T H E O R Y   O F   R A N D O M   F L U C T U A T I O N S

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TO SUSAN

PREFACE

The theory of random fluctuations is concerned with the investigation of the stochastic properties of sums of random variables and the stochastic properties of sample functions of stochastic processes.

The theory of random fluctuations developed concurrently with the advancement of probability theory. The first steps toward fluctuation theory were made by Jakob Bernoulli (1654-1705), who studied the fluctuations of the number of occurrences of a given event in a sequence of repeated random trials, Pierre-Simon Laplace (1749-1827), who studied the fluctuations of sums of independent and identically distributed random variables, and Louis Bachelier (1870 - 1946), who studied the fluctuations of the sample functions of diffusion processes.

The development of the theory of random fluctuations made great progress in the last fifty years. Important general limit theorems were discovered, and various mathematical methods were found to solve specific problems in fluctuation theory. To give a few examples, the weak law of large numbers, the strong law of large numbers, the law of iterated logarithm, the central limit theorem and other limit theorems were proved for sums of random variables and for stochastic processes, and ingenious methods were worked out to solve many problems in the fields of physics (Brownian motion, diffusion), engineering (telephone traffic, dams), industry (storage, production lines), transportation (queues, traffic), insurance and others.

The aim of this book is to give a comprehensive treatment of fluctuation theory and an account of its historical development. The book contains many old and new results which are presented in a simple and unified way. A significant part of the book contains original results which were obtained by the author in the past few years and many of them are published here for the first time.

Comprehension of the material in this book requires only a knowledge of the elements of probability theory and stochastic processes. We give complete proofs for most of the theorems used in the book.

The contents of the book may be briefly described as follows: Chapters I - VI deal with fluctuation problems concerning sums of random variables. Chapters I-III cover finite fluctuation problems. Here the theory of complex variables, algebraic methods, the method of ladder indices, and combinatorial methods are used. Chapter V deals with random walk, ballot, and order statistics. Chapter VI deals with limit theorems and limit distributions for sums of random variables. Stable distributions are thoroughly discussed in this chapter. Chapters VII-X deal with fluctuation problems concerning stochastic processes. Here algebraic, analytic, and combinatorial methods are used for finding various exact and limit distributions. Chapter X deals with queuing processes, insurance risk processes, and storage and dam processes. The Appendix contains several useful auxiliary theorems in the field of probability theory and in some other fields. There are more than one hundred problems for solution and a complete solution is given for each problem.

At the end of each chapter there is a list of references containing the papers and books mentioned in that chapter and numerous other references on related topics.

The ten chapters and the appendix are subdivided into sections. The numbering of the formulas and theorems starts anew in each section. If a reference is made to a formula or theorem in the same section, the section number is not indicated. References to formulas or theorems in another section are indicated by two numbers separated by a dot. The first number indicates the section and the second, the serial number of the formula or the theorem in that section.

Finally, I would like to acknowledge the financial support which I received from the National Science Foundation and from the Office of Naval Research during the writing of the book.

Cleveland

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CONTENTS

SOLUTIONS

INTRODUCTION

The theory of random fluctuations is concerned with the investigation of the stochastic properties of sums of random variables and the stochastic properties of sample functions of stochastic processes. The aim of fluctuation theory is to deduce general theorems which reveal the nature of random fluctuations, and to give various mathematical methods which can be used in solving specific problems. The purpose of this book is to fulfill this aim. We shall deduce general theorems and work out various methods in fluctuation theory. The general results will be applied in the theories of random walk, ballots, order statistics, queues, insurance and storage.

In this book we consider various types of sequences of random variables and various types of stochastic processes and deduce some exact and limit theorems for such sequences and processes.

We shall consider either a finite number of real random variables $\xi_1, \xi_2, \ldots, \xi_m$ or an infinite sequence of real random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ which satisfy one of the following assumptions.

(a) The random variables $\xi_1, \xi_2, \ldots, \xi_m$ are *mutually independent* and *identically distributed* with distribution function $F(x)$, that is,

$$\mathbf{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_m \leq x_m\} = F(x_1)F(x_2)\ldots F(x_m)$$

for all real $x_1, x_2, \ldots, x_m$.

(b) The random variables $\xi_1, \xi_2, \ldots, \xi_m$ are *interchangeable*, that is,

$$\underset{\sim}{P}\{\xi_{i_1} \leq x_1, \xi_{i_2} \leq x_2, \ldots, \xi_{i_m} \leq x_m\} = \underset{\sim}{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_m \leq x_m\}$$

for all the $m!$ permutations $(i_1, i_2, \ldots, i_m)$ of $(1, 2, \ldots, m)$ and for all real $x_1, x_2, \ldots, x_m$.

(c) The random variables $\xi_1, \xi_2, \ldots, \xi_m$ are <u>cyclically interchangeable</u>, that is,

$$\underset{\sim}{P}\{\xi_{i_1} \leq x_1, \xi_{i_2} \leq x_2, \ldots, \xi_{i_m} \leq x_m\} = \underset{\sim}{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_m \leq x_m\}$$

for all the m cyclic permutations $(i_1, i_2, \ldots, i_m)$ of $(1, 2, \ldots, m)$ and for all real $x_1, x_2, \ldots, x_m$.

(d) The random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are <u>mutually independent</u> and <u>identically distributed</u> with distribution function $F(x)$, that is, (a) is satisfied for all $m = 1,2,\ldots$.

(e) The random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are <u>interchangeable</u>, that is, (b) is satisfied for all $m = 2, 3, \ldots$.

We use the notation $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ $(n = 1,2,\ldots)$ and $\zeta_0 = 0$. We shall prove various theorems for the sequence $\{\zeta_n, n = 0,1,2,\ldots\}$ and we shall determine the distributions of various functionals defined on the sequence $\{\zeta_n\}$.

We shall consider various types of real stochastic processes satisfying one of the following assumptions.

(a) The process $\{\xi(u), 0 \leq u < \infty\}$ is <u>homogeneous</u> and has <u>independent increments</u>, that is, $\mathbf{P}\{\xi(0) = 0\} = 1$, $\mathbf{P}\{\xi(u+t) - \xi(u) \leq x\} = \mathbf{P}\{\xi(t) \leq x\}$ for all $u \geq 0$, $t \geq 0$ and real x, and

$$\mathbf{P}\{\xi(t_r)-\xi(t_{r-1}) \leq x_r \text{ for } r = 1,2,\ldots,n\} = \prod_{r=1}^{n} \mathbf{P}\{\xi(t_r)-\xi(t_{r-1}) \leq x_r\}$$

for $0 \leq t_0 < t_1 < \ldots < t_n$, $n = 2,3,\ldots$, and for any real $x_1, x_2, \ldots, x_n$.

If, in particular,

$$\mathbf{P}\{\xi(u) = k\} = e^{-\lambda u} \frac{(\lambda u)^k}{k!}$$

for $u \geq 0$ and $k = 0,1,2,\ldots$, then we say that $\{\xi(u), 0 \leq u < \infty\}$ is a <u>homogeneous Poisson process</u> of density λ.

If, in particular,

$$\mathbf{P}\{\xi(u) \leq xu^{1/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

for $u > 0$ and any real x, then we say that $\{\xi(u), 0 \leq u < \infty\}$ is a <u>Brownian motion process</u>.

(b) The process $\{\xi(u), 0 \leq u \leq t\}$ has <u>interchangeable increments</u>, that is, for all $n = 2,3,\ldots$ the random variables

$$\xi\left(\frac{kt}{n}\right) - \xi\left(\frac{(k-1)t}{n}\right) \quad (k = 1,2,\ldots, n)$$

are interchangeable random variables.

(c) The process $\{\xi(u) , 0 \leqq u \leqq t\}$ is <u>Gaussian</u>, that is, for any $n = 1,2,\ldots$ and $0 < u_1 < u_2 <\ldots< u_n < t$, the random variables $\xi(u_1)$, $\xi(u_2),\ldots, \xi(u_n)$ have an n-dimensional normal distribution.

(d) The process $\{\nu(u), 0 \leqq u < \infty\}$ is a <u>recurrent process</u>, that is, for each $u \geqq 0$ the $\nu(u)$ is a discrete random variable which takes on only nonnegative integers and which satisfies the relation

$$\{\nu(u) < k\} \equiv \{\theta_1 + \theta_2 + \ldots + \theta_k > u\}$$

for all $u \geqq 0$ and $k = 1,2,\ldots$, where $\theta_1, \theta_2,\ldots, \theta_k,\ldots$ is a sequence of mutually independent and identically distributed positive random variables.

If, in particular,

$$\underset{\sim}{P}\{\theta_k \leqq x\} = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

where λ is a positive constant, then $\{\nu(u) , 0 \leqq u < \infty\}$ is a Poisson process of density λ.

(e) The process $\{\chi(u) , 0 \leqq u < \infty\}$ is a <u>compound recurrent process</u>, that is,

$$\chi(u) = \sum_{1 \leqq i \leqq \nu(u)} \chi_i$$

for $u \geqq 0$ where $\chi_1, \chi_2,\ldots, \chi_i,\ldots$ is a sequence of mutually independent and identically distributed real random variables, $\{\nu(u) , 0 \leqq u < \infty\}$ is a recurrent process, and $\{\chi_i\}$ and $\{\nu(u) , 0 \leqq u < \infty\}$ are independent.

If, in particular, $\{\nu(u)\ ,\ 0 \leq u < \infty\}$ is a Poisson process, then $\{\chi(u)\ ,\ 0 \leq u < \infty\}$ is called a compound Poisson process.

We shall prove various theorems for the above mentioned processes and we shall determine the distributions of various functionals defined on these processes.

In what follows we shall give a brief description of the contents of the book.

In Chapter I we consider a space $\underset{\sim}{R}$ of functions $\Phi(s)$ defined for $\mathrm{Re}(s) = 0$ on the complex plane, which can be represented in the form

$$\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$$

where ζ is a complex (or real) random variable with $\underset{\sim}{E}\{|\zeta|\} < \infty$, and η is a real random variable. The norm of $\Phi(s)$ is defined by

$$\| \Phi \| = \inf_{\zeta} \underset{\sim}{E}\{|\zeta|\}$$

where the infimum is taken for all admissible ζ . We define a transform $\underset{\sim}{T}$ on $\underset{\sim}{R}$ as follows. If $\Phi(s) \in \underset{\sim}{R}$ and $\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$, then $\underset{\sim}{T}\{\Phi(s)\} = \Phi^{+}(s)$ where

$$\Phi^{+}(s) = \underset{\sim}{E}\{\zeta e^{-s\eta^{+}}\}$$

and $\eta^{+} = \max(0, \eta)$. The function $\Phi^{+}(s)$ is uniquely determined for $\mathrm{Re}(s) \geq 0$ by $\Phi(s)$.

The main result of this chapter is concerned with the solution of the following problem: Let us suppose that $\gamma(s) \in \underset{\sim}{R}$, $\Gamma_0(s) \in \underset{\sim}{R}$, $\underset{\sim}{T}\{\Gamma_0(s)\} = \Gamma_0(s)$

and define $\Gamma_n(s)$ $(n = 1,2,\ldots)$ by the recurrence formula

$$\Gamma_n(s) = \underset{\sim}{T}\{\gamma(s)\Gamma_{n-1}(s)\}$$

for $n = 1,2,\ldots$. The sequence of functions $\{\Gamma_n(s)\}$ is to be determined. We shall give various methods for finding $\Gamma_n(s)$ for $n = 1,2,\ldots$.

We consider also a space $\underset{\sim}{A}$ of functions $a(s)$ defined for $|s| = 1$ on the complex plane, which can be represented in the form

$$a(s) = \underset{\sim}{E}\{\zeta s^{\eta}\}$$

where ζ is a complex (or real) random variable with $\underset{\sim}{E}\{|\zeta|\} < \infty$, and η is a discrete random variable taking on integers only. The norm of $a(s)$ is defined by

$$\|a\| = \inf_{\zeta} \underset{\sim}{E}\{|\zeta|\}$$

where the infimum is taken for all admissible ζ . We define a transform $\underset{\sim}{\Pi}$ on $\underset{\sim}{A}$ as follows. If $a(s) \in \underset{\sim}{A}$ and $a(s) = \underset{\sim}{E}\{\zeta s^{\eta}\}$, then $\underset{\sim}{\Pi}\{a(s)\} = a^+(s)$ where

$$a^+(s) = \underset{\sim}{E}\{\zeta s^{\eta^+}\}$$

and $\eta^+ = \max(0, \eta)$. The function $a^+(s)$ is uniquely determined for $|s| \leq 1$ by $a(s)$.

For the space $\underset{\sim}{A}$ we obtain similar results as for $\underset{\sim}{R}$.

In Chapter II we consider a sequence of mutually independent and identically distributed real random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ with distribution function $\mathbf{P}\{\xi_n \leq x\} = F(x)$. We write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ $(n = 1,2,\ldots)$, $\zeta_0 = 0$ and $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$. By using the results of Chapter I we determine the joint distribution of η_n and ζ_n for $n = 1,2,\ldots$.

We use the Laplace-Stieltjes transform

$$\phi(s) = \mathbf{E}\{e^{-s\xi_n}\} = \int_{-\infty}^{\infty} e^{-sx}\, dF(x)$$

which is convergent if $\operatorname{Re}(s) = 0$. If ξ_n is a nonnegative random variable, then

$$\phi(s) = \mathbf{E}\{e^{-s\xi_n}\} = \int_{0}^{\infty} e^{-sx}\, dF(x)$$

is convergent for $\operatorname{Re}(s) \geq 0$. Throughout the book we use the above notation for the Laplace-Stieltjes transform of the distribution function of a nonnegative random variable; however, if ambiguity might arise, we write

$$\phi(s) = \int_{-0}^{\infty} e^{-sx}\, dF(x) = \int_{[0,\infty)} e^{-sx}\, dF(x) .$$

If ξ_n is a nonnegative random variable, then

$$\mathbf{E}\{e^{-s\xi_n}\} - \mathbf{P}\{\xi_n = 0\} = \int_{+0}^{\infty} e^{-sx}\, dF(x) = \int_{(0,\infty)} e^{-sx}\, dF(x)$$

for $\operatorname{Re}(s) \geq 0$.

We shall also determine the joint distribution of η_n and ζ_n for $n = 1,2,\ldots$ by using the method of ladder indices. For a sequence of real random variables $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ we say that r $(r = 1,2,\ldots)$ is a ladder index if $\zeta_r > \zeta_n$ for $0 \leqq n < r$ and r $(r = 1,2,\ldots)$ is a weak ladder index if $\zeta_r \geqq \zeta_n$ for $0 \leqq n \leqq r$.

In some particular cases we determine the distribution of η_n $(n = 1,2,\ldots)$ by using combinatorial methods. These methods are based on the following simple theorem: If $k_1, k_2, \ldots, k_n$ are nonnegative integers with sum $k_1 + k_2 + \ldots + k_n = k \leqq n$, then among the n cyclic permutations of $(k_1, k_2, \ldots, k_n)$ there are exactly $n-k$ for which the sum of the first r elements is less than r for all $r = 1,2,\ldots, n$.

In Chapter III methods are given for finding the distribution of the number of positive or nonnegative partial sums for a sequence of mutually independent and identically distributed real random variables. We use analytical methods, based on the results of Chapter I, and combinatorial methods.

In Chapter IV we determine the distribution of the k-th ordered partial sum for n mutually independent and identically distributed real random variables.

In Chapter V the previous results are applied in the theory of random walk, in ballots and in order statistics.

In Chapter VI we prove various limit theorems and limit distributions for a sequence of mutually independent and identically distributed real random variables. In this chapter stable distributions play an important role. We say that a distribution function $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$ where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $c \geq 0$, m is a real number, if the logarithm of the Laplace-Stieltjes transform of $R(x)$ has the following forms

$$\log \psi(s) = -ms - c|s|^{\alpha}(1 + \frac{\beta s}{|s|} \tan \frac{\alpha\pi}{2})$$

for $\alpha \neq 1$ and $Re(s) = 0$, and

$$\log \psi(s) = -ms - c|s| \ (1 - \frac{2\beta s}{\pi|s|} \log |s|)$$

for $\alpha = 1$ and $Re(s) = 0$ where $s/|s| = 0$ for $s = 0$.

The following result is frequently used in this book. Let $\xi_1, \xi_2,\ldots, \xi_n,\ldots$ be mutually independent and identically distributed real random variables with distribution function $F(x)$. Write $\zeta_n = \xi_1+ \xi_2+\ldots+ \xi_n$ for $n = 1,2,\ldots$. If

$$\lim_{x \to \infty} x^{\alpha} F(-x) = a_1$$

and

$$\lim_{x \to \infty} x^{\alpha} [1-F(x)] = a_2$$

exist where $0 < \alpha < 2$ and $a_1+ a_2 > 0$, then we can find normalizing constants A_n and B_n such that

$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\zeta_n - A_n}{B_n} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$. Here α is determined by the above conditions, $\beta = (a_2- a_1)/(a_2+ a_1)$, c is any positive number, and m is any real number. We can choose

$$B_n = (bn)^{1/\alpha}$$

where

$$b = \left(\frac{a_1 + a_2}{c}\right) \frac{\pi}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}} = \left(\frac{a_1 + a_2}{c}\right) \Gamma\,(1-\alpha)\cos\frac{\alpha\pi}{2}$$

$(\,b = (a_1 + a_2)\pi/2c$ if $\alpha = 1)$, and $A_n = -mB_n$ if $0 < \alpha < 1$,

$$A_n = n \int_{|x|<\tau B_n} x\, dF(x) - mB_n - \frac{2\beta c}{\pi}\,[\log\tau - (1-C)]B_n$$

if $\alpha = 1$ where τ is any positive number, and $C = 0.577215\ldots$ is Euler's constant, and

$$A_n = n \int_{-\infty}^{\infty} x\, dF(x) - mB_n$$

if $1 < \alpha < 2$.

In Chapter VII we deduce the basic properties of the Poisson process, compound Poisson process, recurrent process, compound recurrent process, Brownian motion process and the homogeneous stochastic processes with independent increments. We also discuss the topic of weak convergence of stochastic processes.

In Chapter VIII we determine the distributions of the supremum and the first passage time for compound recurrent processes, compound Poisson processes and for homogeneous processes with independent increments.

In Chapter IX we determine the distribution and the asymptotic distribution of the occupation time for a general class of stochastic processes.

In Chapter X the previous results are applied in the theories of queues, insurance and storage.

The Appendix contains numerous useful theorems in the theory of probability, and in the theory of complex variables, and some basic Abelian and Tauberian theorems which have been used in the book.

THEORY OF RANDOM FLUCTUATIONS

Lajos Takács

Typographical Details

The MS consists of 1576 pages. The pages are not numbered consecutively throughout the MS. Each chapter is numbered separately. The contents and numbering are as follows:

Preliminaries	21 pages
Title page (one) Dedication (one) Preface (P1 -3) Contents (C1-5) Introduction (1-11)	
Chapter I (I 1 - 64)	64 "
Chapter II (II 1 - 62)	66 "
(Addition: II 22 a,b,c,d.)	
Chapter III (III 1- 52)	53 "
(Addition: III 47a.)	
Chapter IV (IV 1 -35)	35 "
Chapter V (V 1 - 236)	243 "
(Addition: V 131a, 146a, 198a,b,c,d,e.)	
Chapter VI (VI 1- 353)	361 "
(Addition: VI 29a,b,c,d,e,f, 233a,284a.)	
Chapter VII (VII 1 - 215)	221 "
(Addition: VII 111a,b,c,d,e,f.)	
Chapter VIII (VIII 1 - 96)	96 "
Chapter IX (IX 1 - 54)	54 "
Chapter X (X 1 - 127)	128 "
(Addition: X 14a,38a. No: X 77.)	
Appendix (A 1 - 111)	113 "
(Addition: A 25a,b.)	
Solutions (S 1 - 119)	121 "
(Addition: S 82a, 104a.)	
TOTAL	1576 pages

The ten chapters are divided into 65 sections (numbered 1-65) and the Appendix is divided into 10 sections (numbered 1-10).

The structure of a section is as follows:

Heading. E.g. 39. Order Statistics (Page V - 116) .

⋮

Second-grade heading. E.g. The Comparison of a Theoretical and an Empirical Distribution Function. (Page V-150)

⋮

Third-grade heading. E.g. The distribution of $\delta^{+}_{m,m}$. (Page V - 173)

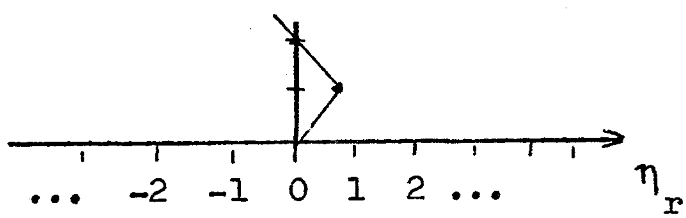
⋮

Other parts are: Theorem 1. Proof. ... Lemma 1. Proof. ..., Note. ... Examples. .... Definition.

*)

Figures.

There is only one figure (page V -43). I enclosed a drawing in the MS; however, η_r on the abscissa is missing and should be printed as follows:



*) In second-grade headings each major word starts with a capital. In third-grade headings simply italics are used.

Symbols

Greek Letters

α, β, γ, δ, ε,	=	l.c. alpha, beta, gamma, delta, epsilon
ζ, η, θ, λ, μ,	=	l.c. zeta, eta, theta, lambda, mu,
ν, ξ, π, ρ, σ,	=	l.c. nu, xi, pi, rho, sigma,
τ, χ, ψ, ω,	=	l.c. tau, chi, psi, omega
ϑ	=	l.c. script theta (ϑ) (open)
ϕ, φ	=	l.c. script phi (φ) (open)
Γ, Δ, Θ, Λ, Π,	=	cap. gamma, delta, theta, lambda, pi
Φ, Ψ, Ω,	=	cap. phi, psi, omega
$\underset{\sim}{\Pi}$, $\underset{\sim}{\xi}$, $\underset{\sim}{\eta}$	=	bold face cap. pi, bold face l.c. xi, bold face l.c. eta

German Letters

$\mathfrak{S}$	=	cap. S

Script Letters

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$,	=	cap. A, B, C, F
$\mathcal{M}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{P}$	=	cap. M, R, S, P

Other Characters

$\underset{\sim}{E}$, $\underset{\sim}{P}$, $\underset{\sim}{R}$, $\underset{\sim}{T}$, $\underset{\sim}{A}$, $\overset{S}{\vee}$ $\underset{\sim}{Q}$, $\underset{\sim}{0}$	=	bold face cap. E, P, R, T, A, $\overset{S}{\vee}$, Q, bold face zero
$\underset{\sim}{\mathrm{Var}}$, $\underset{\sim}{\mathrm{Cov}}$,	=	bold face Var, Cov
Σ, Π, $\subset$, ε,	=	sum, product, subset of, element of ($\in$)
$\times$, $\equiv$, $\leq$, $\geq$, $\sim$	=	product, identity, less than or equal, greater than or equal, equivalent to
$\int$, $\frac{\partial \ldots}{\partial \ldots}$, $\sqrt{\ }$, ∞	=	integral, partial derivative, square root, infinity,
$\Vert \ldots \Vert$, $\cup$, $\cap$, $\Rightarrow$	=	norm, union, intersection, double arrow

<u>Special letters occurring in foreign names in the references</u>

a, á, à, ā, Å	č, ř, š, ž
e, é. è, e, É	Ş, ş, Ç, ç
o, ó, ö, ô	Ł, ł, ń
u, ü, Ü	i, í, ī, ï

<u>Special problems</u>

$\times$ (<u>Cartesian product</u>) occurs in Chapter VII (Pages VII 4-VII 11) and in the Appendix (Pages A 39 - A 47) and is typed as X. In these pages and only here every capital X should be set as a boldface product sign ($\boldsymbol{\times}$).

ϑ (<u>l.c. open theta</u>) occurs in Chapter IX (IX 11 - 15, 39-40) and in Chapter X (X 9 - 12).

Θ (<u>capital theta</u>) occurs in Chapter III, Chapter IV, Chapter VI and in the Appendix.

θ (<u>l.c. closed theta</u>) occurs frequently in Chapters VII, VIII, IX, X.

<u>Note</u>: On pages VII 40 , VI-33, VII 64-110, and X-43 θ means l.c. closed theta. (To correct a typing error I altered capital theta to l.c. theta in this way.)

ϕ (<u>l.c. phi</u>) should be set as open phi (φ) throughout the book.

Π (<u>product</u>). Capital pi is also used as a product sign, but I think it is easy to recognize when the product sign is needed. (See e.g. formulas (84) and (85) on page V-112 and page V-113.)

$\in$ (<u>element of</u>) is typed as l.c. epsilon (ε) throughout the book. If ε occurs in the following context: $\Phi(s) \varepsilon R$, (I 4,5,6), $\omega \varepsilon \Omega$, $A \in \mathcal{B}$, $a \varepsilon S$, that is, if it stands between two letters, then it should be set $\in$. Note: $\notin$

occurs on page VI-101.

Boldface letters

Boldface letters are indicated by wavy underline in black. Throughout the book if P and E are followed by $\{\ldots\}$, then P and E should be set in boldface, that is, $\mathbf{P}\{\ldots\}$ and $\mathbf{E}\{\quad\}$.

In addition to E and P there are boldface R, T, S, A, Q, O (zero), Var, Cov, Π , ξ , η .

$\boldsymbol{\Pi}$ occurs in Chapter I and in the Solutions (Ch.I.).
$\boldsymbol{\eta}$ occurs in Chapter V (Pages V-59, 73, 74).
$\boldsymbol{\xi}$ ($\boldsymbol{\xi}_n$) occurs in Chapter VI (VI-247) and in the Appendix (A-55).

Script letters

Script letters (A, B, C, F, M, P, R, S) are typed as script, and occur in Chapters VI, VII, and in the Appendix.

Suggestions and remarks

1. In the book I would prefer to indicate chapter numbers at the top of even pages, and section numbers at the top of odd pages.

2. There are no footnotes. Material at the bottom of a page or on the margin of a page should be inserted in the text. This is indicated by the sign λ .

3. The sign $\rfloor$ indicates that a new paragraph should be started.

4. In the references the original spellings are followed. E.g. Kolmogoroff, Kolmogorov, Koroljuk, Korolyuk, etc. The title of a paper isreproduced in its original spelling (queuing,queueing, generalization, generalisation, etc.). In each reference volume,

year, and page numbers are indicated. However, if in a volume the numbering of each issue starts anew, then the issue number is indicated too. E.g. 34 No. 2 (1930) 1 -10.

5. Hyphen. If a hyphen in a hyphenated word comes at the margin, I put a hyphen both at the end of the line and at the beginning of the next line.

6. I would prefer the authors' names to be set in italics if the name is followed by a reference number. E.g. M. Kac [100].

7. Brackets. In the MS large and small brackets are equal in size. Please print large brackets when needed. E.g. (^n_k) should be printed $\binom{n}{k}$.

8. Asterisk (*) and convolution symbols ($\ast$) are typed in the same size; however for convolution a larger star ($\ast$) is needed. In superscript position use asterisk (*), in center position convolution symbol ($\ast$). See e.g. VII-103, formula (186), where $\hat{F}(t) \ast F_n(t)$ should be set.

9. The symbols A and a are used on several occasions. E.g. "A set A ..." or " a votes ...", " a steps...". Italics are not indicated in the MS, but they can be recognized from the context.

10. $\bar{A}^*$ should always be set as $\overline{A^*}$, whereas $\bar{\eta}^*$ should be set as typed.

11. Subscripts. There are some complicated subscripts as in the following examples: Page V-107: $N_{(k-s)(m+n)-1}$. Page VI- 65 : $W_{\frac{1}{2},\frac{1}{6}}(\ldots)$. Page A-37: $\mu_{n_k}^{(k)}(A)$.

12. Ellipses. I prefer the points to be set on the line. For example, 1,2,...,n,... .

~~NOTE: Later I will supply the following missing data: In the Preface ...Louis Bachelier (18..-19..), and the page numbers in reference X-[23].~~