}\{\lim_{n\to\infty}\tau_n = \infty\} = 1$, then $\eta(t), \gamma(t), \vartheta(t), \nu(t)$ for $0 \leq t < \infty$ and η_n, γ_n for $0 \leq n < \infty$ are also random variables and the identity (29) holds for almost all realizations of $\{\eta(t), \gamma(t), \vartheta(t), \nu(t) ; 0 \leq t < \infty\}$ and $\{\eta_n, \gamma_n, \tau_n, \chi_n ; 0 \leq n < \infty\}$. The great advantage of the identity (29) is that it is valid for any single-server queue.

Now let us suppose that $\{\tau_n - \tau_{n-1}\}$ and $\{\chi_n\}$ are independent sequences of mutually independent and identically distributed positive random variables for which $\underset{\sim}{P}\{\tau_n - \tau_{n-1} \leq x\} = F(x)$ and $\underset{\sim}{P}\{\chi_n \leq x\} = H(x)$, and that

$\{\tau_n - \tau_{n-1}\}$, $\{\chi_n\}$ and η_0 are independent too. In this case we have the following result.

Theorem 5. If $\mathrm{Re}(q) > 0$, $\mathrm{Re}(v) \geq 0$, $\mathrm{Re}(s+v) \geq 0$, $\mathrm{Re}(w) \geq 0$, $\mathrm{Re}(q+v-w) \geq 0$, $q \neq w$, and $|\rho| \leq 1$, then we have

$$(s+w-q)\int_0^\infty e^{-qt} \mathop{E}\{e^{-s\eta(t)-v\gamma(t)-w\vartheta(t)}\rho^{\nu(t)}\}dt =$$

(32)

$$= \{[1-\rho\phi(w)\psi(s+v)]U(q,s,v,\rho)-\Omega_0(s+v)\} - \frac{s}{q-w}\{[1-\rho\phi(w)\psi(q+v-w)]U(q,q-w,v,\rho)-\Omega_0(q+v-w)\}$$

where

(33) $$U(q,s,v,\rho) = \sum_{n=0}^{\infty} \mathop{E}\{e^{-q\tau_n-s\eta_n-v\gamma_n}\}\rho^n$$

and

(34) $$\Omega_0(s) = \mathop{E}\{e^{-s\eta_0}\} .$$

Proof. Now the identity (29) holds for almost all realizations of $\{\eta(t), \gamma(t), \vartheta(t), \nu(t)\}$ and $\{\eta_n, \gamma_n, \tau_n, \chi_n\}$. If we form the expectation of (29), then we obtain (32). It remains to determine $U(q,s,v,\rho)$ which we shall find in the next theorem.

If we put $v = 0$, $w = 0$, and $\rho = 1$ in (32), then we obtain that

$$(35)\quad (s-q)\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = [1-\psi(s)]U(q,s,0,1) - \Omega_0(s) - \frac{s}{q}\{[1-\psi(q)]U(q,q,0,1) - \Omega_0(q)\}$$

for $Re(q) > 0$ and $Re(s) \geq 0$. By (35) we can find the probability $\underset{\sim}{P}\{\eta(t) \leq x\}$.

If $\sigma(t)$ denotes the total occupation time of the server in the time interval $(0, t)$, then we have obviously

$$(36)\quad \sigma(t) = \gamma(t) - \eta(t)$$

for $t \geq 0$. If we put $s = -v$, $w = 0$ and $\rho = 1$ in (32), then we obtain that

$$(37)\quad (q+v)\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-v\sigma(t)}\}dt = 1 - \frac{v}{q}\{[1-\psi(q+v)]U(q,q,v,1) - \Omega_0(q+v)\}$$

for $Re(q) > 0$ and $Re(v) \geq 0$. By (37) we can find the probability $\underset{\sim}{P}\{\sigma(t) \leq x\}$.

Theorem 6. The generating function

$$(38)\quad U(q,s,v,\rho) = \sum_{n=0}^\infty \underset{\sim}{E}\{e^{-q\tau_n - s\eta_n - v\gamma_n}\}\rho^n$$

is convergent for $Re(q) > 0$, $Re(s+v) \geq 0$, $Re(v) \geq 0$ and $|\rho| \leq 1$, and we have

(39) $$U(q,s,v,\rho) = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(q-s)\psi(s+v)]\}}\underset{\sim}{T}\left\{\frac{\Omega_0(s+v)e^{\underset{\sim}{T}\{\log[1-\rho\phi(q-s)\psi(s+v)]\}}}{1-\rho\phi(q-s)\psi(s+v)}\right\}$$

<u>where</u> $\underset{\sim}{T}$ <u>operates on the variable</u> s .

<u>If, in particular,</u> $\underset{\sim}{P}\{\eta_0 = 0\} = 1$, <u>then</u> $\Omega_0(s) \equiv 1$ <u>and</u> (39) <u>reduces to</u>

(40) $$U(q,s,v,\rho) = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(q-s)\psi(s+v)]\}}$$

<u>Proof.</u> If we take into consideration (28) and that

(41) $$\gamma_{n+1} = \gamma_n + \chi_n$$

for $n = 0,1,2,\ldots$, where $\gamma_0 = \eta_0$, then we obtain that

(42) $$\underset{\sim}{E}\{e^{-q\tau_{n+1}-s\eta_{n+1}-v\gamma_{n+1}}\} = \underset{\sim}{T}\{\phi(q-s)\psi(s+v)\underset{\sim}{E}\{e^{-q\tau_n-s\eta_n-v\gamma_n}\}\}$$

for $n = 0,1,2,\ldots$ and

(43) $$\underset{\sim}{E}\{e^{-q\tau_0-s\eta_0-v\gamma_0}\} = \Omega_0(s+v)$$

where $\underset{\sim}{T}$ operates on the variable s and $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s+v) \geq 0$ and $\mathrm{Re}(v) \geq 0$. Since $\|\phi(q-s)\psi(s+v)\| < 1$ for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(v) \geq 0$, we obtain (39) and (40) by Theorem 4.1 for $|\rho| \leq 1$.

In (39) and in (40) we can determine

$$(44) \qquad \underset{\sim}{T}\{\log[1 - \rho\phi(q-s)\psi(s+v)]\}$$

for $Re(s+v) \geq 0$, $Re(q) > 0$, $Re(v) \geq 0$ and $|\rho| \leq 1$ by the method of factorization. For in this case Theorem 6.1 is applicable and for $Re(s) = 0$ we can write that

$$(45) \qquad 1 - \rho\phi(q-s)\psi(s+v) = \Phi^{+}(s+v, q+v, \rho)\Phi^{-}(s+v, q+v, \rho)$$

where $\Phi^{+}(s+v, q+v, \rho)$ is a regular function of s in the domain $Re(s) > 0$, continuous and free from zeros in $Re(s) \geq 0$, and $\lim_{|s|\to\infty} [\log\Phi^{+}(s+v, q+v, \rho)]/s = 0$ whenever $Re(s) \geq 0$, furthermore $\Phi^{-}(s+v, q+v, \rho)$ is a regular function of s in the domain $Re(s) < 0$, continuous and free from zeros in $Re(s) \leq 0$, and $\lim_{|s|\to\infty} [\log \Phi^{-}(s+v, q+v, \rho)]/s = 0$ whenever $Re(s) \leq 0$. Thus by (6.6) we obtain that

$$(46) \qquad \underset{\sim}{T}\{\log[1-\rho\phi(q-s)\psi(s+v)]\} = \log\Phi^{+}(s+v, q+v, \rho) + \log\Phi^{-}(v, q+v, \rho)$$

for $Re(s) \geq 0$, $Re(q) > 0$, $Re(v) \geq 0$ and $|\rho| \leq 1$.

We observe that $1 - \rho\phi(q-s)\psi(s+v)$ is a regular function of s in the domain $-Re(v) < Re(s) < Re(q)$ and in this domain $|\rho\phi(q-s)\psi(s+v)| < 1$.

Accordingly, $1-\rho\phi(q-s)\psi(s+v)$ has no zeros in the domain $-Re(v) \leq Re(s) \leq Re(q)$.

Thus by analytical continuation we can extend the definition of $\Phi^{+}(s+v, q+v,\rho)$ to the domain $Re(s) \geq -Re(v)$ in such way that the function remains regular in the domain $Re(s) > -Re(v)$ and continuous and free from zeros in $Re(s) \geq Re(v)$. Similarly, by analytical continuation we can extend the definition of $\Phi^{-}(s+v, q+v,\rho)$ to the domain $Re(s) \leq Re(q)$ in such a way that the function remains regular in the domain $Re(s) < Re(q)$ and continuous and free from zeros in $Re(s) \leq Re(q)$.

Finally, by analytical continuation we can conclude that (46) also holds for $Re(s) \geq -Re(v)$.

<u>Examples</u>. First, let us assume that $\phi(s)$, the Laplace-Stieltjes transform of the distribution function of the interarrival times, is a rational function of s . Then we can write that

$$\phi(s) = \frac{\pi_{m-1}(s)}{\prod_{i=1}^{m}(a_i + s)} \tag{47}$$

for $Re(s) \geq 0$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leq m-1$. Since $|\phi(s)| \leq 1$ for $Re(s) \geq 0$, it follows that $Re(a_i) > 0$ for $i = 1,2,\ldots, m$.

If $Re(q) > 0$, $Re(v) \geq 0$ and $|\rho| \leq 1$, then the equation

$$\prod_{i=1}^{m}(a_i + q-s) - \rho\pi_{m-1}(q-s)\psi(s+v) = 0 \tag{48}$$

has exactly m roots $s = \gamma_i(q+v,\rho)-v$ $(i = 1,2,\ldots, m)$ in the domain $\mathrm{Re}(s) \geq 0$. We shall show that

$$(49) \qquad |\rho\pi_{m-1}(q-s)\psi(s+v)| < \left|\prod_{i=1}^{m}(a_i + q-s)\right|$$

if either $0 \leq \mathrm{Re}(s) < \mathrm{Re}(q)$ or $|s| = R$, $\mathrm{Re}(s) \geq 0$ and R is sufficiently large. If $0 \leq \mathrm{Re}(s) < \mathrm{Re}(q)$, then (49) holds because $|\rho| \leq 1$, $|\psi(s+v)| \leq 1$ and $|\phi(q-s)| < 1$. If $|s| = R$ and $\mathrm{Re}(s) \geq 0$, and if we divide both sides of (49) by R^m, and if we let $R \to \infty$, then the left-hand side tends to 0 whereas the right-hand side tends to 1. Thus (49) holds if R is sufficiently large. Therefore we can conclude that (49) cannot have a root either in the domain $0 \leq \mathrm{Re}(s) < \mathrm{Re}(q)$ or in the domain $|s| \geq R$, $\mathrm{Re}(s) \geq 0$ if R is sufficiently large. On the other hand, by Rouché's theorem it follows that (48) has the same number of roots as

$$(50) \qquad \prod_{i=1}^{m}(a_i + q-s) = 0$$

in the domain $|s| < R$, $\mathrm{Re}(s) > 0$ if R is large enough. If R is sufficiently large, then (50) has exactly m roots in this domain. Consequently, (48) has also m roots in the domain $\mathrm{Re}(s) > 0$.

Now in (45) we can write that

$$(51) \qquad \Phi^{+}(s+v,q+v,\rho) = \frac{\prod_{i=1}^{m}(a_i + q-s) - \rho\pi_{m-1}(q-s)\psi(s+v)}{\prod_{i=1}^{m}[\gamma_i(q+v,\rho)-s-v]}$$

for $\operatorname{Re}(s) \geq -\operatorname{Re}(v)$ and

(52) $$\Phi^{-}(s+v,q+v,\rho) = \prod_{i=1}^{m} \left(\frac{\gamma_i(q+v,\rho)-s-v}{a_i+q-s}\right)$$

for $\operatorname{Re}(s) \leq \operatorname{Re}(q)$.

If, in particular, $P\{\eta_0 = 0\} = 1$, then by (40) and (46) we obtain that

(53) $$[1-\rho\phi(q-s)\psi(s+v)]U(q,s,v,\rho) = \frac{\Phi^{-}(s+v,q+v,\rho)}{\Phi^{-}(v,q+v,\rho)} = \prod_{i=1}^{m} \left(\frac{1-\dfrac{s}{\gamma_i(q+v,\rho)-v}}{1-\dfrac{s}{a_i+q}}\right)$$

for $\operatorname{Re}(s+v) \geq 0$, $\operatorname{Re}(q) > 0$, $\operatorname{Re}(v) \geq 0$ and $|\rho| \leq 1$.

As a second example, let us assume that $\psi(s)$, the Laplace-Stieltjes transform of the distribution function of the service times is a rational function of s , that is, we assume that

(54) $$\psi(s) = \frac{\pi_{m-1}(s)}{\prod_{i=1}^{m} (a_i+s)}$$

where the right-hand side has the same properties as (47).

In a similar ways as before we can show that if $\operatorname{Re}(q) > 0$, $\operatorname{Re}(v) \geq 0$ and $|\rho| < 1$, then the equation

$$\prod_{i=1}^{m} (a_i+s+v) - \rho\pi_{m-1}(s+v)\phi(q-s) = 0 \tag{55}$$

has exactly m roots $s = \delta_i(q+v,\rho) - v$ $(i = 1,2,\ldots, m)$ in the domain $\mathrm{Re}(s) \leqq 0$.

In this case in (45) we can write that

$$\Phi^+(s+v,q+v,\rho) = \prod_{i=1}^{m} \left(\frac{\delta_i(q+v,\rho)-s-v}{a_i+ s + v}\right) \tag{56}$$

for $\mathrm{Re}(s) \geqq -\mathrm{Re}(v)$ and

$$\Phi^-(s+v,q+v,\rho) = \frac{\prod_{i=1}^{m} (a_i+s+v) - \rho\pi_{m-1}(s+v)\phi(q-s)}{\prod_{i=1}^{m} [\delta_i(q+v,\rho) - s - v]} \tag{57}$$

for $\mathrm{Re}(s) \leqq \mathrm{Re}(q)$.

If, in particular, $\mathbf{P}\{\eta_0 = 0\} = 1$, then by (40) and (46) we obtain that

$$[1-\rho\phi(q)\psi(v)]U(q,s,v,\rho) = \frac{\Phi^+(v,q+v,\rho)}{\Phi^+(s+v,q+v,\rho)} =$$

(58)

$$= \prod_{i=1}^{m} \left(1 + \frac{s}{a_i+ v}\right)\left(\frac{\delta_i(q+v,\rho) - v}{\delta_i(q+v,\rho) - s - v}\right)$$

for $\mathrm{Re}(s+v) \geqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(v) \geqq 0$ and $|\rho| \leqq 1$.

The following theorem has been found by the author [309].

Theorem 7. *If $b < a < \infty$ and if $F(x)$ is not a lattice distribution function, then the limiting distribution*

(59) $$\lim_{t \to \infty} \mathbf{P}\{\eta(t) \leq x\} = W^*(x)$$

exists, and $W^*(x)$ does not depend on the distribution of η_0. The function $W^*(x)$ is a proper distribution function and we have

(60) $$W^*(x) = (1-\frac{b}{a}) + \frac{b}{a} W(x) * H^*(x)$$

for $x \geq 0$ and $W^*(x) = 0$ for $x < 0$ where $W(x)$ is given in Theorem 2 and

(61) $$H^*(x) = \begin{cases} \frac{1}{b} \int_0^x [1-H(u)]du & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 . \end{cases}$$

Proof. Denote by $\tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots$ all those arrival times in the time interval $(0, \infty)$ when the arriving customer finds the server idle at his arrival. It is easy to see that $\tau_{n+1}^* - \tau_n^*$ $(n = 1,2,\ldots)$ are mutually independent and identically distributed positive random variables. Let $\mathbf{P}\{\tau_{n+1}^* - \tau_n^* \leq x\} = R(x)$ for $n = 1,2,\ldots$.

If $b < a < \infty$, then $R(\infty) = 1$ and

(62) $$\int_0^\infty x dR(x) = \int_0^\infty [1-R(x)]dx = \frac{a}{W(0)}$$

where $W(0) = e^{-M}$ is given in Theorem 2. For $\tau^*_{n+1} - \tau^*_n$ can be represented as a sum of a random number of interarrival times. Since $\lim_{n \to \infty} P\{\eta_n = 0\} = W(0) = e^{-M} > 0$, we can conclude that the number of terms in the above mentioned representation of $\tau^*_{n+1} - \tau^*_n$ is a proper random variable with a finite expectation $1/W(0)$. Each interarrival time has a finite expectation a. Thus by Theorem 6.1 in the Appendix, it follows that $E\{\tau^*_{n+1} - \tau^*_n\} = a/W(0)$. Furthermore, it is easy to see that if $b < a < \infty$, then $P\{\tau^*_1 < \infty\} = 1$ regardless of the distribution of η_0. We observe that if $F(x)$ is not a lattice distribution function, then $P\{\tau^*_{n+1} - \tau^*_n \leq x\}$ is neither a lattice distribution function.

If we denote by $\nu^*(t)$ the number of arrivals in the time interval $(0, t]$ when the arriving customer finds the server idle, then $\{\nu^*(t), 0 \leq t < \infty\}$ is a general recurrent process as we defined in Section 49. (See Note 49.1.) Let

$$m^*(t) = E\{\nu^*(t)\} \tag{63}$$

for $t \geq 0$.

Let us introduce the notation

$$Q^*(t, x) = P\{\eta(t) \leq x \text{ and } \nu^*(t) = 0\} \tag{64}$$

for the general queuing process and let us use the notation $Q(t, x)$ for $Q^*(t, x)$ in the particular case when $P\{\eta_0 = 0\} = 1$.

If we take into consideration that the event $\eta(t) \leq x$ can occur in such a way that $\nu^*(t) = 0,1,2,\ldots$, then we can write that

(65) $$\mathbf{P}\{\eta(t) \leq x\} = Q^*(t, x) + \int_0^t Q(t-u, x)dm^*(u) .$$

Since $Q^*(t, x) \leq \mathbf{P}\{\tau_1^* > t\}$ for any x, it follows that if $b < a$, then

(66) $$\lim_{t \to \infty} Q^*(t, x) = 0$$

for any x.

Now we shall show that for any x the function $Q(u, x)$ is of bounded variation in any finite interval $[0, t]$. The following proof is based on an idea of W. L. Smith [279]. Denote by $\nu(t)$ the number of arrivals in the interval $(0, t]$. Let us suppose that $\eta_0 = 0$, and let $\delta_t = 1$ if $\eta(t) \leq x$ and $\nu^*(t) = 0$, and $\delta_t = 0$ otherwise. Then we can write that

(67) $$Q(t, x) - Q(u, x) = \mathbf{E}\{\delta_t - \delta_u\} = \mathbf{P}\{\delta_u = 0, \delta_t = 1\} - \mathbf{P}\{\delta_u = 1, \delta_t = 0\}$$

for $0 \leq u \leq t$. Hence

(68) $$|Q(t, x) - Q(u, x)| \leq \mathbf{E}\{\delta_t - \delta_u\} + 2\mathbf{P}\{\delta_u = 1, \delta_t = 0\}$$

and obviously

(69) $$\mathbf{P}\{\delta_u = 1, \delta_t = 0\} \leq \mathbf{P}\{\nu(t) - \nu(u) \geq 1\} \leq \mathbf{E}\{\nu(t) - \nu(u)\} .$$

Accordingly, we have the following inequality

$$(70) \qquad |Q(t, x) - Q(u, x)| \leq \underset{\sim}{E}\{\delta_t - \delta_u\} + 2\underset{\sim}{E}\{\nu(t) - \nu(u)\}$$

for $0 \leq u \leq t$. By (70) we obtain that for any subdivision $t_0 = 0 < t_1 < \dots < t_n = t$ of the interval $[0, t]$ we have

$$(71) \qquad \sum_{k=1}^{n} |Q(t_k, x) - Q(t_{k-1}, x)| \leq \underset{\sim}{E}\{\delta_t - \delta_0\} + 2\underset{\sim}{E}\{\nu(t)\} \leq 1+2\underset{\sim}{E}\{\nu(t)\} .$$

Since $E\{\nu(t)\}$ is finite for any $t \geq 0$, it follows that $Q(u, x)$ is of bounded variation in any finite interval $[0, t]$.

If $b < a$, then for any x the function $Q(u, x)$ is integrable over $[0, \infty)$. This follows from (62) and from the inequality

$$(72) \qquad 0 \leq Q(u, x) \leq 1 - R(u) .$$

Finally, by Theorem 49.8 we can conclude that if $b < a < \infty$ and if $F(x)$ is not a lattice distribution function, then the limit

$$(73) \qquad \lim_{t \to \infty} \underset{\sim}{P}\{\eta(t) \leq x\} = W^*(x) = \frac{W(0)}{a} \int_0^\infty Q(u, x)du$$

exists regardless of the distribution of η_0 .

Since obviously $Q(u, x)$ is a nondecreasing function of x and

$$(74) \qquad \lim_{x \to \infty} Q(u, x) = 1 - R(u)$$

for $u \geq 0$, it follows from (62) and (73) that $\lim_{x \to \infty} W^*(x) = 1$.

It remains to prove (60) . Let

$$(75)\qquad \Omega^*(s) = \int_0^\infty e^{-sx}\, dW^*(x)$$

for $Re(s) \geq 0$. By an Abelian theorem of Laplace transforms (Theorem 9.10 in the Appendix) we obtain that

$$(76)\qquad \Omega^*(s) = \lim_{q \to +0} q \int_0^\infty e^{-qt}\, E\{e^{-s\eta(t)}\}dt$$

for $Re(s) \geq 0$ where the right-hand side can be obtained by (35). Since $\lim_{q \to +0} [1-\psi(q)]/q = b$ we obtain that

$$(77)\qquad \Omega^*(s) = 1-b \lim_{q\to +0} q\, U(q,q,0,1) + \frac{1-\psi(s)}{s} \lim_{q \to +0} q\, U(q,s,0,1)$$

for $Re(s) \geq 0$. In (77) $\Omega^*(s)$ does not depend on the distribution of η_0 therefore we may assume without loss of generality that $P\{\eta_0 = 0\} = 1$. If $P\{\eta_0 = 0\} = 1$, then by (40) we have

$$(78)\qquad U(q,s,0,1) = e^{-T\{\log[1-\phi(q-s)\psi(s)]\}} = e^{\sum_{n=1}^{\infty} \frac{1}{n} T\{[\phi(q-s)\psi(s)]^n\}}$$

for $Re(q) > 0$ and $Re(s) \geq 0$, and thus we can write that

$$(79)\qquad qU(q,s,0,1) = \frac{q}{1-\phi(q)}\, e^{-\sum_{n=1}^{\infty} \frac{1}{n}\{[\phi(q)]^n - T\{[\phi(q-s)\psi(s)]^n\}\}} .$$

If $b < a < \infty$ and if we let $q \to +0$ in (79) then we can form the limit term by term in the exponent because the series is uniformly convergent in q for $\mathrm{Re}(q) \geqq 0$. Since $\lim_{q \to +0} [1-\phi(q)]/q = a$, by (15) we obtain that

$$\lim_{q \to +0} q\, U(q,s,0,1) = \Omega(s)/a \tag{80}$$

for $\mathrm{Re}(s) \geqq 0$. If we use (80), then (77) can be expressed as

$$\Omega^*(s) = 1 - \frac{b}{a} + \frac{b}{a}\frac{[1-\psi(s)]}{bs}\Omega(s) \tag{81}$$

for $\mathrm{Re}(s) \geqq 0$. Since

$$\int_0^\infty e^{-sx}\, dH^*(x) = \frac{1-\psi(s)}{bs} \tag{82}$$

for $\mathrm{Re}(s) \geqq 0$ where the right-hand side is 1 for $s = 0$, we obtain (60) by (81). This completes the proof of the theorem.

We observe that by (60) we have

$$W^*(0) = 1 - \frac{b}{a}. \tag{83}$$

We note that if $a \leqq b < \infty$ and $\underset{\sim}{P}\{\xi_n = 0\} < 1$, then $\lim_{t \to \infty} \underset{\sim}{P}\{\eta(t) \leqq x\} = 0$ for every x regardless of the distribution of η_0. This can be deduced from the second part of Theorem 2.

Example. If we suppose that $F(x)$ is given by (25) and that $\lambda b < 1$, then by (26) we obtain that

$$\Omega(s) = (1-\lambda b) + \lambda\frac{[1-\psi(s)]}{s}\Omega(s). \tag{84}$$

Since $a = 1/\lambda$, by (81) we get

(85) $$\Omega^*(s) = \Omega(s)$$

for $\text{Re}(s) \geq 0$ and hence

(86) $$W^*(x) = W(x)$$

for all x.

In this particular case $\Omega^*(s)$ was found in 1930 by H. Cramér [365] in connection with a problem of insurance risk.

All the results which we obtained in this section can be proved in a simpler way if we restrict ourself to discrete queues in which the inter-arrival times and the service times are discrete random variables taking on positive integers only. See the author [322].

Now denote by $\xi(t)$ the queue size at time t and let $\xi_n = \xi(\tau_n - 0)$ for $n = 0, 1, 2, \ldots$, that is, the customer arriving at time τ_n finds exactly ξ_n customers in the system. If we know the limiting distribution of η_n as $n \to \infty$, then we can easily find the limiting distribution of ξ_n as $n \to \infty$. We have the following result.

Theorem 8. If $\underset{\sim}{P}\{\chi_0 = \tau_1\} < 1$ and

(87) $$M = \sum_{n=1}^{\infty} \frac{\underset{\sim}{P}\{\chi_0 + \ldots + \chi_{n-1} > \tau_n\}}{n}$$

is finite, then $\lim_{n \to \infty} \underset{\sim}{P}\{\xi_n \leq k\} = Q_k$ $(k = 0, 1, 2, \ldots)$ exists, is independent of the distribution of the initial queue size, and we have

(88) $$Q_k = \int_0^\infty [1 - F_{k+1}(x)]d[W(x) * H(x)]$$

where $F_{k+1}(x)$ denotes the k+1-st convolution of $F(x)$ with itself and $W(x)$ is given by Theorem 2. If $P\{\chi_0 = \tau_1\} < 1$, and $M = \infty$, then $\lim_{n\to\infty} P\{\xi_n \leq k\} = 0$ for all $k = 0,1,2,\ldots$ regardless of the distribution of the initial queue size.

Proof. The event $\xi_{n+k+1} \leq k$ occurs if and only if the customer who arrives at time τ_n departs before τ_{n+k+1}, that is, if and only if the queue size immediately after the departure of the customer arriving at time τ_n is $\leq k$. Thus for an arbitrary initial queue size ξ_0 we have

(89) $$P\{\xi_{n+k+1} \leq k\} = \int_0^\infty [1-F_{k+1}(x)]d[W_n(x) * H(x)]$$

where

(90) $$W_n(x) = P\{\eta_n \leq x\} .$$

For the queue size immediately after the departure of the customer arriving at time τ_n is equal to the number of arrivals during the waiting time and the service time of this customer, that is, the number of customers arriving in the interval $(\tau_n, \tau_n + \eta_n + \chi_n]$. Thus we obtain (89). If we let $n \to \infty$ in (89), then by Theorem 2 we obtain Theorem 8.

We note that if $b < a < \infty$ and if $F(x)$ is not a lattice distribution function, then $\lim_{t \to \infty} \underset{\sim}{P}\{\xi(t) \leq k\} = Q_k^*$ $(k = 0,1,2,\ldots)$ exists, is independent of the distribution of the initial queue size, and we have

(91) $$Q_k^* = 1 - \frac{b}{a} + \frac{b}{a}\int_0^\infty [1 - F_k(x)]d[W(x) * H^*(x)]$$

where $F_k(x)$ denotes the k-th iterated convolution of $F(x)$ with itself, $W(x)$ is given by Theorem 2 and $H^*(x)$ is defined by (61). If $a \leq b < \infty$ and $\underset{\sim}{P}\{\chi_0 = \tau_1\} < 1$, then $\lim_{t \to \infty} \underset{\sim}{P}\{\xi(t) \leq k\} = 0$ for all $k = 0,1,2,\ldots$ regardless of the distribution of $\xi(0)$. The proof of this last result can be found in reference [309].

The Stochastic Law of the Busy Periods. Let us suppose that in the queuing process defined at the beginning of this section the initial state is given by $\underset{\sim}{P}\{\xi(0) = 0\} = 1$ and $\underset{\sim}{P}\{\eta_0 = 0\} = 1$. Denote by $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ the lengths of the successive busy periods and by $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ the lengths of the successive idle periods. We can easily see that (σ_n, θ_n) $(n = 1,2,\ldots)$ is a sequence of mutually independent and identically distributed vector random variables.

In what follows we shall be concerned with the problem of finding the distribution function

(92) $$\underset{\sim}{P}\{\sigma_1 \leq x\ ,\ \theta_1 \leq y\} = G(x, y)\ .$$

We can write that

(93) $$G(x, y) = \sum_{n=1}^{\infty} G_n(x, y)$$

where $G_n(x, y)$ is the probability that $\sigma_1 \leq x$, $\theta_1 \leq y$ and the first busy period consists of n services.

Let us introduce the following notation

(94) $$\Gamma(w, s) = \int_0^\infty \int_0^\infty e^{-wx-sy} d_x d_y G(x, y)$$

and

(95) $$\Gamma_n(w, s) = \int_0^\infty \int_0^\infty e^{-wx-sy} d_x d_y G_n(x, y)$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(w) \geqq 0$.

Let

(96) $$\gamma_n = \chi_0 + \chi_1 + \dots + \chi_{n-1}$$

for $n = 1,2,\dots$ and $\gamma_0 = 0$. Furthermore, let us denote by $\delta(A)$ the indicator variable of any event A , that is,

(97) $$\delta(A) = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

The following result was found in 1952 by F. Pollaczek [227].

Theorem 9. We have

(98) $$\Gamma(w, s) = 1 - e^{-\sum_{n=1}^{\infty} \frac{1}{n} E\{e^{-w\gamma_n - s(\tau_n - \gamma_n)} \delta(\tau_n \geqq \gamma_n)\}}$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(w) \geqq 0$.

Proof. By definition we have

(99) $$G_n(x, y) = \underset{\sim}{P}\{\gamma_n \leqq x,\ \tau_1 \leqq \gamma_1, \ldots, \tau_{n-1} \leqq \gamma_{n-1},\ \gamma_n < \tau_n \leqq \gamma_n + y\}$$

for $n = 1,2,\ldots$. If we write $\xi_n = \chi_{n-1} - (\tau_n - \tau_{n-1})$ for $n = 1,2,\ldots,$ $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots,$ and $\zeta_0 = 0$, then we can also express (99) as follows:

(100) $$G_n(x, y) = \underset{\sim}{P}\{\gamma_n \leqq x,\ \zeta_1 \leqq 0, \ldots, \zeta_{n-1} \leqq 0,\ 0 < \zeta_n \leqq y\} .$$

By using the terminology of ladder indices, which we defined in Section 19, we can interpret $G_n(x, y)$ as the probability that in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ the first ladder index is n and $\gamma_n \leqq x$ and $\zeta_n \leqq y$.

Denote by $G_n^{(r)}(x, y)$ the probability that in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ the r-th ladder index is n and $\gamma_n \leqq x$ and $\zeta_n \leqq y$. Then $G_n^{(1)}(x, y) = G_n(x, y)$.

Let

(101) $$\Gamma_n^{(r)}(w, s) = \int_0^\infty \int_0^\infty e^{-wx-sy} d_x d_y G_n^{(r)}(x, y)$$

for $\text{Re}(s) \geqq 0$ and $\text{Re}(w) \geqq 0$. Then $\Gamma_n^{(1)}(w, s) = \Gamma_n(w, s)$.

We can easily see that

(102) $$\sum_{n=r}^{\infty} \Gamma_n^{(r)}(w, s)\rho^n = \left(\sum_{n=1}^{\infty} \Gamma_n(w, s)\rho^n \right)^r$$

for $\text{Re}(s) \geqq 0$, $\text{Re}(w) \geqq 0$ and $|\rho| \leqq 1$.

In a similar way as (19.8) or (19.19) we can prove that

$$\sum_{r=1}^{n} \frac{G_n^{(r)}(x, y)}{r} = \frac{1}{n} \underset{\sim}{P}\{\gamma_n \leq x,\ \gamma_n < \tau_n \leq \gamma_n + y\} \tag{103}$$

for $n = 1,2,\ldots$. Hence it follows that

$$\sum_{r=1}^{n} \frac{\Gamma_n^{(r)}(w, s)}{r} = \frac{1}{n} \underset{\sim}{E}\{e^{-w\gamma_n - s(\tau_n - \gamma_n)} \delta(\tau_n \geq \gamma_n)\} \tag{104}$$

for $n = 1,2,\ldots$. If we multiply (104) by ρ^n and add for $n = 1,2,\ldots,$ then we get

$$\sum_{r=1}^{\infty} \frac{1}{r} [\sum_{n=1}^{\infty} \Gamma_n(w, s)\rho^n]^r = \sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-w\gamma_n - s(\tau_n - \gamma_n)} \delta(\tau_n \geq \gamma_n)\} , \tag{105}$$

or equivalently,

$$\sum_{n=1}^{\infty} \Gamma_n(w, s)\rho^n = 1 - e^{-\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-w\gamma_n - s(\tau_n - \gamma_n)} \delta(\tau_n \geq \gamma_n)\}} \tag{106}$$

for $Re(s) \geq 0$, $Re(w) \geq 0$ and $|\rho| \leq 1$. If we put $\rho = 1$ in (106), then we obtain $\Gamma(w, s)$ and this proves (98).

In many cases the Laplace-Stieltjes transform $\Gamma(w, s)$ can easily be obtained by using the method of factorization.

We shall use the following relations. Let $\Psi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\} \in \underset{\sim}{R}$ and write $\Psi^+(s) = \underset{\sim}{T}\{\Psi(s)\} = \underset{\sim}{E}\{\zeta e^{-s\eta^+}\}$ and $\Psi^-(s) = \underset{\sim}{T}\{\Psi(-s)\} = \underset{\sim}{E}\{\zeta e^{-s[-\eta]^+}\}$ for $Re(s) \geq 0$. Then we have

$$\underset{\sim}{E}\{\zeta e^{-s\eta} \delta(\eta > 0)\} = \Psi^+(s) - \Psi^+(\infty) \tag{107}$$

for $\mathrm{Re}(s) \geq 0$ and

(108) $$\underset{\sim}{E}\{\zeta e^{-s\eta}\,\delta(\eta \geq 0)\} = \Psi^{+}(s) - \Psi^{-}(0) + \Psi^{-}(\infty)$$

for $\mathrm{Re}(s) \geq 0$.

<u>Theorem 10</u>. <u>If</u> $|\rho| \leq 1$, $\mathrm{Re}(w) > 0$, <u>and</u>

(109) $$1 - \rho\phi(-s)\psi(w+s) = \Phi^{+}(w+s,\ w,\rho)\Phi^{-}(w+s,w,\rho)$$

<u>for</u> $\mathrm{Re}(s) = 0$ <u>where</u> $\Phi^{+}(w+s,w,\rho)$ <u>and</u> $\Phi^{-}(w+s,w,\rho)$ <u>are defined for</u> $\mathrm{Re}(s) \geq 0$ <u>and</u> $\mathrm{Re}(s) \leq 0$ <u>respectively and satisfy the requirements stated after formula</u> (45), <u>then we have</u>

(110) $$\sum_{n=1}^{\infty} \Gamma_n(w,s)\rho^n = 1 - \Phi^{+}(\infty,w,\rho)\Phi^{-}(w-s,w,\rho)$$

<u>for</u> $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(w) > 0$ <u>and</u> $|\rho| \leq 1$.

<u>Proof</u>. First we note that the factorization (109) always exists. Let us define

(111) $$\Psi(s,w,\rho) = \log[1 - \rho\phi(-s)\psi(w+s)]$$

and write $\Psi^{+}(s,w,\rho) = \underset{\sim}{T}\{\Psi(s,w,\rho)\}$ and $\Psi^{-}(s,w,\rho) = \underset{\sim}{T}\{\Psi(-s,w,\rho)\}$ for $\mathrm{Re}(s) \geq 0$ where $\underset{\sim}{T}$ operates on the variable s .

By (106) and (108) we can write that

$$(112) \qquad \sum_{n=1}^{\infty} \Gamma_n(w,s)\rho^n = 1 - e^{\Psi^-(s,w,\rho)-\Psi^+(0,w,\rho)+\Psi^+(\infty,w,\rho)}$$

for $\mathrm{Re}(s) \geq 0$. Now by (46) we have

$$(113) \qquad \Psi^+(s,w,\rho) = \log\Phi^+(w+s,w,\rho) + \log\Phi^-(w,w,\rho)$$

for $\mathrm{Re}(s) \geq 0$ and

$$(114) \qquad \Psi^-(s,w,\rho) = \log\Phi^-(w-s,w,\rho) + \log\Phi^+(w,w,\rho)$$

for $\mathrm{Re}(s) \geq 0$.

Finally, by (112), (113) and (114) we get (110) which was to be proved.

Example. Let us suppose that $F(x)$ is given by (25), that is.

$$(115) \qquad \phi(s) = \frac{\lambda}{\lambda+s}$$

for $\mathrm{Re}(s) > -\lambda$.

If $\mathrm{Re}(w) > 0$ and $|\rho| \leq 1$, then the equation

$$(116) \qquad \lambda-s = \lambda\rho \ \psi(w+s)$$

has a single root

$$(117) \qquad s = \lambda[1 - \gamma(w,\rho)]$$

in the domain $\mathrm{Re}(s) \geq 0$ where $z = \gamma(w,\rho)$ is the only root of the equation

$$z = \rho\psi(w + \lambda - \lambda z) \tag{118}$$

in the circle $|z| < 1$. For (116) cannot have a root in the domain $\{s : |\lambda - s| > \lambda \text{ and } Re(s) \geq 0\}$ and by Rouché's theorem it follows that (116) has exactly one root in the circle $|\lambda - s| \leq \lambda$ which can be expressed in the form (117). Now we can choose

$$\Phi^{+}(w+s,w,\rho) = \frac{\lambda - s - \lambda\rho\ \psi(w+s)}{\lambda - s - \lambda\gamma(w,\rho)} \tag{119}$$

and

$$\Phi^{-}(w+s,w,\rho) = \frac{\lambda - s - \lambda\gamma(w,\rho)}{\lambda - s} \tag{120}$$

in (109) and by (110) we obtain that

$$\sum_{n=1}^{\infty} \Gamma_n(w,s)\rho^n = \frac{\lambda}{\lambda+s}\ \gamma(w,\rho) \tag{121}$$

for $Re(w) > 0$, $Re(s) \geq 0$ and $|\rho| \leq 1$. By expanding $\gamma(w,\rho)$ into Lagrange's series we get

$$\begin{aligned} \gamma(w,\rho) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^{n-1}\rho^n}{n!}\ \frac{\partial^{n-1}[\psi(w+\lambda)]^n}{\partial w^{n-1}} = \\ &= \sum_{n=1}^{\infty} \frac{\lambda^{n-1}\rho^n}{n!} \int_0^{\infty} e^{-(\lambda+w)x}\ x^{n-1}\ d\ H_n(x) \end{aligned} \tag{122}$$

for $Re(w) > 0$ and $|\rho| \leq 1$ where $H_n(x)$ denotes the n-th iterated convolution of $H(x)$ with itself.

If we interpret $\gamma(w,\rho)$ as that root in z of the equation (118) which has the smallest absolute value, then (121) is also valid for $\mathrm{Re}(w) \geq 0$, $\mathrm{Re}(s) \geq 0$ and $|\rho| \leq 1$, and $\gamma(w,\rho)$ is given by (122) for $\mathrm{Re}(w) \geq 0$ and $|\rho| \leq 1$. If b is defined by (22) and if $\lambda b \leq 1$, then $\gamma(0, 1) = 1$, whereas if $\lambda b > 1$, then $\gamma(0, 1) = \omega$ where $z = \omega$ is the only root of the equation

(123) $$z = \psi(\lambda - \lambda z)$$

in the unit circle $|z| < 1$. The root ω is real and satisfies $0 < \omega < 1$. (See the author [306].)

In a similar way as (122) we obtain by Lagrange's expansion that

(124) $$[\gamma(w,\rho)]^r = r \sum_{n=r}^{\infty} \frac{\lambda^{n-r} \rho^n}{n(n-r)!} \int_0^{\infty} e^{-(\lambda+w)x} x^{n-r} \, dH_n(x)$$

for $r = 1,2,\ldots,$ $\mathrm{Re}(w) \geq 0$ and $|\rho| \leq 1$.

By (121) and (122) we have

(125) $$\Gamma_n(w,s) = \frac{\lambda}{\lambda+s} \int_0^{\infty} e^{-(\lambda+w)x} \frac{(\lambda x)^{n-1}}{n!} \, dH_n(x)$$

for $n = 1,2,\ldots,$ $\mathrm{Re}(w) \geq 0$ and $\mathrm{Re}(s) \geq 0$. Hence it follows that

(126) $$G_n(x, y) = G_n(x) \, F(y)$$

for $x \geq 0$ and $y \geq 0$ where

(127) $$G_n(x) = \frac{\lambda^{n-1}}{n!} \int_0^{x} e^{-\lambda u} u^{n-1} \, dH_n(u)$$

is the probability that a busy period has length $\leq x$ and consists of n services, and

$$F(y) = 1 - e^{-\lambda y} \tag{128}$$

is the probability that and idle period has length $\leq y$.

The probability that a busy has length $\leq x$ is given by

$$G(x) = \sum_{n=1}^{\infty} G_n(x) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^x e^{-\lambda u} u^{n-1} \, dH_n(u) \tag{129}$$

for $x \geq 0$. If

$$\gamma(w) = \int_0^\infty e^{-wx} \, dG(x) \tag{130}$$

for $Re(w) \geq 0$, then by (122) we have

$$\gamma(w) = \gamma(w, 1) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-(\lambda+w)x} x^{n-1} \, dH_n(x) \tag{131}$$

for $Re(w) \geq 0$, and $z = \gamma(w)$ can also be interpreted as that root of the equation

$$z = \psi(w + \lambda - \lambda z) \tag{132}$$

which has the smallest absolute value.

If $\lambda b \leq 1$, then $G(\infty) = \gamma(0) = \gamma(0, 1) = 1$, that is, $G(x)$ is a proper distribution function. If $\lambda b > 1$, then $G(\infty) = \gamma(0) = \gamma(0, 1) = \omega < 1$; that is, the length of a busy period may be infinite with probability $1 - \omega$.

If $G^{(r)}(x)$ denotes the r-th iterated convolution of $G(x)$ with itself, then we have

$$(133) \qquad G^{(r)}(x) = \sum_{n=r}^{\infty} \frac{r}{n} \int_0^x e^{-\lambda u} \frac{(\lambda u)^{n-r}}{(n-r)!} dH_n(u)$$

for $x \geq 0$ and $r = 1,2,\ldots$. This follows from (124) by inversion. (See also Probelm 65. 5.)

We note that if $\phi(s)$ is given by (47), then in Theorem 10, $\Phi^{+}(\infty,w,\rho)$ and $\Phi^{-}(w-s,w,\rho)$ can be obtained by (51) and (52) respectively, and if $\psi(s)$ is given by (54), then in Theorem 10, $\Phi^{+}(\infty,w,\rho)$ and $\Phi^{-}(w-s,w,\rho)$ can be obtained by (56) and (57) respectively.

In the theory of queues it has some importance to find the distribution of the maximal queue size during a busy period and the distribution of the maximal waiting time during a busy period. In what follows we shall consider only single-server queues with Poisson input and general service times. See the author [319], [324].

Contrary to our previous definition we assume here that no customer arrives at time $\tau_0 = 0$ in the queing process. We assume that customers arrive at a counter in the time interval $(0,\infty)$ in accordance with a Poisson process of density λ and are served by a single server in the order of arrival. The service times are mutually independent and identically distributed positive random variables with distribution function $H(x)$ and independent of the arrival times.

The initial state of the process is given either by $\xi(0)$, the initial queue size, or by η_0 , the initial occupation time of the server at time $t = 0$. We suppose either that $\xi(0) = i$ where i is a nonnegative integer or that $\eta_0 = c$ where c is a nonnegative constant.

We denote by σ_0 the length of the initial busy period. If $\xi(0) = 0$ or $\eta_0 = 0$, then $\sigma_0 = 0$. Otherwise σ_0 is a positive random variable which may be ∞ with a positive probability.

In what follows we shall determine the probabilities

$$(134) \qquad P(k,y|i) = \underset{\sim}{P}\{ \sup_{0 \leq t \leq \sigma_0} \xi(t) \leq k \; ; \; \sigma_0 \leq y | \xi(0) = i\}$$

for $0 \leq i \leq k$ and

$$(135) \qquad G(x,y|c) = \underset{\sim}{P}\{ \sup_{0 \leq t \leq \sigma_0} \eta(t) \leq x ; \sigma_0 \leq y | \eta_0 = c\}$$

for $0 \leq c \leq x$.

In (134) $P(k,y|i)$ is the probability that the maximal queue size during the initial busy period is $\leq k$ and the initial busy period has length $\leq y$ given that the initial queue size is i , and in (135) $G(x,y|c)$ is the probability that the maximal virtual waiting time during the initial busy period is $\leq x$ and the initial busy period has length $\leq y$ given that the initial occupation time of the server is c .

If we know these probabilities for the initial busy period, then we can obtain immediately the corresponding probabilities for any other busy period.

For the probability that the maximal queue size during any busy period other than the initial one is $\leq k$ and the length of the busy period is $\leq y$ is evidently

$$P(k, y) = P(k,y|1) , \tag{136}$$

and the probability that the maximal virtual waiting time during any busy period other than the initial one is $\leq x$ and the length of the busy period is $\leq y$ is evidently

$$G(x, y) = \int_0^x G(x,y|c)dH(c) . \tag{137}$$

By knowing (134) and (135) we can easily determine the corresponding probabilities for the initial busy period of the queuing process defined at the beginning of this section.

In what follows we shall determine the Laplace-Stieltjes transforms

$$\Pi(k,s|i) = \int_0^\infty e^{-sy} d_y P(k,y|i) \tag{138}$$

for $0 \leq i \leq k$ and $\text{Re}(s) \geq 0$, and

$$\Gamma(x,w|c) = \int_0^\infty e^{-wy} d_y G(x,y|c) \tag{139}$$

for $0 \leq c \leq x$ and $\text{Re}(w) \geq 0$.

We introduce the notation

$$\psi(s) = \int_0^\infty e^{-sx} dH(x) \tag{140}$$

for $\text{Re}(s) \geq 0$ and

(141) $$\pi_j(s) = \frac{1}{j!}\int_0^\infty e^{-sx-\lambda x}(\lambda x)^j dH(x)$$

for $\mathrm{Re}(s) \geq 0$ and $j = 0,1,2,\ldots$.

The generating function of $\pi_j(s)$ $(j = 0,1,2,\ldots)$ is given by

(142) $$\sum_{j=0}^\infty \pi_j(s)z^j = \int_0^\infty e^{-sx-\lambda(1-z)x}\, dH(x) = \psi(s+\lambda-\lambda z)$$

for $\mathrm{Re}(s) \geq 0$ and $|z| \leq 1$.

Denote by $z = \gamma(s)$ that root of the equation

(143) $$\psi(s + \lambda - \lambda z) = z$$

which has the smallest absolute value. We have $|\gamma(s)| \leq 1$ for $\mathrm{Re}(s) \geq 0$ and $|\gamma(s)| < 1$ for $\mathrm{Re}(s) > 0$.

Theorem 11. If $0 \leq i \leq k$ and $\mathrm{Re}(s) \geq 0$, then we have

(144) $$\Pi(k,s|i) = \frac{Q_{k-i}(s)}{Q_k(s)}$$

where

(145) $$\sum_{k=0}^\infty Q_k(s)z^k = \frac{\psi(s + \lambda - \lambda z)}{\psi(s + \lambda - \lambda z) - z}$$

for $\mathrm{Re}(s) \geq 0$ and $|z| < |\gamma(s)|$.

Proof. First, we observe that

$$P(k,y|k-i) = \int_0^y P(k,y-u|k-j)d_u\, P(j,u|j-i) \tag{146}$$

for $0 \leq i \leq j \leq k$ and $y \geq 0$. If $j = i$, then (146) is obvious because $P(j,u|0) = 1$ for $u \geq 0$. Let $j > i$. In (146), $P(k,y|k-i)$ is the probability that the maximal queue size during the initial busy period is $\leq k$ and the initial busy period has length $\leq y$ given that the initial queue size is $k-i$. This latter event can occur in several mutually exclusive ways: The queue size decreases from $k-j+1$ to $k-j$ for the first time at time u where $0 \leq u \leq y$. The probability that the first transition $k-j+1 \to k-j$ occurs in the interval $(0, u]$ is $P(j,u|j-i)$. For obviously this probability is the same as the probability that in a queuing process the maximal queue size during the initial busy period is $\leq j$ and the initial busy period has length $\leq u$ given that the initial queue size is $j-i$. On the other hand if we measure time from a transition $k-j+1 \to k-j$, then the future behavior of the queuing process is independent of the past and is the same as that of a queuing process with initial queue size $k-j$. On account of these considerations we obtain (146).

If we form the Laplace- Stieltjes transform of (146), then we obtain that

$$\Pi(k,s|k-i) = \Pi(k,s|k-j)\, \Pi(j,s|j-i) \tag{147}$$

for $0 \leq i \leq j \leq k$ and $Re(s) \geq 0$. We note that $\Pi(k,s|0) \equiv 1$. Since $\Pi(k,s|i) \neq 0$ for $0 \leq i \leq k$ and $Re(s) \geq 0$, it follows from (147) that $\Pi(k,s|i)$ can be expressed in the following form

$$\Pi(k,s|i) = \frac{Q_{k-i}(s)}{Q_k(s)} \tag{148}$$

for $0 \leq i \leq k$ and $\mathrm{Re}(s) \geq 0$ where $Q_0(s) \equiv 1$ and $Q_k(s) \neq 0$ for $k \geq 0$ and $\mathrm{Re}(s) \geq 0$.

If we take into consideration that during the first service time in the initial busy period the number of arrivals may be $j = 0,1,2,\ldots,$ then we can write that

$$(149) \qquad \Pi(k+i,s|i) = \sum_{j=0}^{k} \pi_j(s)\Pi(k+i,s|i+j-1)$$

for $i \geq 1$ and $k \geq 0$. If we multiply (149) by $Q_{k+i}(s)$, and if we use (148), then we get the following recurrence formula:

$$(150) \qquad Q_k(s) = \sum_{j=0}^{k} \pi_j(s)\, Q_{k+1-j}(s)$$

for $k = 0,1,2,\ldots$ and $\mathrm{Re}(s) \geq 0$. If we introduce generating functions, then by (142) we obtain (145) for $\mathrm{Re}(s) \geq 0$ and $|z| < |\gamma(s)|$. We can express $Q_k(s)$ explicitly as a polynomial of $1/\pi_0(s)$ and $\pi_j(s)$ $(j = 1,2,\ldots,k-1)$. Knowing $Q_k(s)$, we can determine $P(k,y|i)$ by inversion.

Theorem 12. If $0 \leq c \leq x$ and $\mathrm{Re}(w) \geq 0$, then we have

$$(151) \qquad \Gamma(x,w|c) = \frac{W(x-c,w)}{W(x,w)}$$

where

$$(152) \qquad \int_0^{\infty} e^{-sx}\, d_x W(x,w) = \frac{s}{s - w - \lambda[1 - \psi(s)]}$$

for $\mathrm{Re}(s) > \mathrm{Re}\{w + \lambda[1-\gamma(w)]\}$.

Proof. In this case the process $\{\eta(t), 0 \leq t < \infty\}$ is a homogeneous Markov process and thus we obtain easily that

$$\Gamma(x, w|x-c) = \Gamma(y,w|y-c)\Gamma(x,w|x-y) \tag{153}$$

for $0 \leq c \leq y \leq x$ and $\mathrm{Re}(w) \geq 0$. Since $\Gamma(x,w|c) \neq 0$ for $0 \leq c \leq x$ and $\mathrm{Re}(w) \geq 0$, it follows that $\Gamma(x,w|c)$ can be represented in the following form:

$$\Gamma(x,w|c) = \frac{W(x-c, w)}{W(x,w)} \tag{154}$$

for $0 \leq c \leq x$ and $\mathrm{Re}(w) \geq 0$ where $W(0, w) \equiv 1$, and $W(x, w) \neq 0$ for $x \geq 0$ and $\mathrm{Re}(w) \geq 0$.

If we take into consideration that in the time interval $(0, u)$ one customer arrives with probability $\lambda u + o(u)$, and more than one customer arrives with probability $o(u)$, where $\lim_{u \to 0} o(u)/u = 0$, then we can write for $x \geq 0$ and $y \geq 0$ that

$$\Gamma(x+y,w|y) = (1-\lambda u)e^{-wu}\Gamma(x+y,w|y-u)+\lambda u \int_0^x \Gamma(x+y,w|y+z)dH(z)+ o(u) . \tag{155}$$

If we multiply (155) by $W(x+y,w)$, then we obtain that

$$W(x,w) = (1-\lambda u)e^{-wu} W(x+u,w)+\lambda u \int_0^x W(x-z,w)dH(z)+ o(u) \tag{156}$$

for $x \geq 0$ and $\mathrm{Re}(w) \geq 0$. From (156) it follows that

$$\frac{\partial W(x,w)}{\partial x} = (\lambda+w)\, W(x,w) - \lambda \int_0^x W(x-z ,w)dH(z) \tag{157}$$

for $x > 0$ and $\mathrm{Re}(w) \geq 0$. Let

(158) $$\Omega(s,w) = \int_0^\infty e^{-sx} \, d_x \, W(x, w)$$

for $\mathrm{Re}(w) \geq 0$. If we form the Laplace-Stieltjes transform of (157), then we obtain that

(159) $$s[\Omega(s,w) - W(0,w)] = (\lambda+w)\Omega(s,w) - \lambda\Omega(s,w)\psi(s)$$

for $\mathrm{Re}(w) \geq 0$ and $\mathrm{Re}(s) > \mathrm{Re}\{w+\lambda[1-\gamma(w)]\}$. In (159), $W(0,w) \equiv 1$, and this implies that $\Omega(s,w)$ is equal to the right-hand side of (152). This completes the proof of the theorem. Theorem 12 makes it possible to determine the probability $G(x,y|c)$.

By using Theorem 12 we can also determine $G(x, y)$ defined by (137). Let

(160) $$\Gamma(x,w) = \int_0^\infty e^{-wy} \, d_y \, G(x,y)$$

for $x \geq 0$ and $\mathrm{Re}(w) \geq 0$. By (137), (151), and (157) we obtain easily that

(161) $$\Gamma(x,w) = 1 + \frac{w}{\lambda} - \frac{1}{\lambda} \frac{\partial \log W(x,w)}{\partial x}$$

for $x > 0$ and $\mathrm{Re}(w) \geq 0$.

<u>The Distribution of the Occupation Time.</u> We shall consider the queuing process defined at the beginning of this section, and we shall give some

methods for finding the distribution and the asymptotic distribution of the total occupation time of the server in the time interval $(0, t)$. See references [291], [314], [321], [469]. Accordingly, we suppose that in the time interval $[0, \infty)$ customers arrive at a counter at times $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ where $\tau_0 = 0$ and $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots)$ are mutually independent and identically distributed positive random variables with distribution function $F(x)$. The customers are served by a single server. Denote by χ_n the service time of the customer arriving at time τ_n. We assume that $\chi_0, \chi_1, \ldots, \chi_n, \ldots$ are mutually independent and identically distributed positive random variables with distribution function $H(x)$ and independent of $\{\tau_n\}$. The initial state is given by η_0, the occupation time of the server at time $t = 0$, where η_0 is a nonnegative random variable which is independent of $\{\tau_n\}$ and $\{\chi_n\}$.

Denote by $\sigma(t)$ the total occupation time of the server in the time interval $(0, t)$ and by $\theta(t)$ the total idle time of the server in the time interval $(0, t)$. We have $\sigma(t) + \theta(t) = t$ for all $t \geq 0$. We are interested in determining the distribution and the asymptotic distribution of $\sigma(t)$.

If η_0 is an arbitrary nonnegative random variable, then

$$(162) \qquad \int_0^\infty e^{-qt} \mathbf{E}\{e^{-v\sigma(t)}\}dt$$

is given by (37) for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(v) \geq 0$. Hence $\mathbf{P}\{\sigma(t) \leq x\}$ can be obtained by inversion.

The probability $\underset{\sim}{P}\{\sigma(t) \leq x\} = \underset{\sim}{P}\{\theta(t) \geq t-x\}$ can also be obtained by Theorem 59.1 for $0 < x \leq t$.

Denote by $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ and $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ the lengths of the successive busy periods and idle periods respectively. If $\underset{\sim}{P}\{\eta_0 = 0\} = 1$, then (σ_n, θ_n) $(n = 1,2,\ldots)$ are mutually independent and identically distributed vector random variables whose distribution function $\underset{\sim}{P}\{\sigma_n \leq x ,$ $\theta_n \leq y\} = G(x, y)$ can be obtained by Theorem 9. In this case the probability $\underset{\sim}{P}\{\sigma(t) \leq x\}$ is completely determined by $G(x, y)$ as it can be seen from Theorem 59.1 .

If $\underset{\sim}{P}\{\eta_0 = 0\} = 1$ and if $G(x, y)$ belongs to the domain of attraction of a nondegenerate, two-dimensional, stable distribution function, then there exist a nondegenerate distribution function $R(x)$ and normalizing functions $M_1(t)$ and $M_2(t)$ such that $M_2(t) \to \infty$ as $t \to \infty$, and

(163) $$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\sigma(t) - M_1(t)}{M_2(t)} \leq x \right\} = R(x)$$

in every continuity point of $R(x)$. In many important cases $R(x)$ can be obtained by Theorem 59.2 . We can easily prove that (163) remains valid unchangeably if η_0 has an arbitrary distribution function.

The limiting distribution (163) can be determined in a simple way if

(164) $$a = \int_0^\infty x dF(x)$$

and

(165) $$b = \int_0^\infty x dH(x)$$

exist and $b < a$.

Let us define

(166) $$\chi(t) = \sum_{0<\tau_i \leq t} \chi_i$$

for $t \geq 0$. Then $\{\chi(t), 0 \leq t < \infty\}$ is a compound recurrent process as we defined in Section 49 (Definition 2).

Now we can write that

(167) $$\sigma(t) = \eta_0 + \chi_0 + \chi(t) - \eta(t)$$

for $t \geq 0$.

From (167) we can conclude that if $b < a$ and if

(168) $$\lim_{t \to \infty} P\{ \frac{\chi(t) - M_1(t)}{M_2(t)} \leq x\} = R(x)$$

in every continuity point of $R(x)$ where $M_1(t)$ and $M_2(t)$ are appropriate normalizing functions for which $M_2(t) \to \infty$ as $t \to \infty$, then

(169) $$\lim_{t \to \infty} P\{ \frac{\sigma(t) - M_1(t)}{M_2(t)} \leq x\} = R(x)$$

also holds in every continuity point of $R(x)$. This follows from the fact that if a and b exist and $b < a$, then $\eta(t)/M_2(t) \Rightarrow 0$ as $t \to \infty$ whenever $M_2(t) \to \infty$ as $t \to \infty$. (See Theorem 2 and Theorem 7.)

In many cases the limiting distribution (168) can be obtained by Theorem 45.2 . See also formula (49.205).

<u>Theorem 13.</u> <u>If</u> $0 < b < a$ <u>and</u>

(170) $$\lim_{n \to \infty} \mathbf{P}\left\{ \frac{\tau_n - na}{a_2 n^\alpha} \leq x \right\} = \mathbf{P}\{\tau \leq x\}$$

and

(171) $$\lim_{n \to \infty} \mathbf{P}\left\{ \frac{\chi_1 + \dots + \chi_n - nb}{b_2 n^\beta} \leq x \right\} = \mathbf{P}\{\chi \leq x\}$$

in all continuity points of the limiting distribution functions where τ and χ are independent random variables, then

(172) $$\lim_{t \to \infty} \mathbf{P}\left\{ \frac{\sigma(t) - M_1 t}{M_2 t^\mu} \leq x \right\} = R(x)$$

in every continuity point of $R(x)$ where $R(x)$, M_1, M_2 and μ are given in Table I.

TABLE I

(α, β)	M_1	M_2	μ	$R(x)$
$\alpha > \beta$	b/a	$ba_2/a^{1+\alpha}$	α	$\mathbf{P}\{-\tau \leq x\}$
$\alpha = \beta$	b/a	1	β	$\mathbf{P}\{b_2 a^{-\beta}\chi - ba_2 a^{-(1+\beta)}\tau \leq x\}$
$\alpha < \beta$	b/a	b_2/a^β	β	$\mathbf{P}\{\chi \leq x\}$

Proof. If (170) and (171) are satisfied, then the asymptotic distribution of $\chi(t)$ as $t \to \infty$ is given by (49.205) where the normalizing functions and $R(x)$ are given in Table II. Thus we get (168) whence (163) immediately follows.

<u>Example.</u> Let us suppose that a and b are finite positive numbers satisfying the inequality $b < a$ and let

(173) $$\sigma_a^2 = \int_0^\infty (x - a)^2 dF(x)$$

and

(174) $$\sigma_b^2 = \int_0^\infty (x - b)^2 dH(x)$$

be finite numbers for which $\sigma_a^2 + \sigma_b^2 > 0$.

In this case (170) and (171) are satisfied with $a_2 = \sigma_a$, $b_2 = \sigma_b$, $\alpha = \beta = \frac{1}{2}$ and $P\{\tau \leq x\} = P\{\chi \leq x\} = \Phi(x)$ where $\Phi(x)$ is the normal distribution function. Thus by Theorem 13 we obtain that

(175) $$\lim_{t \to \infty} P\left\{\frac{\sigma(t) - M_1 t}{M_2 t^{1/2}} \leq x\right\} = \Phi(x)$$

where

(176) $$M_1 = \frac{b}{a}$$

and

(177) $$M_2 = \sqrt{(a^2\sigma_b^2 + b^2\sigma_a^2)/a^3}$$

We note that if we know either $P\{\sigma_n \leq x , \theta_n \leq y\} = G(x, y)$ or $E\{\sigma_n\}$, $E\{\theta_n\}$, $Var\{\sigma_n\}$, $Var\{\theta_n\}$ and $Cov\{\sigma_n, \theta_n\}$, then (175) can also be obtained by Theorem 59.2 . (See Problem 65.9.)

Now we shall mention another approach for finding the limiting distribution (163) in the case when the expectations a and b are finite positive numbers and $a = b$.

Write $\bar{\xi}_n = (\tau_n - \tau_{n-1}) - \chi_n$ for $n = 1,2,\ldots$, $\bar{\zeta}_0 = 0$, and $\bar{\zeta}_n = \bar{\xi}_1 + \bar{\xi}_2 + \ldots + \bar{\xi}_n$ for $n = 1,2,\ldots$. In Section 44 we proved that if $\mathbf{P}\{\bar{\xi}_n \leq x\}$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha,\beta,c,0)$ where $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $c > 0$, then there exists a positive function $\rho(t)$ defined for $t > 0$ such that

$$(178) \qquad \lim_{n \to \infty} \mathbf{P}\left\{ \frac{\bar{\zeta}_n}{n^{1/\alpha}\rho(n)} \leq x \right\} = R(x)$$

and

$$(179) \qquad \lim_{t \to \infty} \frac{\rho(\omega t)}{\rho(t)} = 1$$

for all $\omega > 0$. The function $\rho(t)$ can be chosen by Theorem 44.6 and Theorem 44.8 .

If $\bar{\eta}(n) = \max(\bar{\zeta}_0, \bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_n)$ for $n = 1,2,\ldots$, then by Theorem 45.10 it follows that

$$(180) \qquad \lim_{n \to \infty} \mathbf{P}\left\{ \frac{\bar{\eta}(n)}{n^{1/\alpha}\rho(n)} \leq x \right\} = Q(x)$$

where

(181) $$Q(x) = \underset{\sim}{P}\{ \sup_{0 \leq u \leq 1} \xi(u) \leq x\}$$

and $\{\xi(u), 0 \leq u \leq 1\}$ is a separable stable process of type $S(\alpha, \beta, c, 0)$.

By the above results we can easily find the asymptotic distribution of $\theta(t) = t - \sigma(t)$ as $t \to \infty$.

Theorem 14. If $a = b$ is a finite positive number and $\underset{\sim}{P}\{(\tau_n - \tau_{n-1}) - \chi_n \leq x\}$ belongs to the domain of attraction of a stable distribution function of type $S(\alpha, \beta, c, 0)$ where $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $c > 0$, then there exists a positive function $\rho(t)$ defined for $t > 0$ which satisfies (179) such that

(182) $$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{a^{1/\alpha}\theta(t)}{t^{1/\alpha}\rho(t)} \leq x\right\} = Q(x)$$

where $Q(x)$ is defined by (181).

Proof. First we observe that

(183) $$\theta(t) = \sup\{0 \text{ and } u - \chi(u) - \eta_0 - \chi_0 \text{ for } 0 \leq u \leq t\}$$

for $t \geq 0$ where $\chi(u)$ is defined by (166). Hence we can easily see that $\theta(t)$ has the same asymptotic distribution as

(184) $$\sup_{0 \leq u \leq t} [u - \chi(u)]$$

regardless of the distribution of η_0. On the other hand if we denote by $\nu(t)$ the number of arrivals in the time interval $(0, t]$, then (184) has the same asymptotic distribution as $\overline{\eta}(\nu(t))$. Since by the weak law of large numbers

(185) $$\frac{\nu(t)}{t} \Rightarrow \frac{1}{a}$$

as $t \to \infty$, (Problem 53.1), it follows from (180) that

(186) $$\lim_{t\to\infty} \mathbf{P}\left\{ \frac{a^{1/\alpha}}{t^{1/\alpha}} \frac{\bar{\eta}(\nu(t))}{\rho(t/a)} \leq x \right\} = Q(x) .$$

To prove (186) we can use the same method as we used in proving Theorem 45.4. By (179) $\rho(t)/\rho(t/a) \to 1$ as $t \to \infty$, and hence (186) implies (182).

Examples. First, let us suppose that $a = b$ is a finite positive number and $0 < \sigma_a^2 + \sigma_b^2 < \infty$ where σ_a^2 and σ_b^2 are defined by (173) and (174) respectively. In this case (178) is satisfied with $\alpha = 2$, $\rho(n) = \sqrt{\sigma_a^2 + \sigma_b^2}$, and $R(x) = \Phi(x)$ where $\Phi(x)$ is the normal distribution function, that is, $R(x)$ is a stable distribution function of type $S(2,0,\frac{1}{2},0)$. Now by Theorem 45.6 or by (45.220) we can conclude that (180) holds with

(187) $$Q(x) = \begin{cases} 2\Phi(x) - 1 & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 . \end{cases}$$

Thus by Theorem 14 it follows that

(188) $$\lim_{t\to\infty} \mathbf{P}\left\{ \frac{\theta(t)\sqrt{a}}{\sqrt{(\sigma_a^2 + \sigma_b^2)t}} \leq x \right\} = Q(x)$$

where $Q(x)$ is given by (187).

Second let us suppose that $a = b$ is a finite positive number,

$$(189) \qquad \lim_{x \to \infty} x^{\alpha}[1 - H(x)] = h$$

where h is a positive number, $1 < \alpha < 2$, and

$$(190) \qquad \lim_{x \to \infty} x^{\alpha}[1 - F(x)] = 0 .$$

In this case

$$(191) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{na-(\chi_1+\dots+\chi_n)}{(nh)^{1/\alpha}} \leqq x\right\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha,-1,\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2},0)$. Since by (190)

$$(192) \qquad \frac{\tau_n - na}{n^{1/\alpha}} \Rightarrow 0$$

as $n \to \infty$, it follows from (191) that (178) is satisfied with $\rho(n) = h^{1/\alpha}$, with the above α, and with the above $R(x)$. Consequently, (180) also holds and

$$(193) \qquad Q(x) = \begin{cases} 1 - \dfrac{1-R(x)}{1-R(0)} & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

where $R(0) = (\alpha-1)/\alpha$. This follows from (45.223) or from (56.38). Thus by Theorem 14 we obtain that

$$(194) \qquad \lim_{t \to \infty} \underset{\sim}{P}\left\{\frac{a^{1/\alpha}\theta(t)}{(th)^{1/\alpha}} \leqq x\right\} = Q(x)$$

where $Q(x)$ is given by (193) and $R(x)$ is a stable distribution function of type $S(\alpha,-1,\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}, 0)$. We note that we can also write that

(195) $$Q(x) = G_{1/\alpha}\left(\frac{x}{(-\Gamma(1-\alpha))^{1/\alpha}}\right)$$

where the distribution function $G_\alpha(x)$ is defined by (42.178) for $0 < \alpha < 1$. According to (42.178) we have

(196) $$G_{1/\alpha}(x) = 1 - R(x^{-\alpha}; \frac{1}{\alpha}, 1, \cos\frac{\pi}{2\alpha}, 0)$$

for $x \geqq 0$ and $\alpha > 1$ where on the right-hand side we have a stable distribution function of type $S(\frac{1}{\alpha}, 1, \cos\frac{\pi}{2\alpha}, 0)$. The representation (195) follows from (42.184) and (42.192). For by these formulas we have

(197) $$G_{1/\alpha}(x) = \alpha[R(x ; \alpha,-1,-\cos\frac{\alpha\pi}{2}, 0) - \frac{\alpha-1}{\alpha}]$$

for $x \geqq 0$ and $1 < \alpha \leqq 2$ where on the right-hand side we have a stable distribution function of type $S(\alpha,-1,-\cos\frac{\alpha\pi}{2}, 0)$.

We note that if in the last example instead of (189) we assume that

(198) $$x^\alpha[1 - H(x)] = h(x)$$

where $1 < \alpha < 2$ and

(199) $$\lim_{x\to\infty} \frac{h(\omega x)}{h(x)} = 1,$$

for $\omega > 0$, then we have

(200) $$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{na - (\chi_1+\ldots+\chi_n)}{n^{1/\alpha} \rho(n)} \leq x\} = R(x)$$

where $R(x)$ has the same meaning as in (191) and $\rho(t)$ can be chosen in such a way that

(201) $$\lim_{t \to \infty} t[1 - H(t^{1/\alpha}\rho(t))] = 1 .$$

In a similar way as in the last example, (200) implies that

(202) $$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{a^{1/\alpha}\theta(t)}{t^{1/\alpha}\rho(t)} \leq x\} = Q(x)$$

where $Q(x)$ is given by (193).

In conclusion, we shall mention some results for single-server queues with Poisson input and general service times. For simplicity we assume that the initial occupation time of the server is 0 and no customer arrives at time $t = 0$. We assume that customers arrive at a counter in the time interval $(0, \infty)$ in accordance with a Poisson process of denisty λ and are served by a single server. The service times are mutually independent and identically distributed positive random variables with distribution function $H(x)$ and independent of the arrival times.

In this case the lengths of the successive idle periods, $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ and the successive busy periods, $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ are independent sequences of mutually independent and identically distributed positive random variables. We have

(203) $$\underset{\sim}{P}\{\theta_n \leq x\} = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

and

$$P\{\sigma_n \leq x\} = G(x) \tag{204}$$

where $G(x)$ is given by (129).

Now by (59.9) we obtain that

$$P\{\sigma(t) \leq x\} = \sum_{r=0}^{\infty} e^{-\lambda(t-x)} \frac{[\lambda(t-x)]^r}{r!} G^{(r)}(x) \tag{205}$$

for $0 \leq x < t$ where $G^{(r)}(x)$ denotes the r-th iterated convolution of $G(x)$ with itself and $G^{(0)}(x) = 1$ for $x \geq 0$. For $r \geq 1$ the distribution function $G^{(r)}(x)$ is given by (133).

The distribution function (205) can also be obtained in the following way. Denote by $\chi(u)$ the total service time of all those customers who arrive in the time interval $(0, u]$. Then $\{\chi(u), 0 \leq u < \infty\}$ is a compound Poisson process for which

$$P\{\chi(u) \leq x\} = \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} H_n(x) \tag{206}$$

where $H_n(x)$ denotes the n-th iterated convolution of $H(x)$ with itself and $H_0(x) = 1$ for $x \geq 0$ and $H_0(x) = 0$ for $x < 0$. Since in this case

$$\theta(t) = \sup_{0 \leq u \leq t} [u - \chi(u)] \tag{207}$$

for $t \geq 0$ by Theorem 55.9 we obtain that

$$P\{\theta(t) \leq x\} = 1 - \int_x^t \frac{x}{u} d_u P\{\chi(u) \leq u - x\} \tag{208}$$

for $0 < x \leq t$. If we take into consideration that $\sigma(t) = t - \theta(t)$ for $t \geq 0$, then (205) can be obtained by (208).

Now let us determine the asymptotic distribution of $\sigma(t)$ in various cases.

First, let us suppose that $\lambda b < 1$ and that

$$\sigma_b^2 = \int_0^\infty (x - b)^2 dH(x) \tag{209}$$

is finite. Then $G(\infty) = 1$,

$$E\{\sigma_n\} = \frac{b}{1-\lambda b} , \tag{210}$$

$$Var\{\sigma_n\} = \frac{(\sigma_b^2 + \lambda b^3)}{(1 - \lambda b)^3} \tag{211}$$

and obviously (212) $E\{\theta_n\} = 1/\lambda$ and $Var\{\theta_n\} = 1/\lambda^2$. In this case by the 11-th statement of Theorem 59.2 we obtain that

$$\lim_{t \to \infty} P \{\frac{\sigma(t) - \lambda bt}{\sqrt{\lambda(\sigma_b^2 + b^2)t}} \leq x\} = \Phi(x) \tag{213}$$

where $\Phi(x)$ is the normal distribution function. (See also the author [291].) The same result can be obtained by (175) if we put $a = 1/\lambda$ and $\sigma_a^2 = 1/\lambda^2$ in (176) and in (177).

Second, let us suppose that $\lambda b = 1$ and that $\sigma_b^2 < \infty$. Then we have

(214) $$\lim_{x \to \infty} x^{\frac{1}{2}} [1 - G(x)] = \left(\frac{2}{\pi\lambda^3(\sigma_b^2 + b^2)}\right)^{1/2}$$

This result is due to S. M. Brodi [39]. If we take into consideration that $\gamma(s)$, the Laplace-Stieltjes transform of $G(x)$, can be obtained for $Re(s) > 0$ as the only root in z of the equation

(215) $$z = \psi(s + \lambda - \lambda z)$$

in the unit circle $|z| < 1$, and that

(216) $$1 - \psi(s) = bs - \frac{(\sigma_b^2 + b^2)s^2}{2} + o(s^2)$$

as $s \to +0$, then we can easily prove that

(217). $$\lim_{s \to +0} s^{-\frac{1}{2}}[1 - \gamma(s)] = \left(\frac{2}{\lambda^3(\sigma_b^2 + b^2)}\right)^{1/2}$$

and this implies (214).

Now

(218) $$\lim_{n \to \infty} P\{ \frac{\theta_1 + \ldots + \theta_n - \frac{n}{\lambda}}{\sqrt{n}/\lambda} \leq x\} = \Phi(x)$$

and by (214)

(219) $$\lim_{n \to \infty} P\{ \frac{\sigma_1 + \ldots + \sigma_n}{n^2/\lambda^3(\sigma_b^2 + b^2)} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\frac{1}{2}, 1, 1, 0)$, that is,

(220)
$$R(x) = \begin{cases} 2[1 - \Phi(\frac{1}{\sqrt{x}})] & \text{for } x > 0 , \\ 0 & \text{for } x \leqq 0 . \end{cases}$$

In this case by the 7-th statement of Theorem 59.2 we obtain that

(221)
$$\lim_{t \to \infty} P\{ \frac{\theta(t)}{\sqrt{\lambda(\sigma_b^2 + b^2)t}} \leqq x\} = 1 - R(\frac{1}{x^2})$$

for $x > 0$. Here $b = 1/\lambda$ and hence

(222)
$$\lim_{t \to \infty} P\{ \frac{\theta(t)\sqrt{\lambda}}{\sqrt{(1+\lambda^2\sigma_b^2)t}} \leqq x\} = 2\Phi(x)-1$$

for $x \geqq 0$. The same result can be obtained by (188) if we put $a = 1/\lambda$ and $\sigma_a^2 = 1/\lambda^2$ in it.

Third, let us suppose that $\lambda b = 1$ and that

(223)
$$x^{\alpha}[1 - H(x)] = h(x)$$

where $1 < \alpha < 2$ and

(224)
$$\lim_{x \to \infty} \frac{h(\omega x)}{h(x)} = 1 \quad \text{for} \quad \omega > 0.$$

In this case $G(x)$ belongs to the domain of attraction of a nondegenerate stable distribution function of type $S(\frac{1}{\alpha}, 1, c, 0)$ where $c > 0$. (See D. L. Iglehart [133].) Indeed we have

(225)
$$x^{1/\alpha}[1 - G(x)] \sim \frac{D(\alpha, \lambda)}{[h(x^{1/\alpha})]^{1/\alpha}}$$

as $x \to \infty$ where

(226) $$D(\alpha, \lambda) = \frac{1}{\lambda^{1+\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})[-\Gamma(1-\alpha)]^{\frac{1}{\alpha}}} .$$

For in this case we have

(227) $$1-\psi(s) = bs - s \int_0^\infty (1 - e^{-sx})h(x)x^{-\alpha}dx =$$
$$= bs + \Gamma(1-\alpha)s^\alpha h(\tfrac{1}{s}) + o(s^\alpha)$$

as $s \to 0$ and if we take into consideration that $z = \gamma(s)$ satisfies (215) for $Re(s) > 0$, then we obtain easily that

(228) $$1-\gamma(s) \sim \frac{D(\alpha, \lambda)\Gamma(1-\frac{1}{\alpha})s^{1/\alpha}}{[h(\frac{1}{s^{1/\alpha}})]^{1/\alpha}}$$

as $s \to +0$. Hence by a Tauberian theorem (Theorem 9.14 in the Appendix) it follows that (225) holds.

In the particular case where $\lim_{x \to \infty} h(x) = h$ and h is a positive number by (225) we have

(229) $$\lim_{x \to \infty} x^{1/\alpha}[1-G(x)] = g$$

where

(230) $$g = \frac{D(\alpha, \lambda)}{h^{1/\alpha}} .$$

Thus it follows that

(231) $$\lim_{n \to \infty} P\{ \frac{\sigma_1 + \sigma_2 + \dots + \sigma_n}{(gn)^{\alpha}} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\frac{1}{\alpha}, 1, \Gamma(1-\frac{1}{\alpha})\cos\frac{\pi}{2\alpha}, 0)$. Since in this case (218) holds, by the 7-th statement of Theorem 59.2 we obtain that

(232) $$\lim_{t \to \infty} P\{ \frac{\theta(t)\lambda g}{t^{1/\alpha}} \leq x\} = 1 - R(\frac{1}{x^{\alpha}})$$

for $x > 0$, or equivalently,

(233) $$\lim_{t \to \infty} P\{ \frac{\theta(t)D(\alpha,\lambda)\lambda}{(ht)^{1/\alpha}} \leq x\} = G_{\frac{1}{\alpha}}(\Gamma(1-\frac{1}{\alpha})x)$$

where $D(\alpha, \lambda)$ is given by (226) and $G_{1/\alpha}(x)$ is defined by (42.178). (See also (196) and (197).) This result is a particular case of (194). If we put $a = 1/\lambda$ in (194), then we get

(234) $$\lim_{t \to \infty} P\{ \frac{\theta(t)}{(\lambda ht)^{1/\alpha}} \leq x\} = G_{\frac{1}{\alpha}}(\frac{x}{[-\Gamma(1-\alpha)]^{1/\alpha}})$$

which is in agreement with (233).

In the general case when $H(x)$ satisfies (223) with (224), we can find a function $\rho^*(t)$ such that

(235) $$\lim_{t \to \infty} t\,[1 - G(t^{\alpha}\rho^*(t))] = 1$$

and

(236) $$\lim_{n \to \infty} P\{ \frac{\sigma_1 + \sigma_2 + \dots + \sigma_n}{n^{\alpha}\rho^*(n)} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\frac{1}{\alpha}, 1, \Gamma(1-\frac{1}{\alpha})\cos\frac{\pi}{2\alpha}, 0)$. Since (218) holds, in exactly the same way as we proved (232) we can conclude that

(237) $$\lim_{t\to\infty} P\{\frac{\theta(t)\lambda[\rho^*(t^{1/\alpha})]^{1/\alpha}}{t^{1/\alpha}} \leq x\} = G_{\frac{1}{\alpha}}(\Gamma(1-\frac{1}{\alpha})x) .$$

If we put $a = 1/\lambda$ in (202), then we obtain that

(238) $$\lim_{t\to\infty} P\{\frac{\theta(t)}{(\lambda t)^{1/\alpha}\rho(t)} \leq x\} = G_{\frac{1}{\alpha}}(\frac{x}{[-\Gamma(1-\alpha)]^{1/\alpha}}).$$

where $\rho(t)$ should be chosen in such a way that (201) is satisfied. A comparison of (237) and (238) shows that

(239) $$\rho^*(t) = [\frac{D(\alpha, \lambda)}{\rho(t^\alpha)}]^\alpha$$

is an appropriate choice in (235). This can be verified by using the fact that

(240) $$\lim_{t\to\infty}\frac{\rho(\omega t)}{\rho(t)} = 1$$

for all $\omega > 0$. (See Problem 46. 12 .) For by (223) and (201) it follows that

(241) $$h(t^{1/\alpha}\rho(t)) \sim [\rho(t)]^\alpha$$

as $t \to \infty$ and by (225) and (229) we have

(242) $$h(t[\rho^*(t)]^{1/\alpha})\rho^*(t) \sim [D(\alpha, \lambda)]^\alpha$$

as $t \to \infty$. If we put (239) in (242), and if we replace t by $t^{1/\alpha}$ in (242), then we obtain that

(243) $$h(t^{1/\alpha}/\rho(t)) \sim [\rho(t)]^{\alpha}$$

as $t \to \infty$, and this is indeed true by (240) and (241).

Finally, we note that if $\lambda b > 1$, then

(244) $$\lim_{t \to \infty} P\{\theta(t) \leq x\} = 1 - e^{-\omega^* x}$$

for $x \geq 0$ where ω^* is the largest real root of the equation

(245) $$\lambda[1 - \psi(\omega^*)] = \omega^* .$$

In this case the distribution of $\theta(t)$ is given by (208). If we let $t \to \infty$ in (208), then by Theorem 55.10 we get (244).

63. Risk Processes. One of the most important tasks in the mathematical theory of insurance risk is to study the fluctuations of the risk reserve process. The first results concerning risk reserve processes were obtained in 1903 by F. Lundberg [373], [374]. The theory was further developed between 1926 and 1955 by F. Lundberg [375], [376], [377], H. Cramér [364], [365], [366], [367], F. Escher [369], C.-O. Segerdahl [384], [385], [386], [387], S. Täcklind [390], T. Saxén [382], [383], H. Ammeter [353], G. Arfwedson [355], [356], [357], [358], [359] and others. Some recent results can be found in the papers mentioned in the references.

Let us suppose that a company deals with insurance and annuities. In the time interval $(0, \infty)$ the company continuously receives risk premiums from the policyholders at a constant rate. If a claim occurs, the company pays the risk sum of the claim to the policyholder. The company also pays annuities continuously to the policyholders at a constant rate. We may consider the annuities as negative risk premiums. If a policy terminates the corresponding reserve is placed at the disposal of the company, thus implying a payment from the policyholder to the company, or a payment of a negative amount by the company to the policyholder.

Accordingly, we shall assume that the company continuously receives risk premiums, which may be positive or negative, at a constant rate, and the amount paid by the company in settlement of a claim may take positive or negative values.

Suppose that the total risk premium received in the time interval $(0, u)$ is cu where c is a positive or negative constant. Suppose that in the time interval $(0, \infty)$ claims occur at random times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ and the corresponding risk sums are $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ which are random variables taking on positive or negative values. Suppose that at time $u = 0$ the company has at its disposal a certain initial capital $x \geq 0$ which is available for covering the losses due to random fluctuations. In this case the size of the risk reserve at time u is

(1) $$x + cu - \sum_{0<\tau_n\leq u} \chi_n$$

for $u \geq 0$. Let us introduce the notation

$$\xi(u) = \sum_{0<\tau_n \leq u} \chi_n - cu \tag{2}$$

for $u \geq 0$.

One of the fundamental problems of the theory of insurance risk is to find the probability that in the time interval $(0, t]$ the risk reserve does not become negative, or in other words, that no ruin occurs in the time interval $(0, t]$. This probability is evidently given by

$$\underset{\sim}{P}\{\sup_{0 \leq u \leq t} \xi(u) \leq x\} \ . \tag{3}$$

The probability that in the time interval $(0, \infty)$ ruin never occurs is given by

$$\underset{\sim}{P}\{\sup_{0 \leq u < \infty} \xi(u) \leq x\} \ . \tag{4}$$

In the mathematical theory of risk reserve processes it is important to find the probabilities (3) and (4) for various processes $\{\xi(u) , 0 \leq u < \infty\}$.

General methods for finding the probabilities (3) and (4) were given in 1954 by H. Cramér [366] in the case where claims occur according to a Poisson process and $\{\chi_n\}$ are mutually independent and identically distributed random variables, and in 1970 by the author [389] in the case where claims occur according to a recurrent process and $\{\chi_n\}$ are mutually independent and identically distributed random variables.

In what follows we shall consider the case where $\{\xi(u) , 0 \leq u < \infty\}$ is a general compound recurrent process as we defined in Section 54 and we shall give methods for finding the probabilities (3) and (4).

We suppose that $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots ; \tau_0 = 0)$ is a sequence of mutually independent and identically distributed positive random variables with distribution function $\underset{\sim}{P}\{\tau_n - \tau_{n-1} \leq x\} = F(x)$ and χ_n $(n = 1,2,\ldots)$ is a sequence of mutually idnependent and identically distributed random variables with distribution function $\underset{\sim}{P}\{\chi_n \leq x\} = H(x)$. We suppose also that the two sequences $\{\tau_n\}$ and $\{\chi_n\}$ are independent.

Let us introduce the following notation

(5) $$\phi(s) = \int_0^\infty e^{-sx}\, dF(x)$$

for $\mathrm{Re}(s) \geq 0$, and

(6) $$\psi(s) = \int_{-\infty}^\infty e^{-sx}\, dH(x)$$

for $\mathrm{Re}(s) = 0$.

We are interested in finding the distribution and the limiting distribution of the random variable

(7) $$\eta(t) = \sup_{0 \leq u \leq t} \xi(u)$$

where $\xi(u)$ is defined by (2) for $u \geq 0$.

<u>Theorem 1.</u> <u>If</u> $c \geq 0$, $\mathrm{Re}(q) > 0$ <u>and</u> $\mathrm{Re}(s) \geq 0$, <u>then we have</u>

(8) $$q \int_0^\infty e^{-qt}\, \underset{\sim}{E}\{e^{-s\eta(t)}\}dt = [1-\phi(q)]e^{-\underset{\sim}{T}\{\log[1-\phi(q-cs)\psi(s)]\}}$$

where $\underset{\sim}{T}$ operates on the variable s.

Proof. This theorem is a particular case of Theorem 54.1. If we put $v = s$ in (54.28), then we get (8). In the above form of (8)′ we have made use of a rather obvious fact, namely that if $\Phi(s) \in \underset{\sim}{R}$, then

(9) $$[\underset{\sim}{T}\{\Phi(v-s)\}]_{v=s} = \underset{\sim}{T}\{\Phi(s)\}$$

for $\mathrm{Re}(s) \geq 0$.

Theorem 2. If $c \leq 0$, $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$, then we have

(10) $$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = Q(s,s,q) + \frac{cs}{q-cs}\,Q(s-\frac{q}{c}, s, q)$$

where

(11) $$Q(s,v,q) = 1 - \phi(q-cv)\underset{\sim}{T}\{[1-\psi(v-s)]e^{-\underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\}}\}$$

for $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$ and $\underset{\sim}{T}$ operates on the variable s. If $c = 0$, then the second term on the right-hand side of (10) is 0.

Proof. This theorem is a particular case of Theorem 54.2. If we put $v = s$ in (54.36), then we get (10).

In both cases, if either $c \geq 0$, or $c \leq 0$, we can use the method of factorization to obtain

(12) $$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)}\}dt$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$.

Let us suppose that

(13) $$1 - \phi(q-cs)\psi(s) = \Phi^{+}(s,q,c)\Phi^{-}(s,q,c)$$

for $Re(s) = 0$ and $Re(q) > 0$ where $\Phi^{+}(s,q,c)$ and $\Phi^{-}(s,q,c)$ satisfy the requirements stated after formula (54.51). Such a factorization always exists and by (54.52) we have

(14) $$\underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\} = \log\Phi^{+}(v,q,c) + \log\Phi^{-}(v-s,q,c)$$

for $Re(s) \geqq Re(v) \geqq 0$ and $Re(q) > 0$.

If $c \geqq 0$, $Re(q) > 0$ and $Re(s) \geqq 0$, then by (8) and (13) we obtain that

(15) $$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \frac{[1-\phi(q)]}{\Phi^{+}(s,q,c)\Phi^{-}(0,q,c)}.$$

If $c \leqq 0$, $Re(q) > 0$ and $Re(s) \geqq Re(v) \geqq 0$, then in (10) we can write that

(16) $$Q(s,v,q) = 1 - \frac{\phi(q-cv)}{\Phi^{+}(v,q,c)}\underset{\sim}{T}\{\frac{[1-\psi(v-s)]}{\Phi^{-}(v-s,q,c)}\}.$$

By Theorem 54.4 it follows that

(17) $$W(x) = \underset{\sim}{P}\{\sup_{0\leqq u<\infty}\xi(u) \leqq x\}$$

is a proper distribution function if and only if

(18) $$\sum_{n=1}^{\infty}\frac{1}{n}\underset{\sim}{P}\{\chi_1+\ldots+\chi_n > c\,\tau_n\} < \infty .$$

If the series in (18) is divergent, then $W(x) = 0$ for all x.

We note that if $\underset{\sim}{E}\{\chi_1 - c\tau_1\} < 0$ or $\underset{\sim}{P}\{\chi_1 = c\tau_1\} = 1$, then $W(x)$ is a proper distribution function, whereas if $\underset{\sim}{E}\{\chi_1 - c\tau_1\} \geq 0$ and $\underset{\sim}{P}\{\chi_1 = c\tau_1\} = 1$, then $W(x) = 0$ for all x.

The Laplace-Stieltjes transform

$$\Omega(s) = \int_0^\infty e^{-sx}\, dW(x) \tag{19}$$

can be obtained by Theorem 54.3 for $\mathrm{Re}(s) \geq 0$. We can also obtain $\Omega(s)$ by the method of factorization. If we suppose that $\underset{\sim}{P}\{\chi_1 = c\tau_1\} < 1$ and that (18) is satisfied, then we can write that

$$1 - \phi(-cs)\psi(s) = \Phi^+(s,c)\Phi^-(s,c) \tag{20}$$

for $\mathrm{Re}(s) = 0$ where $\Phi^+(s,c)$ satisfies the requirements A_1, A_2, A_3 and $\Phi^-(s,c)$ satisfies the requirements B_1, B_2, B_3 after formula (43.131). In this case by Theorem 43.15 we obtain that for $\mathrm{Re}(s) \geq 0$

$$\Omega(s) = \frac{\Phi^+(0,c)}{\Phi^+(s,c)} \tag{21}$$

whenever $c \geq 0$ and

$$\Omega(s) = \phi(-cs)\,\frac{\Phi^-(0,c)}{\Phi^-(-s,c)} \tag{22}$$

whenever $c \leq 0$.

Note. If we suppose in Theorem 2 that the random variables $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ are nonpositive with probability 1 , then (10) can be simplified. In this case $T\{\psi(v-s)\} = \psi(v-s)$ for $\mathrm{Re}(s) \geq \mathrm{Re}(v)$ and (11) reduces to

$$(23) \qquad Q(s,v,q) = 1 - \phi(q-cv)[1-\psi(v-s)]V(s,v,q)$$

where

$$(24) \qquad V(s,v,q) = e^{-T\{\log[1-\phi(q+cs-cv)\psi(v-s)]\}}$$

for $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$. By (9) it follows immediately that

$$(25) \qquad V(s,s,q) = e^{-T\{\log[1-\phi(q-cs)\psi(s)]\}}$$

for $\mathrm{Re}(s) \geq 0$, and by (24) we can easily prove that

$$(26) \qquad V(s,s+\frac{q}{c}, q) = e^{-T\{\log[1-\phi(-cs)\psi(s+\frac{q}{c})]\}}$$

for $\mathrm{Re}(s) \geq 0$. Thus (10) reduces to the following form

$$(27) \qquad q\int_0^\infty e^{-qt} E\{e^{-s\eta(t)}\}dt = \frac{q}{q-cs} - \frac{cs}{q-cs}\phi(q-cs)[1-\psi(\tfrac{q}{c})]V(s-\tfrac{q}{c}, s, q)$$

where $V(s-\frac{q}{c}, s, q)$ is determined by (26) s being replaced by $s-\frac{q}{c}$ in it.

In this particular case (16) reduces to

$$(28) \qquad Q(s,v,q) = 1 - \frac{\phi(q-cv)[1-\psi(v-s)]}{\Phi^+(v,q,c)\Phi^-(v-s,q,c)}$$

and by (10) we obtain that

$$(29)\qquad q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \frac{q}{q-cs} - \frac{cs[1-\psi(\frac{q}{c})]\phi(q-cs)}{(q-cs)\Phi^-(\frac{q}{c},q,c)\Phi^+(s,q,c)}$$

for $\text{Re}(q) > 0$ and $\text{Re}(s) \geq 0$.

The above results make it possible to find the probabilities (3) and (4) for every x if $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a general compound recurrent process. If we know these probabilities, then we may decide which precautions (reinsurance, etc.) should be taken in order to make the probability of ruin so small that in practice no ruin is to be expected.

In the particular case when $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a compound Poisson process we can use the results of Section 54 to find the probabilities (3) and (4). If we suppose that $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a compound Poisson process and that either only positive risk sums or only negative risk sums occur, then the probabilities (3) and (4) can be determined explicitly by Theorems 6, 7, 9, 10 in Section 54. In these theorems $c = 1$ or $c = -1$ which can always be achieved by choosing a suitable monetary unit. See also the author [316 pp. 147-161.]

64. Storage and Dam Processes. The first mathematical investigations of the theory of storage and dam processes started in 1954 by P. A. P. Moran [445]. In the past two decades the theory developed tremendously. Numerous papers have been published on this subject some of which are mentioned in the references.

In what follows we shall study the mathematical laws governing the fluctuations of the stock level in a store or the content of a dam. We suppose that we know the stochastic properties of the input (supply) and the stochastic properties of the demand, and we want to determine the mathematical laws governing the fluctuations of the stock level or the content of the dam. It is important to know these laws if we want to provide efficient service which satisfy the demand consistently with high probability.

We shall consider various mathematical models for stores and dams and give methods for finding the distribution of the stock level or the content of the dam , and the distribution of the total empty time in a given time interval.

Dams of Unlimited Capacity. First, we shall consider the case of water storage (dams, reservoirs), liquid storage (oil, gasoline), or gas storage (natural gas, compressed air). In what follows we shall use the terminology of dams; however, the results can be applied for general storage processes too.

Let us consider the following mathematical model of infinite dams. In the time interval $(0, \infty)$ water is flowing into a dam (reservoir). Denote by $\chi(u)$ the total quantity of water flowing into the dam in the time interval $(0, u]$. Denote by $\eta(0)$ the initial content of the dam at time $u = 0$. Let us suppose that in the time interval $(0, \infty)$ there is a continuous release at a constant unit rate when the dam is not empty.

If we denote by $\eta(t)$ the content of the dam at time t , then we can write that

(1) $$\eta(t) = \sup\{\eta(0) + \chi(t) - t \text{ and } \chi(t) - \chi(u) - (t-u) \text{ for } 0 \leq u \leq t\}$$

for $t \geq 0$. This formula can be proved as follows: If in the interval $(0, t]$ the dam never becomes empty, then $\eta(t) = \eta(0) + \chi(t) - t$ and (1) holds. If in the interval $(0, t]$ the dam becomes empty and u is the last time when the dam is empty, then $\eta(t) = \chi(t) - \chi(u) - (t-u)$; and (1) holds in this case too.

The total time in the interval $(0, t)$ during which the dam is empty is given by

(2) $$\theta(t) = \sup\{0 \text{ and } u - \chi(u) - \eta(0) \text{ for } 0 \leq u \leq t\}$$

for $t \geq 0$. This can be proved directly, or it can be deduced from (1) by using the obvious relation

(3) $$\theta(t) = \eta(t) + t - \chi(t) - \eta(0)$$

which holds for all $t \geq 0$.

Define also σ_0 as the time of the first emptiness in time interval $(0, \infty)$, that is,

(4) $$\sigma_0 = \inf\{u : \eta(0) + \chi(u) - u \leq 0 \text{ and } 0 \leq u < \infty\}$$

and $\sigma_0 = \infty$ if $\eta(0) + \chi(u) - u > 0$ for all $u \geq 0$.

We note that for any input process $\{\chi(u), 0 \leq u < \infty\}$ we have

(5) $$\underset{\sim}{P}\{\theta(t) = 0 | \eta(0) = c\} = \underset{\sim}{P}\{\sigma_0 \geq t | \eta(0) = c\}$$

for $t > 0$ and $c \geq 0$, and

$$(6) \qquad \underset{\sim}{P}\{\theta(t) \geqq x \mid \eta(0) = c\} = \underset{\sim}{P}\{\sigma_0 \leqq t \mid \eta(0) = c + x\}$$

for $0 < x \leqq t$ and $c \geqq 0$. These relations immediately follow from (2) and (4).

Our aim is to give methods for finding the distributions of the random variables $\eta(t)$, $\theta(t)$ and σ_0 for various input processes $\{\chi(u), 0 \leqq u < \infty\}$.

We observe that if we consider a single server queue in which the initial virtual waiting time (immediately after $u = 0$) is $\eta(0)$ and the total service time of all those customers who arrive in the time interval $(0, u]$ is $\chi(u)$, then $\eta(t)$ can be interpreted as the virtual waiting time at time t, provided that service is in order of arrival, $\theta(t)$ can be interpreted as the total idle time of the server in the time interval $(0, t)$ and σ_0 as the length of the initial busy period.

If we suppose that $\{\chi(u), 0 \leqq u < \infty\}$ is a compound recurrent process, that is,

$$(7) \qquad \chi(u) = \sum_{0<\tau_n \leqq u} \chi_n$$

for $u \geqq 0$ where $\tau_1, \tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}, \ldots$ and $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ are independent sequences of mutually independent and idnetically distributed positive random variables, and that $\eta(0)$ is a nonnegative random variable which is independent of the process $\{\chi(u), 0 \leqq u < \infty\}$, then we can apply the results of Section 62 to find the distributions and the limiting distributions of $\eta(t)$ and $\theta(t)$.

Actually, in the queuing process discussed in Section 62 we assumed that there is an arrival at time $\tau_0 = 0$ and the service time of the customer arriving at time $\tau_0 = 0$ is χ_0. Thus the initial virtual waiting time in the queuing process is $\eta(0) = \eta_0 + \chi_0$ where η_0 is the initial occupation time of the server. If we consider a dam process with initial content $\eta(0)$ where $\eta(0)$ is the same as the initial virtual waiting time in the queuing process, then the queuing process $\{\eta(t), 0 \leq t < \infty\}$ and the dam process $\{\eta(t), 0 \leq t < \infty\}$ become identical.

If we suppose that $\eta(0)$ is a nonnegative random variable which is independent of the process $\{\chi(u), 0 \leq u < \infty\}$ defined by (7), then the distribution of $\eta(t)$ can be obtained by the following theorem. We use the following notation

(8) $$U_0^*(s) = \underset{\sim}{E}\{e^{-s\eta(0)}\}$$

for $Re(s) \geq 0$,

(9) $$\psi(s) = \underset{\sim}{E}\{e^{-s\chi_n}\}$$

and

(10) $$\phi(s) = \underset{\sim}{E}\{e^{-s(\tau_n - \tau_{n-1})}\}$$

for $Re(s) \geq 0$ and $n = 1,2,\ldots$ $(\tau_0 = 0)$.

<u>Theorem 1.</u> <u>If</u> $Re(q) > 0$ <u>and</u> $Re(s) \geq 0$, <u>then we have</u>

(11) $$(s-q)\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \{[1-\psi(s)]U^*(q,s,0) - U_0^*(s)\} - \frac{s}{q}\{[1-\psi(q)]U^*(q,q,0) - U_0^*(q)\}$$

where

$$U^*(q,s,v) = e^{-\underset{\sim}{T}\{\log[1-\phi(q-s)\psi(s+v)]\}}.$$

(12)

$$\cdot \underset{\sim}{T}\left\{\frac{\underset{\sim}{T}\{U_0^*(s+v)\phi(q-s)\}e^{-\underset{\sim}{T}\{\log[1-\phi(q-s)\psi(s+v)]\}}}{1-\phi(q-s)\psi(s+v)}\right\}$$

for $\operatorname{Re}(q) > 0$, $\operatorname{Re}(s+v) \geq 0$ and $\operatorname{Re}(v) \geq 0$. If, in particular, $\underset{\sim}{P}\{\eta(0) = 0\} = 1$, then $U_0^*(s) \equiv 1$ and (12) reduces to

(13) $$U^*(q,s,v) = \phi(q)e^{-\underset{\sim}{T}\{\log[1-\phi(q-s)\psi(s+v)]\}}.$$

Proof. Let us define $\gamma_1 = \eta(0)$ and $\gamma_n = \eta(0)+\chi_1+\dots+\chi_{n-1}$ for $n = 2,3,\dots$ and $\eta_1 = [\eta(0)-\tau_1]^+$ and

(14) $$\eta_n = [\eta_{n-1}+\chi_{n-1}-(\tau_n-\tau_{n-1})]^+$$

for $n = 2,3,\dots$. Then by Theorem 62.4 for almost all realizations of the process $\{\eta(t)\ ,\ 0 \leq t < \infty\}$ we have

(15)
$$(s-q)\int_0^\infty e^{-qt-s\eta(t)-v\chi(t)-v\eta(0)}\,dt =$$

$$= \left\{\sum_{n=1}^\infty e^{-q\tau_n-s\eta_n-v\gamma_n}\left(1-e^{-(s+v)\chi_n}\right) - e^{-(s+v)\eta(0)}\right\}$$

$$-\frac{s}{q}\left\{\sum_{n=1}^\infty e^{-q\tau_n-q\eta_n-v\gamma_n}\left(1-e^{-(q+v)\chi_n}\right) - e^{-(q+v)\eta(0)}\right\}$$

where $\operatorname{Re}(q) > 0$, $\operatorname{Re}(v) \geq 0$ and $\operatorname{Re}(s+v) \geq 0$. If we put $w = 0$, $\rho = 1$ and $\eta_0 + \chi_0 = \eta(0)$ in (62.29), then we obtain (14). By forming the expectation of (15) we get

$$(s-q)\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)-v\chi(t)-v\eta(0)}\}dt =$$

(16)

$$= \{[1-\psi(s+v)]U^*(q,s,v) - U_0^*(s+v)\} -$$

$$-\frac{s}{q}\{[1-\psi(q+v)]U^*(q,q,v) - U_0^*(q+v)\}$$

for $Re(q) > 0$, $Re(v) \geqq 0$ and $Re(s+v) \geqq 0$ where

(17) $$U^*(q,s,v) = \sum_{n=1}^{\infty} \underset{\sim}{E}\{e^{-q\tau_n - s\eta_n - v\gamma_n}\}.$$

If we write

(18) $$U_n^*(q,s,v) = \underset{\sim}{E}\{e^{-q\tau_n - s\eta_n - v\gamma_n}\} ,$$

then $U_1^*(q,s,v) = \underset{\sim}{T}\{U_0^*(s+v)\phi(q-s)\}$ and

(19) $$U_{n+1}^*(q,s,v) = \underset{\sim}{T}\{U_n^*(q,s,v)\phi(q-s)\psi(s+v)\}$$

for $n = 1,2,\ldots$. In exactly the same way as we proved Theorem 62.6 we obtain that $U^*(q,s,v)$ can be expressed by (12). If we put $v=0$ in (16), then we obtain (11) which was to be proved.

If $U_0^*(s) \equiv 1$, then $U_1^*(q,s,v) = \phi(q)$, and thus in this case (12) reduces to (13).

In a similar way as in Section 62 we can also use the method of factorization in finding the distribution of $\eta(t)$.

Theorem 62.7 is valid unchangeably for the limiting distribution of $\eta(t)$

as $t \to \infty$.

By (16) we can also determine the distribution of $\theta(t)$. If we replace q by $q+s$ and v by $-s$ in (16), then we obtain that

$$(20) \qquad q \int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\theta(t)}\}dt = 1 - \frac{s}{q+s} U_0^*(q) + $$

$$+ \frac{s}{q+s} [1 - \psi(q)]U^*(q+s, q+s, -s)$$

for $\operatorname{Re}(q) \geq 0$ and $-\operatorname{Re}(q) < \operatorname{Re}(s) \leq 0$. By analytical continuation we can extend the definition of $U^*(q+s, q+s, -s)$ for $\operatorname{Re}(q) \geq 0$ and $\operatorname{Re}(q+s) > 0$, and thus (20) will be valid for $\operatorname{Re}(q) \geq 0$ and $\operatorname{Re}(q+s) > 0$.

In the particular case when $U_0^*(s) \equiv 1$ we obtain from (13) that

$$(21) \qquad U^*(q+s, q+s, -s) = \phi(q+s)V(q+s, q)$$

for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(q+s) \geq 0$ where

$$(22) \qquad V(s,q) = e^{-\underset{\sim}{T}\{\log[1-\phi(s)\psi(q-s)]\}}$$

for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(q) > 0$. Thus if $U_0^*(s) \equiv 1$, then

$$(23) \qquad q \int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\theta(t)}\}dt = \frac{q}{q+s} + \frac{s}{q+s} [1 - \psi(q)]\phi(q+s)V(q+s, q)$$

for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(q) > 0$ where $V(s,q)$ is given by (22). This last result can also be obtained from (63.26) if put $c = -1$ in it and if we replace $\psi(s)$ by $\psi(-s)$.

The limiting distribution and the asymptotic distribution of $\theta(t)$ as $t \to \infty$ can be studied in a similar way as in the case of the corresponding queuing process.

Finally, the distribution of σ_0 can be determined by the relations (5) and (6).

If we suppose that $\{\chi(u) , 0 \leq u < \infty\}$ is a compound Poisson process, or more generally, a separable homogeneous process with independent increments whose sample functions are nondecreasing step functions which vanish at the origin with probability 1 , then the distributions of the random variables $\eta(t)$, $\theta(t)$ and σ_0 can be determined explicitly.

If $\{\chi(u) , 0 \leq u < \infty\}$ is a homogeneous stochastic process with independent increments for which $P\{\chi(0) = 0\} = 1$, then for every $t > 0$ the processes $\{\chi(u) , 0 \leq u \leq t\}$ and $\{\chi(t) - \chi(t-u) , 0 \leq u \leq t\}$ have identical finite dimensional distributions. Thus if the process $\{\chi(u) , 0 \leq u < \infty\}$ is separable, then by (1) we can write that

$$(24) \qquad P\{\eta(t) \leq x\} = P\{\chi(u) - u \leq x \text{ for } 0 \leq u \leq t \text{ and } \eta(0) + \chi(t) - t \leq x\}$$

and by (3) we have

$$(25) \qquad P\{\theta(t) \leq x\} = P\{u - \chi(u) - \eta(0) \leq x \text{ for } 0 \leq u \leq t\}$$

for $x \geq 0$. The probabilities (24) and (25) can easily be obtained by using Theorem 51.8 . In the particular case where $P\{\eta(0) = 0\} = 1$ these probabilities are given by Theorem 55.6 and by Theorem 55.9 for a compound Posson process and by Theorem 56.5 and by Theorem 56.7 for a separable homogeneous process $\{\chi(u), 0 \leq u < \infty\}$ having independent increments and nondecreasing sample functions which increase only in jumps and which vanish at the origin with probability 1 .

Dams of Finite Capacity. We shall use again the terminology of dams; however, the results can be applied for general storage processes too.

Let us consider the following mathematical model of finite dams. In the time interval $(0, \infty)$ water is flowing into a dam (reservoir). Denote by $\chi(u)$ the total quantity of water flowing into the dam in the time interval $(0, u]$. The capacity of the dam is a finite positive number m . If the dam becomes full, the excess water overflows. Denote by $\eta^*(0)$ the initial content of the dam at time $u = 0$. Let us suppose that in the time interval $(0, \infty)$ there is a continuous release at a constant unit rate when the dam is not empty.

Denote by $\eta^*(t)$ the content of the dam at time t . Our aim is to give methods for finding the distribution and the limiting distribution of $\eta^*(t)$ for various input processes $\{\chi(u), 0 \leq u < \infty\}$. See references [463], [464].

If $m = \infty$, that is, if the dam has unlimited capacity, denote by $\eta(t)$ the content of the dam at time t . We assume that $\eta(0) = \eta^*(0)$ and we shall consider the two processes $\{\eta^*(t), 0 \leq t < \infty\}$ and $\{\eta(t), 0 \leq t < \infty\}$ simultaneously.

First, let us suppose that the input process $\{\chi(u), 0 \leq u < \infty\}$ is a compound Poisson process and $\mathbf{P}\{\eta^*(0) = m\} = 1$.

In this case

$$\chi(u) = \sum_{0<\tau_n\leq u} \chi_n \tag{26}$$

for $u \geq 0$ where $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots, \tau_0 = 0)$ are mutually independent and identically distributed random variables with distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 , \end{cases} \tag{27}$$

and χ_n $(n = 1,2,\ldots)$ are mutually independent and identically distributed positive random variables with distribution function $\mathbf{P}\{\chi_n \leq x\} = H(x)$ and the two sequences $\{\tau_n\}$ and $\{\chi_n\}$ are independent too. Let

$$\psi(s) = \int_0^\infty e^{-sx}\, dH(x) \tag{28}$$

for $\text{Re}(s) \geq 0$.

If $\mathbf{P}\{\eta(0) = m\} = 1$, then by Theorem 1 we can prove that

$$\int_0^\infty e^{-qt}\, \mathbf{E}\{e^{-s\eta(t)}\}dt = \frac{se^{-m\omega(q)} - \omega(q)e^{-ms}}{\omega(q)[s-q-\lambda+\lambda\psi(s)]} \tag{29}$$

for $\text{Re}(q) > 0$ and $\text{Re}(s) \geq 0$ where $z = \omega(q)$ is the only root of the equation

$$z = q + \lambda[1 - \psi(z)] \tag{30}$$

in the domain $\text{Re}(z) > 0$. We can write that

$$\omega(q) = q + \lambda[1 - \gamma(q)] \tag{31}$$

where $z = \gamma(q)$ is the only root of the equation

$$z = \psi(q + \lambda - \lambda z) \tag{32}$$

in the unit circle $|z| < 1$. We already considered the function $\gamma(q)$ in Section 62. See formula (62.130). We note that the limit $\lim_{q \to +0} \omega(q) = \omega_0$ exists and $\omega_0 = 0$ if $\lambda \mathbf{E}\{\chi_n\} \leq 1$, and $\omega_0 > 0$ if $\lambda \mathbf{E}\{\chi_n\} > 1$.

In exactly the same way as we proved Theorem 55.6 we can obtain $\mathbf{P}\{\eta(t) \leq x\}$ explicitly. If we use the representation (24) and if $\mathbf{P}\{\eta(0) = m\} = 1$, then we have

$$\mathbf{P}\{\eta(t) = 0\} = \int_0^{t-m} \left(1 - \frac{y}{t}\right) d_y \mathbf{P}\{\chi(t) \leq y\} \tag{33}$$

for $t \geq m$ and $\mathbf{P}\{\eta(t) = 0\} = 0$ for $t < m$, and

$$\mathbf{P}\{\eta(t) \leq x\} = \mathbf{P}\{\chi(t) \leq t+x-m\} - \int_{+0}^{t} \mathbf{P}\{\eta(t-u) = 0\} d_u \mathbf{P}\{\chi(u) \leq u+x\} \tag{34}$$

for all x and $t \geq 0$.

By using the above results we can determine the distribution of $\eta^*(t)$ by the following theorem.

<u>Theorem 2</u>. <u>If</u> $\mathbf{P}\{\eta^*(0) = m\} = 1$ <u>and</u> $\mathbf{P}\{\eta(0) = m\} = 1$, <u>then</u>

$$q \int_0^\infty e^{-qt} \mathbf{P}\{\eta^*(t) \leq x\} dt = \frac{\int_0^\infty e^{-qt} \mathbf{P}\{\eta(t) \leq x\} dt}{\int_0^\infty e^{-qt} \mathbf{P}\{\eta(t) \leq m\} dt} \tag{35}$$

for $0 \leq x \leq m$ and $\mathrm{Re}(q) > 0$.

Proof. Denote by $m^*(t)$ the expected number of transitions $m \to m - 0$ occurring in the interval $[0, t]$ in the process $\{\eta^*(t), 0 \leq t < \infty\}$, and by $m(t)$ the same expectation for the process $\{\eta(t), 0 \leq t < \infty\}$. Let

(36) $$G^*(t,x) = \underset{\sim}{P}\{\eta^*(u) < m \text{ for } 0 < u \leq t \text{ and } \eta^*(t) \leq x\}$$

and

(37) $$G(t,x) = \underset{\sim}{P}\{\eta(u) < m \text{ for } 0 < u \leq t \text{ and } \eta(t) \leq x\}$$

for $t > 0$. Obviously $G^*(t,x) = G(t,x)$.

For $0 \leq x \leq m$ we have

(38) $$\underset{\sim}{P}\{\eta^*(t) \leq x\} = \int_{-0}^{t} G(t-u,x)dm^*(u)$$

and

(39) $$\underset{\sim}{P}\{\eta(t) \leq x\} = \int_{-0}^{t} G(t-u,x)dm(u) .$$

Let

(40) $$\mu^*(q) = \int_{-0}^{\infty} e^{-qt}\, dm^*(t)$$

and

(41) $$\mu(q) = \int_{-0}^{\infty} e^{-qt}\, dm(t)$$

for $\mathrm{Re}(q) > 0$.

If we form the Laplace transforms of (38) and (39) and form their ratio, then we get

$$(42) \qquad \int_0^\infty e^{-qt} \underset{\sim}{P}\{\eta^*(t) \leq x\}dt = \frac{\mu^*(q)}{\mu(q)} \int_0^\infty e^{-qt} \underset{\sim}{P}\{\eta(t) \leq x\}dt$$

for $\operatorname{Re}(q) > 0$ and $0 \leq x \leq m$. If we put $x = m$ in (42), then $\underset{\sim}{P}\{\eta^*(t) \leq m\} = 1$ and therefore

$$(43) \qquad \frac{1}{q} = \frac{\mu^*(q)}{\mu(q)} \int_0^\infty e^{-qt} \underset{\sim}{P}\{\eta(t) \leq m\}dt$$

for $\operatorname{Re}(q) > 0$. If we divide (42) by (43), then we get (35) which was to be proved.

By Theorem 2 we can determine the limiting distribution of $\eta^*(t)$ as $t \to \infty$.

Theorem 3. If $\{\chi(u), 0 \leq u < \infty\}$ is a compound Poisson process defined by (26), if $\{\chi(u), 0 \leq u < \infty\}$ and $\eta^*(0)$ are independent and $\underset{\sim}{P}\{\eta(0) \leq m\} = 1$, then

$$(44) \qquad \lim_{t \to \infty} \underset{\sim}{P}\{\eta^*(t) \leq x\} = \frac{W(x)}{W(m)}$$

exists for $0 \leq x \leq m$ and is independent of the distribution of $\eta^*(0)$. We have $W(x) = 0$ for $x < 0$, $W(x)$ is nondecreasing and continuous in the interval $(0, \infty)$ and

$$(45) \qquad \int_{-0}^\infty e^{-sx}\, dW(x) = \frac{s}{s-\lambda[1-\psi(s)]}$$

for $\operatorname{Re}(s) > \omega_0$.

Proof. The process $\{\eta^*(t), 0 \leq t < \infty\}$ is a Markov process and we can easily prove that the limiting distribution function $\lim_{t\to\infty} P\{\eta^*(t) \leq x\}$ exists and is independent of the distribution of $\eta^*(0)$. Thus in finding the limit (44) we may assume without loss of generality that $P\{\eta^*(0) = m\} = 1$. If $P\{\eta(0) = m\} = 1$, then by (29) it follows that

$$(46) \qquad \lim_{q \to 0} \omega(q) \int_0^\infty e^{-qt} E\{e^{-s\eta(t)}\}dt = \frac{se^{-m\omega_0} - \omega_0 e^{-ms}}{s - \lambda[1 - \psi(s)]}$$

for $\mathrm{Re}(s) > \omega_0$. Hence by (45) we can conclude that

$$(47) \qquad \lim_{q \to 0} \omega(q) \int_0^\infty e^{-qt} P\{\eta(t) \leq x\}dt = e^{-m\omega_0} W(x) - \omega_0 \int_0^{x-m} W(u)du .$$

Thus if $0 \leq x \leq m$, then

$$(48) \qquad \lim_{q \to 0} \omega(q) \int_0^\infty e^{-qt} P\{\eta(t) \leq x\}dt = e^{-m\omega_0} W(x) .$$

If we multiply both the numerator and denominator on the right-hand side of (35) by $\omega(q)$ and let $q \to +0$, then by (48) we obtain that

$$(49) \qquad \lim_{q \to +0} q \int_0^\infty e^{-qt} P\{\eta^*(t) \leq x\}dt = \frac{W(x)}{W(m)}$$

for $0 \leq x \leq m$. Hence (44) follows by an Abelian theorem for Laplace transforms. (See Theorem 9.10 in the Appendix).

If

$$(50) \qquad b = \int_0^\infty x \, dH(x)$$

and $\lambda b \leq 1$, then $\omega_0 = 0$ and if $\lambda b > 1$, then $\omega_0 > 0$ in the above formulas.

If $\lambda b \neq 1$, then we have

$$(51) \qquad W(x) = \frac{e^{\omega_0 x}}{1+\lambda\psi'(\omega_0)} - \int_{+0}^{\infty} d_u P\{\chi(u) \leqq u+x\}$$

for every x .

If b is a finite positive number, and if we define

$$(52) \qquad H^*(x) = \frac{1}{b}\int_0^x [1-H(u)]du$$

for $x \geqq 0$ and $H^*(x) = 0$ for $x < 0$, then we have

$$(53) \qquad W(x) = \sum_{n=0}^{\infty} (\lambda b)^n H_n^*(x)$$

for every x where $H_n^*(x)$ denotes the n-th iterated convolution of $H^*(x)$ with itself, and $H_0^*(x) = 1$ for $x \geqq 0$ and $H_0^*(x) = 0$ for $x < 0$.

Next let us suppose that $\{\chi(u) , 0 \leqq u < \infty\}$ is a separable, homogeneous process with independent increments almost all of whose sample functions are nondecreasing step functions vanishing at $u = 0$. Then we have

$$(54) \qquad E\{e^{-s\chi(u)}\} = e^{-u\Phi(s)}$$

for $u \geqq 0$ and $Re(s) \geqq 0$ where

$$(55) \qquad \Phi(s) = \int_{+0}^{\infty} (1 - e^{-sx})dN(x)$$

and $N(x)$ $(0 < x < \infty)$ is a nondecreasing function for which $\lim_{x \to \infty} N(x) = 0$ and

(56) $$\int_{+0}^{1} x dN(x) < \infty .$$

If we approximate the process $\{\chi(u), 0 \leq u < \infty\}$ by a sequence of suitably chosen compound Poisson processes in such a way that the finite dimensional distribution functions of the approximating processes converge to the corresponding finite dimensional distribution functions of the process $\{\chi(u), 0 \leq u < \infty\}$, then by Theorem 52.3 we can conclude that Theorem 2 remains valid for the more general process $\{\chi(u), 0 \leq u < \infty\}$ defined above.

If $\mathbf{P}\{\eta(0) = m\} = 1$, then by (29) we can conclude that now we have

(57) $$\int_0^\infty e^{-qt} \mathbf{E}\{e^{-s\eta(t)}\} dt = \frac{se^{-m\omega(q)} - \omega(q)e^{-ms}}{\omega(q)[s - q - \Phi(s)]}$$

for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s) \geq 0$ and $z = \omega(q)$ is the only root of the equation

(58) $$z - q = \Phi(z)$$

in the domain $\operatorname{Re}(z) > 0$. Formulas (34) and (35) remain valid unchangeably for the more general process $\{\chi(u), 0 \leq u < \infty\}$.

Theorem 3 remains also valid for the more general process $\{\chi(u), 0 \leq u < \infty\}$ with the modification that now

(59) $$\int_{-0}^{\infty} e^{-sx} dW(x) = \frac{s}{s - \Phi(s)}$$

for $\operatorname{Re}(s) > \omega_0$ where $\omega_0 = \lim_{q \to +0} \omega(q)$. If $\mathbf{E}\{\chi(1)\} \leq 1$, then $\omega_0 = 0$, whereas, if $\mathbf{E}\{\chi(1)\} > 1$, then $\omega_0 > 0$.

We note that if $\underset{\sim}{E}\{\chi(1)\} \neq 1$, then we have

$$(60) \qquad W(x) = \frac{e^{\omega_0 x}}{1-\Phi'(\omega_0)} - \int_{+0}^{\infty} d_u \underset{\sim}{P}\{\chi(u) \leq u + x\}$$

for every x .

If $\underset{\sim}{E}\{\chi(1)\} = \rho$ is a finite positive number, then there exists a distribution function $H^*(x)$ of a nonnegative random variable such that

$$(61) \qquad \int_0^{\infty} e^{-sx}\, dH^*(x) = \frac{\Phi(s)}{\rho s}$$

for $\mathrm{Re}(s) > 0$. By the aid of $H^*(x)$ we can write that

$$(62) \qquad W(x) = \sum_{n=0}^{\infty} \rho^n H_n^*(x)$$

for every x where $H_n^*(x)$ denotes the n-th iterated convolution of $H^*(x)$ with itself, and $H_0^*(x) = 1$ for $x \geq 0$ and $H_0^*(x) = 0$ for $x < 0$.

Examples. First, let us suppose that $\{\chi(u)\ ,\ 0 \leq u < \infty\}$ is a gamma input, that is,

$$(63) \qquad \underset{\sim}{P}\{\chi(u) \leq x\} = \frac{1}{\Gamma(u)} \int_0^{\mu x} e^{-y}\, y^{u-1} dy$$

for $x \geq 0$ where μ is a positive constant. In this case

$$(64) \qquad \Phi(s) = \log(1 + \frac{s}{\mu})$$

and $\rho = \underset{\sim}{E}\{\chi(1)\} = 1/\mu$.

If $\mu > 1$, then

$$(65) \qquad W(x) = \frac{\mu}{\mu-1} - \mu \int_0^\infty e^{-\mu(y+x)} \frac{[\mu(y+x)]^{y-1}}{\Gamma(y)} dy$$

for $x \geq 0$ and if $\mu < 1$, then

$$(66) \qquad W(x) = \frac{\mu^* e^{(\mu^*-\mu)x}}{\mu^* - 1} - \mu \int_0^\infty e^{-\mu(y+x)} \frac{[\mu(y+x)]^{y-1}}{\Gamma(y)} dy$$

for $x \geq 0$ where $\mu^* > 1$ and $\mu^* e^{-\mu^*} = \mu e^{-\mu}$.

Second, let us suppose that $\{\chi(u), 0 \leq u < \infty\}$ is a stable process of type $S(\alpha, 1, 1, 0)$ where $0 < \alpha < 1$. In this case

$$(67) \qquad \Phi(s) = s^\alpha$$

for $Re(s) \geq 0$ and $\rho = E\{\chi(1)\} = \infty$. By (59) we obtain that

$$(68) \qquad W(x) = \sum_{n=0}^\infty \frac{x^{n(1-\alpha)}}{\Gamma(n(1-\alpha)+1)}$$

for $x \geq 0$, that is $W(x) = E_{1-\alpha}(x^{1-\alpha})$ for $x \geq 0$ where $E_{1-\alpha}(z)$ is the Mittag-Leffler function defined by (42.180) for $0 < \alpha < 1$.

We note that we can prove directly that

$$(69) \qquad \lim_{t\to\infty} P\{\eta^*(t) \leq x\} = P\{\chi(u) \leq u + x \text{ for } 0 \leq u \leq \sigma(m-x)\}$$

for $0 \leq x \leq m$ where

$$(70) \qquad \sigma(m-x) = \inf\{u : \chi(u) - u \leq x-m \text{ for } 0 \leq u < \infty\}$$

and $\sigma(m-x) = \infty$ if $\chi(u) - u > x - m$ for all $u \geq 0$. Se reference [463].

Note. Finally, we mention a further generalization of Theorem 3. Let us suppose that $\{\xi(u), 0 \leq u < \infty\}$ is a separable, homogeneous process with independent increments for which the sample functions have no negative jumps and vanish at $u = 0$ with probability 1. Then

(71) $$E\{e^{-s\xi(u)}\} = e^{u\Psi(s)}$$

exists for $Re(s) \geq 0$ and

(72) $$\Psi(s) = as + \frac{1}{2}\sigma^2 s^2 + \int_0^\infty (e^{-sx} - 1 + \frac{sx}{1+x^2})dN(x)$$

where a is a real constant, σ^2 is a nonnegative constant, $N(x)$, $0 < x < \infty$, is a nondecreasing function of x satisfying the requirements $\lim_{x \to \infty} N(x) = 0$ and

(73) $$\int_0^1 x^2 dN(x) < \infty .$$

To exclude some trivial cases we suppose that either $\sigma^2 > 0$ or $a > 0$ and $N(x) \not\equiv 0$.

Now let us consider a dam in which the level of the dam may vary in the interval $(-\infty, \infty)$ and let $\eta(t) = \eta(0) + \xi(t)$ be the level of the dam at time t where $\eta(0)$ is a random variable which is independent of $\{\xi(u), 0 \leq u < \infty\}$ and for which $P\{0 \leq \eta(0) \leq m\} = 1$. Let us also define another dam process in which the initial content is $\eta^*(0) = \eta(0)$ and in which the level varies according to the process $\{\xi(u), 0 \leq u < \infty\}$, only in the interval $[0, m]$ where m is a positive constant. That is, the dam has capacity m, and the excess water overflows, and if necessary auxiliary water is used to ensure that the level never decrease below 0. Denote by $\eta^*(t)$ the level of the finite dam at time t.

In reference [464] we proved that the limiting distribution

$$\lim_{t \to \infty} P\{\eta^*(t) \leq x\} = \frac{W(x)}{W(m)} \tag{74}$$

exists for $0 \leq x \leq m$ and is independent of the distribution of $\eta^*(0)$. We have $W(x) = 0$ for $x < 0$, $W(x)$ is nondecreasing and continuous on the right in the interval $(0, \infty)$ and

$$\int_{-0}^{\infty} e^{-sx}\, dW(x) = \frac{s}{\Psi(s)} \tag{75}$$

for $\operatorname{Re}(s) > \omega_0$ where ω_0 is the largest nonnegative real root of $\Psi(s) = 0$.

Examples. First, let us suppose that $\{\xi(u), 0 \leq u < \infty\}$ is a separable Brownian motion process for which $E\{\xi(u)\} = \alpha u$ and $\operatorname{Var}\{\xi(u)\} = \sigma^2 u$ where $\sigma^2 > 0$. Then

$$P\left\{\frac{\xi(u) - \alpha u}{\sqrt{\sigma^2 u}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \tag{76}$$

and

$$\Psi(s) = -\alpha s + \frac{1}{2}\sigma^2 s^2 . \tag{77}$$

Now $\omega_0 = 0$ if $\alpha \leq 0$ and $\omega_0 = 2\alpha/\sigma^2$ if $\alpha > 0$. Since

(78) $$\int_0^\infty e^{-sx}\,dW(x) = \frac{2}{s(\sigma^2 s-2\alpha)}$$

for $\mathrm{Re}(s) > \omega_0$, we get by inversion that for $x \geqq 0$

(79) $$W(x) = \frac{1}{\alpha}\,(e^{2\alpha x/\sigma^2} - 1)$$

whenever $\alpha \neq 0$, and

(80) $$W(x) = \frac{2x}{\sigma^2}$$

whenever $\alpha = 0$.

As a second example, let us suppose that $\{\xi(u),\ 0 \leqq u < \infty\}$ is a separable stable process of type $S(1, 1, \frac{\pi}{2}, 1 - C - a)$ where $C = 0.5772157\ldots$ is Euler's constant. In this case

(81) $$\Psi(s) = as + \int_0^\infty (e^{-sx}-1 + \frac{sx}{1+x^2})\,\frac{dx}{x^2} = (a-1+C+\log s)s$$

for $\mathrm{Re}(s) \geqq 0$ and $\omega_0 = e^{1-C-a}$.

By (73) we obtain that

(82) $$W(x) = J(\omega_0\, x)$$

for $x \geqq 0$ where

(83) $$J(x) = \int_0^\infty \frac{x^u}{\Gamma(u+1)}\,du$$

for $x \geqq 0$. By a result of G. H. Hardy [428 p. 196] we can also write that

(84) $$J(x) = e^x - \int_0^\infty \frac{e^{-ux}}{u[\pi^2 + (\log u)^2]}\,du$$

for $x \geqq 0$.

65. Problems

65.1. Let us consider a single server queue with recurrent input and general service times. Denote by $\theta(t)$ the total idle time of the server in the time interval $(0, t)$. Give a method for finding the distribution of $\theta(t)$.

65.2. Let us consider a single server queue with recurrent input and general service times. Let us suppose that $P\{\eta_0 = 0\} = 1$ and χ_n $(n = 0,1,2,\ldots)$ and $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots;\ \tau_0 = 0)$ are independent sequences of mutually independent and identically distributed positive random variables. Denote by $\theta(t)$ the total idle time of the server in the time interval $(0, t)$. Find the limiting distribution of $\theta(t)$ as $t \to \infty$ in the case when $E\{\tau_n - \tau_{n-1}\} = a$ and $E\{\chi_n\} = b$ exist and $a < b$.

65.3. Let us consider a single server queue with recurrent input and general service times. Denote by $\tau_0 = 0$, $\tau_1, \tau_2, \ldots$ the arrival times, $\chi_0, \chi_1, \chi_2, \ldots$ the service times, and η_0 the initial occupation time of the server. Let us suppose that service is in order of arrival and denote by η_n the waiting time of the customer arriving at time τ_n. Let

$$\chi(u) = \sum_{0<\tau_n \leqq u} \chi_n$$

for $u \geqq 0$. Prove that if $P\{\tau_1 = \chi_1\} < 1$, then

$$\lim_{n \to \infty} P\{\eta_n \leqq x\} = P\{\sup_{0 \leqq u < \infty} [\chi(u) - u] \leqq x\} .$$

65.4. Let us consider a single server queue in which service is in order of arrival. Denote by $\tau_0 = 0, \tau_1, \tau_2, \ldots$ the arrival times, $\chi_0, \chi_1, \ldots$ the service times, and η_n the waiting time of the customer arriving at time τ_n. Define the inverse queue as a single server queue in which the arrival times are $0, \chi_0, \chi_0 + \chi_1, \ldots,$ the service times are $\tau_1, \tau_2 - \tau_1, \ldots$ and the initial occupation time of the server is η_0^*. Denote by ρ_0^* the number of customers served in the initial busy period in the inverse queue. Prove that

$$\underset{\sim}{P}\{\eta_n \leqq x \mid \eta_0 = 0\} = 1 - \underset{\sim}{P}\{\rho_0^* \leqq n \mid \eta_0^* = x\}$$

for $x > 0$ and $n = 0,1,2,\ldots$.

65.5. Prove formula (62.133) .

65.6. Let us consider a single server queue with recurrent input and general service times. Denote by a the expectation and σ_a^2 the variance of the interarrival times, and b the expectation and σ_b^2 the variance of the service times. Denote by η_n the waiting time of the n-th customer and by $\eta(t)$ the virtual waiting time at time t . Let us suppose that $a = b$ is a finite positive number and $0 < \sigma_a^2 + \sigma_b^2 < \infty$. Find the asymptotic distribution of η_n as $n \to \infty$ and the asymptotic distribution of $\eta(t)$ as $t \to \infty$. (See S. M. Brodi [39].)

65.7. Let us consider a single server queue with recurrent input and general service times. Denote by $F(x)$ the distribution function of the interarrival times, and $H(x)$ the distribution function of the service times. Let us suppose that

$$x^{\alpha}[1-H(x)] = h(x) \quad \text{and} \quad \lim_{x \to \infty} x^{\alpha}[1-F(x)] = 0$$

where $1 < \alpha < 2$ and $\lim_{x \to \infty} h(\omega x)/h(x) = 1$ for any $\omega > 0$, furthermore that

$$\int_0^{\infty} x dH(x) = \int_0^{\infty} x dF(x) = a$$

is a positive number. Find the asymptotic distribution of η_n, the waiting time of the n-th customer, and the asymptotic distribution of $\eta(t)$, the virtual waiting time at time t.

65.8. Let us consider a single server queue with recurrent input and general service times. Denote by $F(x)$ the distribution function of the interarrival times, and $H(x)$ the distribution function of the service times. Let us suppose that

$$\lim_{x \to \infty} [1-F(x)]x^{\alpha_1} = a_1 \quad \text{and} \quad \lim_{x \to \infty} [1-H(x)]x^{\alpha_2} = a_2$$

where a_1 and a_2 are finite positive numbers and $0 < \alpha_2 < \alpha_1 < 1$. Determine the asymptotic distribution of $\eta(t)$, the virtual waiting time at time t, as $t \to \infty$.

65.9. Prove (62.175) by using Theorem 59.3.

65.10. Let us consider a single server queue with recurrent input and general service times. Denote by a the expectation and σ_a^2 the variance of the interarrival times, and b the expectation and σ_b^2 the variance of the service times. Let us suppose that $a = b$ is a finite positive number and $0 < \sigma_a^2 + \sigma_b^2 < \infty$. Find the asymptotic distribution of $\theta(t)$, the total idle time of the server in the time interval $(0, t)$, as $t \to \infty$.

65.11. Prove (62.194) by using Theorem 59.3.

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