CHAPTER IV

ORDERED PARTIAL SUMS

28. Preliminaries. Let $x_0, x_1, \ldots, x_n$ be real numbers and arrange them in increasing order of magnitude. Let us assume that x_i precedes x_j if either $x_i < x_j$ or $x_i = x_j$ and $i < j$. Denote by

$$x_k^* = R_{n,k}(x_0, x_1, \ldots, x_n) \tag{1}$$

the k-th $(k = 0,1,\ldots, n)$ number in the ordered sequence. Then $x_0^* \leq x_1^* \leq \cdots \leq x_n^*$.

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables. Define $\zeta_n = \xi_1 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Write

$$\eta_{n,k} = R_{n,k}(\zeta_0, \zeta_1, \ldots, \zeta_n) \tag{2}$$

for $k = 0,1,\ldots, n$ and $n = 0,1,2,\ldots$. We say that $\eta_{n,k}$ is the k-th $(k = 0,1,\ldots, n)$ ordered partial sum in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$. We have $\eta_{n0} \leq \eta_{n1} \leq \cdots \leq \eta_{nn}$.

Our aim in this chapter is to give mathematical methods for finding the distribution of the random variable $\eta_{n,k}$ for $0 \leq k \leq n$. Such methods were given in 1952 by F. Pollaczek [5], in 1960 by J. G. Wendel [7] and in 1962 by D. A. Darling [2].

In what follows we shall introduce some auxiliary random variables which will be useful in solving our problem.

Let $(\alpha_{n0}, \alpha_{n1}, \ldots, \alpha_{nn})$ be that permutation of $(0,1,\ldots, n)$ for which $\eta_{nk} = \zeta_{\alpha_{nk}}$ for $k = 0,1,\ldots, n$. In other words α_{nk} $(k = 0,1,\ldots, n)$ is the subscript of the k-th ordered partial sum in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$. We have $\alpha_{nk} = j$ if and only if $\eta_{nk} = \zeta_j$.

Denote by $(\beta_{n0}, \beta_{n1}, \ldots, \beta_{nn})$ the inverse of the permutation $(\alpha_{n0}, \alpha_{n1}, \ldots, \alpha_{nn})$, that is, $\beta_{nj} = k$ if and only if $\alpha_{nk} = j$. In other words, β_{nk} $(k = 0,1,\ldots, n)$ is the rank of the k-th partial sum in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$. We have $\beta_{nk} = j$ if and only if $\eta_{nj} = \zeta_k$.

We note that $\eta_{nn} = \eta_n^*$ as we defined in Section 14 and $\eta_{n0} = -\bar{\eta}_n$ as we defined in Section 15. Furthermore, we have $\beta_{nn} = \rho_n^*$ as we defined in Section 22.

We are interested in finding the distribution of η_{nk} for $0 \leq k \leq n$; however, to achieve our goal we shall solve first a more general problem. Let us define the following expectation

$$(3) \qquad A_{nk}(s,v,z) = \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n} z^{\alpha_{nk}}\}$$

which exists if $\text{Re}(s) = \text{Re}(v) = 0$ and $0 \leq k \leq n$. We shall determine the generating function

$$(4) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n \omega^k$$

for $Re(s) = Re(v) = 0$, $|z| \leq 1$, $|\omega| \leq 1$ and $|\rho| < 1$. If $v = 0$ and $z = 1$ in (4) , then we obtain the generating function

(5) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} E\{e^{-s\eta_{nk}}\}\rho^n \omega^k$$

for $Re(s) = 0$, $|\omega| \leq 1$ and $|\rho| < 1$. From (5) we can obtain $E\{e^{-s\eta_{nk}}\}$ and $P\{\eta_{nk} \leq x\}$ can be obtained by inversion.

The determination of the generating function (4) makes it possible to solve another problem which we discussed at the end of Section 24. Denote by $\theta_n(x)$ the number of partial sums $\zeta_0, \zeta_1, \ldots, \zeta_n$ which are $\leq x$. According to Theorem 24.3 we have

(6) $$\sum_{n=0}^{\infty} \rho^n \int_{-\infty}^{\infty} e^{-sx} d_x E\{e^{-v\zeta_n} \omega^{\theta_n(x)}\} = -(1-\omega) \sum_{n=0}^{\infty} \sum_{k=0}^{n} E\{e^{-s\eta_{nk} - v\zeta_n}\}\rho^n \omega^k$$

for $Re(s) = Re(v) = 0$, $|\omega| < 1$ and $|\rho| < 1$. The right-hand side of (6) can be obtained by (4) in the particular case when $z = 1$. By (6) we can find the joint distribution of ζ_n and $\theta_n(x)$ for $n = 1,2,\ldots$ and for any real x .

Taking into account what we said above we can state our goal as the determination of the generating function (4).

If we would be able to provide a simple proof for the following relation

(7) $$A_{nk}(s,v,z) = A_{kk}(s,v,z)A_{n-k,0}(s,v,z)$$

which holds for $0 \leq k \leq n$, then we could obtain (4) immediately by using Theorem 24.1 . Although the relation (7) is simple its proof is far from evident. Actually, we shall conclude that (7) is true, only after (4) has been found.

To find (4) we shall introduce another expectation

$$B_{nj}(s,v,\omega) = \underset{\sim}{E}\{e^{-s\zeta_j - v\zeta_n}\omega^{\beta_{nj}}\} \tag{8}$$

which exists if $Re(s) = Re(v) = 0$ and $0 \leq j \leq n$. We shall show that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = \sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(s,v,\omega)\rho^n z^j \tag{9}$$

for $Re(s) = Re(v) = 0$, $|\rho z\omega| < 1$, $|\rho\omega| < 1$, $|\rho z| < 1$ and $|\rho| < 1$. Furthermore, we shall show that

$$B_{nj}(s,v,\omega) = B_{jj}(s,v,\omega)B_{n-j,0}(s,v,\omega) \tag{10}$$

for $0 \leq j \leq n$ which can easily be proved. Finally, $B_{nn}(s,v,\omega)$ and $B_{n0}(s,v,\omega)$ for $n = 0,1,2,\ldots$ can be obtained by Theorem 24.1 .

29. <u>The Determination of</u> $B_{nj}(s,v,\omega)$. For any event A define $\delta(A) = 1$ if A occurs and $\delta(A) = 0$ if A does not occur. The random variable $\delta(A)$ is called the indicator variable of the event A .

For any given j $(j = 0,1,\ldots, n)$ let us write $\bar{\zeta}_0 = \zeta_j - \zeta_j$, $\bar{\zeta}_1 = \zeta_{j+1} - \zeta_j, \ldots, \bar{\zeta}_{n-j} = \zeta_n - \zeta_j$ and define $\bar{\alpha}_{n-j,k}$ $(k = 0,1,\ldots, n-j)$ and $\bar{\beta}_{n-j,k}$ $(k = 0,1,\ldots, n-j)$ for the sequence $\bar{\zeta}_0, \bar{\zeta}_1,\ldots, \bar{\zeta}_{n-j}$ in exactly the same way as we defined α_{nk} and β_{nk} for $\zeta_0, \zeta_1,\ldots, \zeta_n$.

<u>Theorem 1. We have</u>

(1)
$$\delta(\alpha_{nk} = j) = \sum_{\max(0,j+k-n)\leq r\leq\min(j,k)} \delta(\alpha_{jr} = j)\delta(\bar{\alpha}_{n-j,k-r} = 0)$$

<u>for</u> $0 \leq k \leq n$, $0 \leq j \leq n$ <u>and</u> $n = 0,1,2,\ldots$.

<u>Proof</u>. The event $\{\alpha_{nk} = j\}$ can occur in several mutually exclusive ways: There is an r $[\max(0,j+k-n) \leq r \leq \min(j,k)]$ such that in the sequence $\zeta_0, \zeta_1,\ldots, \zeta_{j-1}$ precisely r elements precede ζ_j (that is, $\zeta_i \leq \zeta_j$ holds for precisely r subscripts $i = 0,1,\ldots, j-1$) and in the sequence $\zeta_{j+1},\ldots, \zeta_n$ precisely $k-r$ elements precede ζ_j (that is, $\zeta_i < \zeta_j$ holds for precisely $k-r$ subscripts $i = j+1,\ldots, n$) . Thus (1) follows.

Since $\{\alpha_{nk} = j\} \equiv \{\beta_{nj} = k\}$ we can write equivalently that

(2)
$$\delta(\beta_{nj} = k) = \sum_{\max(0,j+k-n)\leq r\leq\min(j,k)} \delta(\beta_{jj} = r)\delta(\bar{\beta}_{n-j,0} = k-r)$$

for $0 \leq k \leq n$, $0 \leq j \leq n$ and $n = 0,1,2,\ldots$.

Theorem 2. If $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(v) = 0$, then we have

(3) $$B_{nj}(s,v,\omega) = B_{jj}(s,v,\omega)B_{n-j,0}(s,v,\omega)$$

for $0 \leq j \leq n$.

Proof. Let us multiply (2) by $e^{-s\zeta_j - v\zeta_n}\omega^k$. Then we obtain that

(4) $$e^{-s\zeta_j - v\zeta_n}\omega^k \delta(\beta_{nj} = k) = \sum_{\max(0,j+k-n)\leq r\leq \min(j,k)} [e^{-s\zeta_j - v\zeta_j}\omega^r \delta(\beta_{jj} = r)] \cdot$$
$$\cdot [e^{-v(\zeta_n - \zeta_j)}\omega^{k-r}\delta(\overline{\beta}_{n-j,0} = k-r)] .$$

The two factors in brackets on the right-hand side of (4) are independent and the second factor has the same distribution as $e^{-v\zeta_{n-j}}\omega^{k-r}\delta(\beta_{n-j,0} = k-r)$. If we form the expectation of (4), then we get (3) which was to be proved.

Let us define

(5) $$B_n^+(v,\omega) = E\{e^{-v\zeta_n}\omega^{\beta_{nn}}\}$$

and

(6) $$B_n^-(v,\omega) = E\{e^{-v\zeta_n}\omega^{\beta_{n0}}\}$$

for $\mathrm{Re}(v) = 0$ and $n = 0,1,2,\dots$. Then we can write that

(7) $$B_{nn}(s,v,\omega) = B_n^+(s+v,\omega)$$

and

(8) $$B_{n0}(s,v,\omega) = B_n^-(v,\omega)$$

and thus

(9) $$B_{nj}(s,v,\omega) = B_j^+(s+v,\omega)B_{n-j}^-(v,\omega)$$

for $Re(s) = 0$, $Re(v) = 0$ and $0 \leqq j \leqq n$.

Theorem 3. We have

(10) $$B_n^-(v,\omega) = \omega^n B_n^+ (v, \frac{1}{\omega})$$

and

(11) $$B_n^+(v,\omega) = \omega^n B_n^- (v, \frac{1}{\omega})$$

for $n = 0,1,2,\ldots$ and $Re(v) = 0$.

Proof. We shall prove that the joint distribution of ζ_n and β_{nn} is the same as the joint distribution of ζ_n and $n-\beta_{n0}$. Hence both (10) and (11) follow.

Let us write $\bar{\zeta}_i = \zeta_n - \zeta_{n-i}$ for $i = 0,1,\ldots, n$ and define $\bar{\beta}_{n0}$ for $\bar{\zeta}_0, \bar{\zeta}_1,\ldots, \bar{\zeta}_n$ in the same way as we defined β_{n0} for $\zeta_0, \zeta_1,\ldots, \zeta_n$. Then we have $\bar{\beta}_{n0} = n - \beta_{nn}$. For if $\beta_{nn} = k$, then $\zeta_i \leqq \zeta_n$ for exactly k subscripts $i = 0,1,\ldots, n-1$ or, equivalently, $\bar{\zeta}_i = \zeta_n - \zeta_{n-i} \leqq 0$ for $n-k$ subscripts $i = 1,2,\ldots, n$, that is, $\bar{\beta}_{n0} = n-k$. Thus $\bar{\zeta}_n = \zeta_n$ and $\bar{\beta}_{n0} = n- \beta_{nn}$. Since evidently $(\bar{\zeta}_n, \bar{\beta}_{n0})$ and (ζ_n, β_{n0}) have identical two-dimensional distributions, it follows that

(12) $$E\{e^{-v\zeta_n}\omega^{\beta_{n0}}\} = E\{e^{-v\zeta_n}\omega^{n-\beta_{nn}}\}$$

for $Re(v) = 0$. This implies both (10) and (11).

Let us introduce the generating functions

(13) $$G^{+}(v,\rho,\omega) = \sum_{n=0}^{\infty} B_n^{+}(v,\omega)\rho^n$$

and

(14) $$G^{-}(v,\rho,\omega) = \sum_{n=0}^{\infty} B_n^{-}(v,\omega)\rho^n$$

for $\mathrm{Re}(v) = 0$, $|\omega| \leq 1$ and $|\rho| < 1$.

Theorem 4. We have

(15) $$G^{+}(v,\rho,\omega)G^{-}(v,\rho,\omega) = \frac{1}{[1-\rho\phi(v)][1-\rho\omega\phi(v)]}$$

for $\mathrm{Re}(v) = 0$, $|\rho\omega| < 1$ and $|\rho| < 1$ where

(16) $$\phi(v) = E\{e^{-v\xi_n}\}$$

for $\mathrm{Re}(v) = 0$.

Proof. By (9) we can write that

$$\sum_{j=0}^{n} B_j^{+}(v,\omega)B_{n-j}^{-}(v,\omega) = \sum_{j=0}^{n} B_{nj}(0,v,\omega) = \sum_{j=0}^{n} E\{e^{-v\zeta_n}\omega^{\beta_{nj}}\}$$

(17) $$= [1+\omega+\omega^2+\ldots+\omega^n][\phi(v)]^n$$

for $\mathrm{Re}(s) = 0$. If we multiply (17) by ρ^n and add for $n = 0,1,2,\ldots,$ then we get (15).

Theorem 5. If $\text{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$, then we have

(18) $$G^+(v,\rho,\omega) = G^+(v,\rho,0)G^-(v,\rho\omega,0)$$

and

(19) $$G^-(v,\rho,\omega) = G^+(v,\rho\omega,0)G^-(v,\rho,0)$$

where

(20) $$G^+(s,\rho,0) = \frac{e^{T\{\log[1-\rho\phi(s)]\}+\sum_{n=1}^{\infty}\frac{\rho^n}{n}P\{\zeta_n < 0\}}}{1-\rho\phi(s)} = \exp\left\{\sum_{n=1}^{\infty}\frac{\rho^n}{n}E\{e^{-s\zeta_n}\delta(\zeta_n < 0)\}\right\}$$

and

(21) $$G^-(s,\rho,0) = e^{-T\{\log[1-\rho\phi(s)]\}-\sum_{n=1}^{\infty}\frac{\rho^n}{n}P\{\zeta_n < 0\}} = \exp\left\{\sum_{n=1}^{\infty}\frac{\rho^n}{n}E\{e^{-s\zeta_n}\delta(\zeta_n \geq 0)\}\right\}$$

for $\text{Re}(s) = 0$ and $|\rho| < 1$.

Proof. Since $\beta_{nn} = \rho_n^*$ (the subscript of the last maximal element in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$), and since by Theorem 22.1 (ζ_n, ρ_n^*) and (ζ_n, Δ_n^*) have the same two-dimensional distribution, it follows that

(22) $$G^+(s,\rho,\omega) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} V_{nk}^*(s)\rho^n\omega^k$$

for $\text{Re}(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$ where $V_{nk}^*(s)$ is defined by (23.3). The right-hand side of (22) is determined by (23.10) and by Theorem 24.1. Thus we obtain that

(23) $$G^+(s,\rho,\omega) = \exp\{\sum_{n=1}^{\infty}[\frac{(\rho\omega)^n}{n} E\{e^{-s\zeta_n}\delta(\zeta_n \geq 0)\} + \frac{\rho^n}{n} E\{e^{-s\zeta_n}\delta(\zeta_n < 0)\}]\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$. By (15) and (23) we obtain that

(24) $$G^-(s,\rho,\omega) = \exp\{\sum_{n=1}^{\infty}[\frac{(\rho\omega)^n}{n} E\{e^{-s\zeta_n}\delta(\zeta_n < 0)\} + \frac{\rho^n}{n} E\{e^{-s\zeta_n}\delta(\zeta_n \geq 0)\}]\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

By Theorem 24.2 we can write that

(25) $$G^+(s,\rho,\omega) = \frac{e^{-T\{\log[1-\rho\omega\phi(s)]\}+T\{\log[1-\rho\phi(s)]\}}}{1-\rho\phi(s)} \exp\{\sum_{n=1}^{\infty}\frac{\rho^n(1-\omega^n)}{n} P\{\zeta_n < 0\}\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$. By (15) and (25) we obtain that

(26) $$G^-(s,\rho,\omega) = \frac{e^{-T\{\log[1-\rho\phi(s)]\}+T\{\log[1-\rho\omega\phi(s)]\}}}{1-\rho\omega\phi(s)} \exp\{-\sum_{n=1}^{\infty}\frac{\rho^n(1-\omega^n)}{n} P\{\zeta_n < 0\}\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

Formulas (23), (24) and (25), (26) prove Theorem 5.

Theorem 6. If $Re(s) = Re(v) = 0$, $|\rho\omega z| < 1$, $|\rho\omega| < 1$, $|\rho z| < 1$, and $|\rho| < 1$, then we have

(27) $$\sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(s,v,\omega)\rho^n z^j = G^+(s+v,\rho z,\omega)G^-(v,\rho,\omega)$$

and

(28) $$\sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(s,v,\frac{1}{\omega})(\rho\omega)^n z^j = G^-(s+v,\rho z,\omega)G^+(v,\rho,\omega)$$

where the right-hand sides can be obtained by (23) and (24) or by (25) and (26).

Proof. If we multiply (9) by $\rho^n z^j$ and add for $j = 0,1,\ldots, n$ and $n = 0,1,2,\ldots,$ then by (13) and (14) we obtain (27).

By (9), (10) and (11) we can write also that

$$\omega^n B_{nj}(s,v,\frac{1}{\omega}) = B_j^-(s+v,\omega)B_{n-j}^+(v,\omega) \tag{29}$$

for $0 \leq j \leq n$. If we multiply (29) by $\rho^n z^j$ and add for $j = 0,1,\ldots, n$ and $n = 0,1,2,\ldots,$ then by (13) and (14) we obtain (28).

30. The Determination of $A_{nk}(s,v,\omega)$. By using the results of Section 29 we are in the position to determine the generating function (28.4). Next we shall prove that (28.9) holds indeed.

Theorem 1. If $Re(s) = 0$, $Re(v) = 0$, $|\rho\omega z| < 1$, $|\rho\omega| < 1$, $|\rho z| < 1$ and $|\rho| < 1$, then we have

(1) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = G^+(s+v,\rho z,\omega)G^-(v,\rho,\omega)$$

and

(2) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^{n-k} = G^-(s+v,\rho z,\omega)G^+(v,\rho,\omega)$$

where the right-hand sides can be obtained by (23) and (24) or by (25) and (26).

Proof. Since we have the obvious relation

(3) $$\sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n}z^{\alpha_{nk}}\}\omega^k = \sum_{j=0}^{n} \underset{\sim}{E}\{e^{-s\zeta_j-v\zeta_n}\omega^{\beta_{nk}}\}z^j$$

for $Re(s) = Re(v) = 0$ and for any z and ω, or, equivalently,

(4) $$\sum_{k=0}^{n} A_{nk}(s,v,z)\omega^k = \sum_{j=0}^{n} B_{nj}(s,v,\omega)z^j ,$$

it follows that

(5) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = \sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(s,v,\omega)\rho^n z^j$$

for $Re(s) = Re(v) = 0$, $|\rho\omega z| < 1$, $|\rho\omega| < 1$, $|\rho z| < 1$ and $|\rho| < 1$.

This proves that (28.9) is indeed true. Formula (1) follows from (29.27) and (5).

If we replace ρ by $\rho\omega$ and ω by $1/\omega$ in (5) and if we use (29.28), then we obtain (2).

By (29.23) and (29.24) we can express (1) in the following equivalent way

$$(6)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} E\{e^{-s\eta_{nk}-v\zeta_n} z^{\alpha_{nk}}\}\rho^n\omega^k =$$

$$= \exp\Big\{\sum_{n=1}^{\infty}\Big[\frac{(\rho\omega z)^n}{n} E\{e^{-(s+v)\zeta_n}\delta(\zeta_n \geq 0)\} + \frac{(\rho z)^n}{n} E\{e^{-(s+v)\zeta_n}\delta(\zeta_n < 0)\} +$$

$$+ \frac{(\rho\omega)^n}{n} E\{e^{-v\zeta_n}\delta(\zeta_n < 0)\} + \frac{\rho^n}{n} E\{e^{-v\zeta_n}\delta(\zeta_n \geq 0)\}\Big]\Big\}$$

whenever $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$, $|\rho\omega z| < 1$, $|\rho z| < 1$, $|\rho\omega| < 1$ and $|\rho| < 1$.

If we use (29.25) and (29.26), then by (1) we can write that

$$(7)\quad [1-\rho z\phi(s+v)][1-\rho\omega\phi(v)]\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k =$$

$$= e^{-\Phi(s+v,\omega\rho z)+\Phi(s+v,\rho z)-\Phi(v,\rho)+\Phi(v,\rho\omega)}\exp\Big\{-\sum_{n=1}^{\infty}\frac{\rho^n(1-\omega^n)(1-z^n)}{n} P\{\zeta_n < 0\}\Big\}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$, $|\rho\omega z| < 1$, $|\rho z| < 1$, $|\rho\omega| < 1$ and $|\rho| < 1$, where

$$(8)\qquad \Phi(s,\rho) = T\{\log[1-\rho\phi(s)]\}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho| < 1$.

We can also express (2) in a similar form as (6) or (7).

We note that if $\omega = 1$ in (1), then we get

$$(9) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n = \frac{1}{[1-\rho\phi(v)][1-\rho z\phi(s+v)]}$$

for $\mathrm{Re}(s) = \mathrm{Re}(v) = 0$, $|\rho z| < 1$ and $|\rho| < 1$. This follows from (7) or it can be proved directly as follows:

$$(10) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n} z^{\alpha_{nk}}\}\rho^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \underset{\sim}{E}\{e^{-s\zeta_j-v\zeta_n}\}z^j\rho^n =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} [\phi(s+v)]^j[\phi(v)]^{n-j}z^j\rho^n = \frac{1}{[1-\rho\phi(v)][1-\rho z\phi(s+v)]}$$

for $\mathrm{Re}(s) = \mathrm{Re}(v) = 0$, $|\rho z| < 1$ and $|\rho| < 1$.

We mentioned at the beginning of Section 28 that the functions $A_{nk}(s,v,z)$ defined for $0 \leq k \leq n$ by (28.3) satisfy a simple relation, namely, (28.7). Now we shall prove that this relation is indeed true.

Theorem 2. The functions

$$(11) \qquad A_{nk}(s,v,z) = \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n} z^{\alpha_{nk}}\}$$

defined for $\mathrm{Re}(s) = \mathrm{Re}(v) = 0$ and $0 \leq k \leq n$ satisfy the relation

$$(12) \qquad A_{nk}(s,v,z) = A_{kk}(s,v,z)A_{n-k,0}(s,v,z)$$

for $0 \leq k \leq n$.

Proof. This theorem follows immediately from Theorem 1. If we put $\omega = 0$ in (1), then we obtain that

$$\sum_{n=0}^{\infty} A_{n0}(s,v,z)\rho^n = G^+(s+v,\rho z,0)G^-(v,\rho,0) \tag{13}$$

and if we put $\omega = 0$ in (2), then we obtain that

$$\sum_{n=0}^{\infty} A_{nn}(s,v,z)\rho^n = G^-(s+v,\rho z,0)G^+(v,\rho,0) \ . \tag{14}$$

If we replace ρ by $\rho\omega$ in (14) and if we form the product of (13) and (14), then we obtain $G^+(s+v,\rho z,\omega)G^-(v,\rho,\omega)$. This follows from Theorem 29.5 . Accordingly by (1), (13) and (14) we can conclude that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = \Big(\sum_{n=0}^{\infty} A_{n0}(s,v,z)\rho^n\Big)\Big(\sum_{n=0}^{\infty} A_{nn}(s,v,z)(\rho\omega)^n\Big) \tag{15}$$

for $Re(s) = Re(v) = 0$, $|\rho\omega z| < 1$, $|\rho\omega| < 1$; $|\rho z| < 1$ and $|\rho| < 1$. If we form the coefficient of $\rho^n\omega^k$ for $0 \leq k \leq n$ in (15), then we obtain (12) which was to be proved.

It is interesting to point out that we have the identity

$$A_{nk}(0,v,z) = B_{nk}(0,v,z) \tag{16}$$

for $Re(v) = 0$ and $0 \leq k \leq n$. This follows from (29.27) and (30.1). For by (29.27)

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(0,v,z)\rho^n\omega^j = G^+(v,\rho\omega,z)G^-(v,\rho,z) \tag{17}$$

and by (30.1)

$$(18)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(0,v,z)\rho^n\omega^k = G^+(v,\rho z,\omega)G^-(v,\rho,\omega) .$$

The right-hand sides of (17) and (18) are equal which can be seen by using (29.18) and (29.19). Accordingly (16) is indeed true.

In what follows we shall provide a direct proof for (16). The identity (16) can also be expressed in the following form.

Theorem 3. If $\text{Re}(v) = 0$ and $0 \leq k \leq n$, then we have

$$(19)\qquad E\{e^{-v\zeta_n}\omega^{\alpha_{nk}}\} = E\{e^{-v\zeta_n}\omega^{\beta_{nk}}\} .$$

Proof. First we shall prove (19) in the particular case of $k = 0$, and then we shall show that the general case can be reduced to this particular case.

Let $k = 0$. Let us define $\bar{\beta}_{n0}$ and $\bar{\alpha}_{n0}$ for the sequence $\bar{\zeta}_i = \zeta_n - \zeta_{n-i}$ $(i = 0,1,\ldots, n)$ in exactly the same way as we defined β_{n0} and α_{n0} for the sequence $\zeta_0, \zeta_1,\ldots, \zeta_n$. We have $\bar{\beta}_{n0} = n - \beta_{nn}$, $\bar{\alpha}_{n0} = n - \alpha_{nn}$ and $\bar{\zeta}_n = \zeta_n$. In proving Theorem 29.3 we have already shown that if $\beta_{nn} = k$, then $\bar{\beta}_{n0} = n-k$ for $k = 0,1,\ldots, n$. If $\alpha_{nn} = k$ where $k = 0,1,\ldots, n$, then $\zeta_i \leq \zeta_k$ for $0 \leq i \leq k$ and $\zeta_i < \zeta_k$ for $k < i \leq n$, or equivalently, $\bar{\zeta}_{n-i} \geq \bar{\zeta}_{n-k}$ for $0 \leq i \leq k$ and $\bar{\zeta}_{n-i} > \bar{\zeta}_{n-k}$ for $k < i \leq n$, that is, $\bar{\alpha}_{n0} = n-k$.

Accordingly, β_{n0} and ζ_n have the same joint distribution as $n- \beta_{nn}$ and ζ_n , and similarly, α_{n0} and ζ_n have the same joint distribution as $n- \alpha_{nn}$ and ζ_n .

Now let us prove (19) for $k = 0$. We shall prove by mathematical induction that β_{n0} and ζ_n have the same joint distribution as α_{n0} and ζ_n for $n = 0,1,2,\ldots$. If $n = 0$, then $\alpha_{00} = \beta_{00} = 0$, and the statement is true. Suppose that for n, where $n = 1,2,\ldots$, the variables $(\beta_{n-1,0}, \zeta_{n-1})$ and $(\alpha_{n-1,0}, \zeta_{n-1})$ have the same two-dimensional distribution. Since $\beta_{n-1,0}$, $\alpha_{n-1,0}$, and ζ_{n-1} do not depend on ξ_n and $\zeta_n = \zeta_{n-1} + \xi_n$, it follows that both $(\beta_{n-1,0}, \zeta_n)$ and $(\alpha_{n-1,0}, \zeta_n)$ have exactly the same two-dimensional distribution.

If $\zeta_n \geqq 0$, then evidently $\beta_{n0} = \beta_{n-1,0}$ and $\alpha_{n0} = \alpha_{n-1,0}$. Thus if $x \geqq 0$ and $j = 0,1,\ldots, n-1$, then

$$(20)\qquad \underset{\sim}{P}\{\beta_{n0} = j,\ \zeta_n \geqq x\} = \underset{\sim}{P}\{\beta_{n-1,0} = j,\ \zeta_n \geqq x\}$$

and

$$(21)\qquad \underset{\sim}{P}\{\alpha_{n0} = j,\ \zeta_n \geqq x\} = \underset{\sim}{P}\{\alpha_{n-1,0} = j,\ \zeta_n \geqq x\}\ .$$

By the induction hypothesis the right-hand sides of (20) and (21) are equal and hence

$$(22)\qquad \underset{\sim}{P}\{\beta_{n0} = j,\ \zeta_n \geqq x\} = \underset{\sim}{P}\{\alpha_{n0} = j,\ \zeta_n \geqq x\}$$

for $j = 0,1,\ldots, n$ and $x \geqq 0$. If $j = n$, then both sides of (22) are 0.

If $\zeta_n < 0$, then evidently $\beta_{nn} = \beta_{n-1,n-1}$ and $\alpha_{nn} = \alpha_{n-1,n-1}$ and thus for $x \leqq 0$ and $j = 1,2,\ldots, n$

$$\text{(23)}\quad \begin{aligned} &\underset{\sim}{P}\{\beta_{n0} = j\ ,\ \zeta_n < x\} = \underset{\sim}{P}\{\beta_{nn} = n-j,\ \zeta_n < x\} = \\ &= \underset{\sim}{P}\{\beta_{n-1,n-1} = n-j,\ \zeta_n < x\} = \underset{\sim}{P}\{\beta_{n-1,0} = j-1,\ \zeta_n < x\} \end{aligned}$$

and

$$\text{(24)}\quad \begin{aligned} &\underset{\sim}{P}\{\alpha_{n0} = j,\ \zeta_n < x\} = \underset{\sim}{P}\{\alpha_{nn} = n-j,\ \zeta_n < x\} = \\ &= \underset{\sim}{P}\{\alpha_{n-1,n-1} = n-j,\ \zeta_n < x\} = \underset{\sim}{P}\{\alpha_{n-1,0} = j-1,\ \zeta_n < x\}\ . \end{aligned}$$

By the induction hypothesis the right-hand sides of (23) and (24) are equal and hence

$$\text{(25)}\quad \underset{\sim}{P}\{\beta_{n0} = j,\ \zeta_n < x\} = \underset{\sim}{P}\{\alpha_{n0} = j,\ \zeta_n < x\}$$

for $x \leqq 0$ and $j = 0,1,\ldots, n$. If $j = 0$, then both sides of (25) are 0 .

Since (22) holds for all $x \geqq 0$ and (25) holds for all $x \leqq 0$, it follows that (β_{n0}, ζ_n) and (α_{n0}, ζ_n) have identical two-dimensional distributions. Thus by mathematical induction it follows that (19) is true for $k = 0$ and all $n = 0,1,2,\ldots$.

Since $(\bar{\beta}_{n0}, \bar{\zeta}_n) = (n-\beta_{nn}, \zeta_n)$ has the same two-dimensional distribution as (β_{n0}, ζ_n) , and $(\bar{\alpha}_{n0}, \bar{\zeta}_n) = (n-\alpha_{nn}, \zeta_n)$ has the same two-dimensional distribution as (α_{n0}, ζ_n) , it follows that (19) is also true for $k = n$ and all $n = 0,1,2,\ldots$.

Finally, it remains to prove that (19) is true for all $0 \leqq k \leqq n$ and $n = 0,1,2,\ldots,$ that is,

$$\text{(26)}\quad A_{nk}(0,v,\omega) = B_{nk}(0,v,\omega)$$

holds for all $0 \leq k \leq n$ and $n = 0,1,2,\ldots$. We shall use Theorem 29.2, Theorem 29.6 and Theorem 30.1 . Accordingly, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = \sum_{n=0}^{\infty}\sum_{j=0}^{n} B_{nj}(s,v,\omega)\rho^n z^j = \tag{27}$$

$$= G^+(s+v,\rho z,\omega)G^-(v,\rho,\omega) .$$

where

$$G^+(s+v,\rho z,\omega) = \sum_{n=0}^{\infty} B_{nn}(s,v,\omega)(\rho z)^n , \tag{28}$$

and

$$G^-(v,\rho,\omega) = \sum_{n=0}^{\infty} B_{n0}(0,v,\omega)\rho^n . \tag{29}$$

If we put $s = 0$ and $\omega = 0$ in (27), then we obtain that

$$\sum_{n=0}^{\infty} A_{n0}(0,v,z)\rho^n = G^+(v,\rho z,0)G^-(v,\rho,0) . \tag{30}$$

On the other hand since (26) is true for $k = 0$, it follows that

$$\sum_{n=0}^{\infty} A_{n0}(0,v,z)\rho^n = \sum_{n=0}^{\infty} B_{n0}(0,v,z)\rho^n = G^-(v,\rho,z) . \tag{31}$$

By comparing (30) and (31) we obtain that

$$G^-(v,\rho,z) = G^+(v,\rho z,0)G^-(v,\rho,0) \tag{32}$$

which is in agreement with (29.19).

If we replace ρ by $\rho\omega$ and ω by $1/\omega$ in (27) and if we put $s=0$ and $\omega = 0$ in it, then we obtain that

(33) $$\sum_{n=0}^{\infty} A_{nn}(0,v,z)\rho^n = G^-(v,\rho z,0)G^+(v,\rho,0) .$$

On the other hand since (26) is true for $k = n$, it follows that

(34) $$\sum_{n=0}^{\infty} A_{nn}(0,v,z)\rho^n = \sum_{n=0}^{\infty} B_{nn}(0,v,z)\rho^n = G^+(v,\rho,z) .$$

By comparing (33) and (34) we obtain that

(35) $$G^+(v,\rho,z) = G^+(v,\rho,0)G^-(v,\rho z,0)$$

which is in agreement with (29.18).

Thus by (27), (32) and (35)

(36) $$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k &= G^+(s+v,\rho z,\omega)G^-(v,\rho,\omega) = \\ &= G^+(s+v,\rho z,0)G^-(s+v,\rho z\omega,0)G^+(v,\rho\omega,0)G^-(v,\rho,0) .\end{aligned}$$

If $s = 0$ in (36), and if we interchange z and ω , then the right-hand side remains unchanged. Accordingly, we have the following identity

(37) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(0,v,z)\rho^n\omega^k = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(0,v,\omega)\rho^n z^k$$

and by (27) we can express the right-hand side of (37) as

(38) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{nk}(0,v,\omega)\rho^n z^k = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{nk}(0,v,z)\rho^n\omega^k .$$

Consequently by (37) and (38)

(39) $$A_{nk}(0,v,z) = B_{nk}(0,v,z)$$

for $\operatorname{Re}(v) = 0$ and $0 \leq k \leq n$ and $n = 0,1,2,\ldots$. This completes the proof of Theorem 3.

In conclusion, we mention that Theorem 3 implies Theorem 2, that is, the identity

$$A_{nk}(s,v,z) = A_{kk}(s,v,z)A_{n-k,0}(s,v,z) \tag{40}$$

for $0 \leq k \leq n$ and $\operatorname{Re}(s) = \operatorname{Re}(v) = 0$.

By (36) it follows that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{nk}(s,v,z)\rho^n\omega^k = \Big(\sum_{n=0}^{\infty} A_{n0}(s,v,z)\rho^n\Big)\Big(\sum_{n=0}^{\infty} A_{nn}(s,v,z)(\rho\omega)^n\Big) . \tag{41}$$

The two factors on the right-hand side of (41) can be obtained from (36) by putting $\omega = 0$ in (36) first, and then by replacing ρ by $\rho\omega$ and ω by $1/\omega$ in (36) and by putting $\omega = 0$ in it. If we form the coefficient of $\rho^n\omega^k$ in (41) for $0 \leq k \leq n$ then we obtain (40).

If we start with Theorem 3, then the problem of finding $A_{nk}(s,v,z)$ for $0 \leq k \leq n$ can be reduced to finding $G^+(v,\rho,z)$ and $G^-(v,\rho,z)$. These two functions can also be obtained directly from Theorem 29.4 by using the method of factorization.

31. <u>The Distribution of the</u> k-th <u>Ordered Partial Sum</u>. Let

(1) $$\Psi_{nk}(s,v) = \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n}\}$$

for $Re(s) = 0$, $Re(v) = 0$, $k = 0,1,\ldots, n$ and $n = 0,1,2,\ldots$. By (28.3) we can write also that

(2) $$\Psi_{nk}(s,v) = A_{nk}(s,v,1) .$$

<u>Theorem 1. If</u> $Re(s) = 0$, $Re(v) = 0$, $|\rho| < 1$ <u>and</u> $|\rho\omega| < 1$, <u>then we have</u>

(3) $$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \Psi_{nk}(s,v)\rho^n\omega^k = \frac{e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}}{[1-\rho\phi(v)]\,[1-\rho\phi(s+v)]}$$

<u>where</u> $\underset{\sim}{T}$ <u>operates on the variable</u> s .

<u>Proof</u>. Formula (3) is a particular case of (30.7) . If we put $z = 1$ in (30.7), then we can obtain (3).

We shall give, however, a separate proof for (3) based on the identity

(4) $$\Psi_{nk}(s,v) = \Psi_{kk}(s,v)\Psi_{n-k,0}(s,v)$$

which holds for $0 \leq k \leq n$ and $Re(s) = Re(v) = 0$. The identity (4) is a particular case of (30.12) or (30.40).

If we take into consideration that $\eta_{nn} = \eta_n^*$ defined in Section 17, then by Theorem 17.1 we obtain that

(5) $$\sum_{n=0}^{\infty} \Psi_{nn}(s,v)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. If we take into consideration that $\eta_{n0} = -\bar{\eta}_n$ defined in Section 17, then by Theorem 17.3 we obtain that

$$(6) \qquad \sum_{n=0}^{\infty} \Psi_{n0}(s,v)\rho^n = \frac{e^{\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}}{[1-\rho\phi(v)][1-\rho\phi(s+v)]}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$.

Since by (4)

$$(7) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Psi_{nk}(s,v)\rho^n\omega^k = \Big(\sum_{n=0}^{\infty} \Psi_{nn}(s,v)(\rho\omega)^n\Big)\Big(\sum_{n=0}^{\infty} \Psi_{n0}(s,v)\rho^n\Big)$$

for $|\rho\omega| < 1$, $|\rho| < 1$ and $\mathrm{Re}(s) = \mathrm{Re}(v) = 0$, we obtain (3) by (5) and (6).

Let $\eta_{nk}^{+} = \max(0, \eta_{nk})$. Our next aim is to give a method for finding the distribution of η_{nk}^{+} for $0 \leq k \leq n$.

Define

$$(8) \qquad \Phi_{nk}(s,v) = \underset{\sim}{E}\{e^{-s\eta_{nk}^{+}-v\zeta_n}\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$, $0 \leq k \leq n$ and $n = 0,1,2,\ldots$.

<u>Theorem 2.</u> <u>We have</u>

$$(9) \qquad (1-\omega)[1-\rho\phi(v)] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Phi_{nk}(s,v)\rho^n\omega^k = 1 - \omega\, e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}$$

<u>for</u> $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$, $|\rho| < 1$ <u>and</u> $|\rho\omega| < 1$.

Proof. By (3) we can write that

$$(10)\quad (1-\omega)[1-\rho\phi(v)] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Psi_{nk}(s,v)\rho^n\omega^k = \\ = \left[\frac{1-\rho\omega\phi(s+v)}{1-\rho\phi(s+v)} - \omega\right] e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}$$

for $Re(s) = 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$. We can see easily that (10) considered as a function of s belongs to the space $\underset{\sim}{R}$ introduced in Section 2. On the left-hand side of (10) the functions $\Psi_{nk}(s,v)$ belong to $\underset{\sim}{R}$, and we can apply the operator $\underset{\sim}{T}$ term by term in the double sum. Since

$$(11)\quad \underset{\sim}{T}\{\Psi_{nk}(s,v)\} = \Phi_{nk}(s,v)$$

for $Re(s) \geq 0$, $Re(v) = 0$ and $0 \leq k \leq n$, it follows that if we apply the operator $\underset{\sim}{T}$ to the left-hand side of (10), then we obtain the left-hand side of (9). We shall show that if we apply $\underset{\sim}{T}$ to the right-hand side of (10), then we get the right-hand side of (9) which implies the theorem. It is sufficient to show that

$$(12)\quad \underset{\sim}{T}\left\{ \frac{1-\rho\omega\phi(s+v)}{1-\rho\phi(s+v)} e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}} \right\} = 1 .$$

This is true because obviously

$$(13)\quad \underset{\sim}{T}\left\{ \frac{e^{\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}}{1-\rho\phi(s+v)} \right\} = \\ = \underset{\sim}{T}\{e^{-\log[1-\rho\phi(s+v)]+\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}\} = 1$$

and

$$(14)\qquad \underset{\sim}{T}\{[1-\rho\omega\phi(s+v)]e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\}}\} = \\ = \underset{\sim}{T}\{e^{\log[1-\rho\omega\phi(s+v)]-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\}}\} = 1 .$$

For if $\underset{\sim}{T}\{\Phi_1(s)\} = 1$ and $\underset{\sim}{T}\{\Phi_2(s)\} = 1$, then $\underset{\sim}{T}\{\Phi_1(s)\Phi_2(s)\} = 1$.

In what follows we shall consider, some particular cases of Theorem 2 separately.

First, we note that $\eta_{nn} \geqq 0$ and therefore $\eta_{nn}^{+} = \eta_{nn}$ and $\Phi_{nn}(s,v) = \Psi_{nn}(s,v)$. Thus by (5) we have

$$(15)\qquad \sum_{n=0}^{\infty} \Phi_{nn}(s,v)\rho^n = \sum_{n=0}^{\infty} \Psi_{nn}(s,v)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}$$

for $Re(s) \geqq 0$, $Re(v) = 0$ and $|\rho| < 1$. The same result can be obtained from (9) if we replace ρ by $\rho\omega$ and ω by $1/\omega$ in it and put $\omega = 0$ in it.

Second, we note that

$$(16)\qquad \sum_{n=0}^{\infty} \Phi_{n0}(s,v)\rho^n = \frac{1}{1-\rho\phi(v)}$$

for $Re(v) = 0$ and $|\rho| < 1$. Since $\eta_{n0} \leqq 0$, it follows that $\eta_{n0}^{+} = 0$ and $\Phi_{n0}(s,v) = [\phi(v)]^n$ for $n = 0,1,\ldots$.

Third, we have

$$(17)\qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Phi_{nk}(s,v)\rho^n = \frac{1}{1-\rho\phi(v)} \underset{\sim}{T}\{ \frac{1}{1-\rho\phi(s+v)}\}$$

for $Re(s) \geqq 0$, $Re(v) = 0$ and $|\rho| < 1$.

Since

(18)
$$\sum_{k=0}^{n} \Psi_{nk}(s,v) = \sum_{k=0}^{n} E\{e^{-s\eta_{nk}-v\zeta_n}\} = \sum_{j=0}^{n} E\{e^{-s\zeta_j-v\zeta_n}\} = $$
$$= \sum_{j=0}^{n} [\phi(s+v)]^j[\phi(v)]^{n-j}$$

for $Re(s) = Re(v) = 0$, it follows that

(19)
$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \Psi_{nk}(s,v)\rho^n = \frac{1}{[1-\rho\phi(v)][1-\rho\phi(s+v)]}$$

for $Re(s) = 0$, $Re(v) = 0$ and $|\rho| < 1$. If we apply T to (19), then we get (17). The same result can also be obtained from (9) if we let $\omega \to 1$ in it.

32. <u>A Generalization of the Previous Results</u>. Let us suppose that $\bar{\xi}$ is a real random variable. Let $\bar{\xi}$ and the sequence $\{\xi_n, n = 1,2,\ldots\}$ be independent. Define

(1) $$\bar{\phi}(s) = \underset{\sim}{E}\{e^{-s\bar{\xi}}\}$$

for $\mathrm{Re}(s) = 0$.

In what follows we shall be concerned with the random variables

(2) $$\bar{\eta}_{nk} = R_{nk}(\zeta_0 + \bar{\xi}, \zeta_1 + \bar{\xi}, \ldots, \zeta_n + \bar{\xi})$$

where $0 \leq k \leq n$. If $\bar{\xi} = 0$, then (2) reduces to η_{nk} defined by (28.2) . In the previous section we dealt with various problems connected with the variables η_{nk} where $0 \leq k \leq n$. In this section we consider analogous problems for $\bar{\eta}_{nk}$ where $0 \leq k \leq n$.

First, let us define

(3) $$\bar{\Psi}_{nk}(s,v) = \underset{\sim}{E}\{e^{-s\bar{\eta}_{nk} - v(\zeta_n + \bar{\xi})}\}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$ and $0 \leq k \leq n$ as a generalization of (31.1) . Since obviously

(4) $$\bar{\eta}_{nk} = \eta_{nk} + \bar{\xi}$$

for $0 \leq k \leq n$, it follows that

(5) $$\bar{\Psi}_{nk}(s,v) = \bar{\phi}(s+v)\Psi_{nk}(s,v)$$

for $\mathrm{Re}(s) =$, $\mathrm{Re}(v) = 0$ and $0 \leq k \leq n$. That is, if $\Psi_{nk}(s,v)$ is known, which can be obtained by Theorem 31.1, then $\bar{\Psi}_{nk}(s,v)$ is determined by (5).

Let $\bar{\eta}_{nk}^{+} = \max(0, \bar{\eta}_{nk})$. Our next aim is to find the distribution of $\bar{\eta}_{nk}^{+}$ for $0 \leq k \leq n$.

Define

$$(6) \qquad \bar{\Phi}_{nk}(s,v) = \underset{\sim}{E}\{e^{-s\bar{\eta}_{nk}^{+}-v(\zeta_n+\bar{\xi})}\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $0 \leq k \leq n$.

If we know $\bar{\Psi}_{nk}(s,v)$ for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(v) = 0$, then (6) can be obtained by

$$(7) \qquad \bar{\Phi}_{nk}(s,v) = \underset{\sim}{T}\{\bar{\Psi}_{nk}(s,v)\}$$

for $\mathrm{Re}(s) \geq 0$ and $R(v) = 0$. Here $\underset{\sim}{T}$ operates on the variable s whereas v is a parameter. By (5) we can write that

$$(8) \qquad \bar{\Phi}_{nk}(s,v) = \underset{\sim}{T}\{\bar{\phi}(s+v)\Psi_{nk}(s,v)\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $0 \leq k \leq n$. By Theorem 31.1 we can deduce the following theorem.

Theorem 1. We have

$$(9) \qquad [1-\rho\phi(v)] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \bar{\Phi}_{nk}(s,v)\rho^n\omega^k = \\ = \underset{\sim}{T}\left\{\frac{\bar{\phi}(s+v)}{1-\rho\phi(s+v)}\, e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}\right\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

Proof. If we multiply (31.3) by $\bar{\phi}(s+v)$ and apply the transformation $\underset{\sim}{T}$, then we obtain (9).

Note. It is interesting to note that if we know (9) in the particular case when $\bar{\xi} = c$ (constant), then $\Psi_{nk}(s,v)$ can be obtained from (7) by the following limiting procedure:

$$(10) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Psi_{nk}(s,v)\rho^n \omega^k = \lim_{c\to\infty} e^{c(s+v)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \bar{\Phi}_{nk}(s,v)\rho^n \omega^k$$

for $\text{Re}(s) = 0$, $\text{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

This follows from the following observation. If $\Phi(s) \in \underset{\sim}{R}$, then we have

$$(11) \qquad \Phi(s) = \lim_{c\to\infty} e^{cs} \underset{\sim}{T}\{e^{-cs}\Phi(s)\}$$

for $\text{Re}(s) = 0$. For if $\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$ where $\underset{\sim}{E}\{|\zeta|\} < \infty$, then

$$(12) \qquad \underset{\sim}{E}\{\zeta e^{-s\eta}\} = \lim_{c\to\infty} \underset{\sim}{E}\{\zeta e^{-s\{[\eta+c]^+ - c\}}\} = \lim_{c\to\infty} e^{cs} \underset{\sim}{E}\{\zeta e^{-s[\eta+c]^+}\}$$

for $\text{Re}(s) = 0$.

If

$$(13) \qquad \eta_{nk} = R_{nk}(\zeta_0, \zeta_1, \ldots, \zeta_n),$$

then

$$(14) \qquad \eta_{nk} + c = R_{nk}(\zeta_0 + c, \zeta_1 + c, \ldots, \zeta_n + c)$$

and

$$(15) \qquad [\eta_{nk} + c]^+ = R_{nk}([\zeta_0 + c]^+, [\zeta_1 + c]^+, \ldots, [\zeta_n + c]^+).$$

Hence

$$(16)\qquad \eta_{nk} = \lim_{c \to \infty} \{[\eta_{nk} + c]^+ - c\} = \lim_{c \to \infty} \{R_{nk}([\zeta_0 + c]^+, [\zeta_1 + c]^+, \ldots, [\zeta_n + c]^+) - c\} .$$

If we take into consideration that in (9)

$$(17)\qquad \begin{aligned} &\lim_{c \to \infty} e^{c(s+v)} \underset{\sim}{T}\left\{ \frac{e^{-c(s+v)}}{1-\rho\phi(s+v)} e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}} \right\} \\ &= \frac{e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}}{1 - \rho\phi(s+v)} \end{aligned}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$, then we can prove Theorem 31.1 by Theorem 1 if we apply it to $\overline{\phi}(s+v) = e^{-c(s+v)}$.

33. An Alternative Method. In this section we shall mention briefly an alternative method for proving Theorem 31.2 . Accordingly, our aim is again to find

(1) $$\Phi_{nk}(s,v) = \underset{\sim}{E}\{e^{-s\eta_{nk}^{+} - v\zeta_n}\}$$

for $Re(s) \geq 0$, $Re(v) = 0$ and $0 \leq k' \leq n$. Obviously, we have

(2) $$\Phi_{n0}(s,v) = \underset{\sim}{E}\{e^{-v\zeta_n}\} = [\phi(v)]^n$$

for $Re(v) = 0$ and $n = 0,1,2,\ldots$.

By using Dirichlet's discontinuity factors F . Pollaczek [5] deduced a recurrence formula for $\Phi_{nj}(s,v)$. From Pollaczek's formula we can easily deduce the following equivalent recurrence formula:

(3) $$\sum_{j=k}^{n} \Phi_{nj}(s,v) = \sum_{r=0}^{n-k} \underset{\sim}{T}\{[\phi(s+v)]^r \Phi_{n-r,k}(s,v)\}$$

for $Re(s) \geq 0$, $Re(v) = 0$, $0 \leq k \leq n$ and $n = 0,1,2,\ldots$.

This recurrence formula makes it possible to find $\Phi_{nk}(s,v)$ for $0 \leq k \leq n$ and $n = 0,1,2,\ldots$.

Let us introduce the generating function

(4) $$U(s,v,\rho,\omega) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Phi_{nk}(s,v)\rho^n\omega^k$$

for $Re(s) \geq 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

First, we observe that

$$(5)\quad U(s,v,\rho,1) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta^{+}_{nk}-v\zeta_n}\}\rho^n = \underset{\sim}{T}\{\sum_{n=0}^{\infty}\sum_{j=0}^{n} \underset{\sim}{E}\{e^{-s\zeta_j-v\zeta_n}\}\rho^n\} =$$

$$= \underset{\sim}{T}\{\sum_{n=0}^{\infty}\sum_{j=0}^{n} [\phi(s+v)]^j[\phi(v)]^{n-j}\rho^n\} = \frac{1}{1-\rho\phi(v)}\underset{\sim}{T}\{\frac{1}{1-\rho\phi(s+v)}\},$$

that is,

$$(6)\quad [1-\rho\phi(v)]U(s,v,\rho,1) = \underset{\sim}{T}\{\frac{1}{1-\rho\phi(s+v)}\}$$

for $Re(s) \geqq 0$, $Re(v) = 0$ and $|\rho| < 1$.

Next we obtain that

$$(7)\quad (1-\omega)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=k}^{n}\Phi_{nj}(s,v)\rho^n\omega^k = (1-\omega)\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\Phi_{nj}(s,v)\rho^n\omega^k =$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\Phi_{nj}(s,v)\rho^n(1-\omega^{j+1}) = U(s,v,\rho,1) - \omega U(s,v,\rho,\omega) =$$

$$= \frac{1}{1-\rho\phi(v)}\underset{\sim}{T}\{\frac{1}{1-\rho\phi(s+v)}\} - \omega U(s,v,\rho,\omega)$$

for $Re(s) \geqq 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

On the other hand by (3) we can write that

$$(8)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=k}^{n}\Phi_{nj}(s,v)\rho^n\omega^k = \underset{\sim}{T}\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}[\phi(s+v)]^r\Phi_{n-r,k}(s,v)\} =$$

$$= \underset{\sim}{T}\{\sum_{n=0}^{\infty}\rho^n\sum_{r=0}^{n}\sum_{k=0}^{n-r}\Phi_{n-r,k}(s,v)[\phi(s+v)]^r\omega^k\} =$$

$$= \underset{\sim}{T}\{\sum_{n=0}^{\infty}\sum_{r=0}^{n}[\rho\phi(s+v)]^r\rho^{n-r}\sum_{k=0}^{n-r}\Phi_{n-r,k}(s,v)\omega^k\} =$$

$$= \underset{\sim}{T}\{(\sum_{r=0}^{\infty} [\rho\phi(s+v)]^r)(\sum_{n=0}^{\infty}\sum_{k=0}^{n} \Phi_{n,k}(s,v)\rho^n\omega^k)\} =$$

$$= \underset{\sim}{T}\{\frac{U(s,v,\rho,\omega)}{1-\rho\phi(s+v)}\}$$

for $Re(s) \geq 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

By (7) and (8) we obtain that

(9) $$\omega U(s,v,\rho,\omega)+(1-\omega)\underset{\sim}{T}\{\frac{U(s,v,\rho,\omega)}{1-\rho\phi(s+v)}\} = \frac{1}{1-\rho\phi(v)}\underset{\sim}{T}\{\frac{1}{1-\rho\phi(s+v)}\},$$

or equivalently,

(10) $$\underset{\sim}{T}\{\frac{1-\rho\omega\phi(s+v)}{1-\rho\phi(s+v)}[(1-\omega)U(s,v,\rho,\omega) - \frac{1}{1-\rho\phi(v)}]\} = -\frac{\omega}{1-\rho\phi(v)}$$

for $Re(s) \geq 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

The solution of (9) or (10) can be expressed as

(11) $$U(s,v,\rho,\omega) = \frac{1-\omega e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s+v)]\}+ \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}}{(1-\omega)[1-\rho\phi(v)]}$$

for $Re(s) \geq 0$, $Re(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$. This can easily be proved. If $U(s,v,\rho,\omega)$ is given by (11), then $\underset{\sim}{T}\{U(s,v,\rho,\omega)\} = U(s,v,\rho,\omega)$ and $U(s,v,\rho,\omega)$ satisfies (9) and (10). If we expand $U(s,v,\rho,\omega)$ into a power series according to (4), then it follows from (9) that the coefficients $\Phi_{nk}(s,v)$ satisfy (2) and (3). This proves that (11) is indeed the correct formula. Formula (11) is in agreement with (31.9).

34. Problems

34.1. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables. Let $\zeta_n = \xi_1 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Denote by α_{nk} $(k = 0,1,\ldots,n)$ the subscript of the k-th ordered partial sum in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$. Find the probability $\mathbf{P}\{\alpha_{nk} = j\}$ for $j = 0,1,\ldots, n$.

34.2. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and idnetically distributed random variables for which $\mathbf{E}\{e^{-s\xi_n}\} = \lambda\psi(s)/(\lambda - s)$ for $0 \leq \mathrm{Re}(s) < \lambda$ where $\psi(s)$ is the Laplace-Stieltjes transform of a nonnegative random variable. Find $\Phi_{nk}(s) = \mathbf{E}\{e^{-s\eta^+_{n,k}}\}$ for $0 \leq k \leq n$ and $\mathrm{Re}(s) \geq 0$ where $\eta_{n,k}$ is defined by (28.2).

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