SOLUTIONS

CHAPTER I

13.1. We have $|\Phi_m(s) - \Phi_n(s)| \leqq \|\Phi_m - \Phi_n\|$ for $\text{Re}(s) = 0$. Thus for every $\varepsilon > 0$ there exists an N such that $|\Phi_m(s) - \Phi_n(s)| < \varepsilon$ whenever $m > N$, $n > N$ and $\text{Re}(s) = 0$. Accordingly, the sequence $\{\Phi_n(s)\}$ is a uniformly convergent sequence of continuous functions for $\text{Re}(s) = 0$. Thus $\lim_{n \to \infty} \Phi_n(s) = \Phi(s)$ exists for $\text{Re}(s) = 0$ and $\Phi(s)$ is a continuous function of s for $\text{Re}(s) = 0$. (We have $|\Phi(s) - \Phi_n(s)| < \varepsilon$ if $n > N$ and $\text{Re}(s) = 0$.)

First, we shall prove that $\Phi(s) \varepsilon \underset{\sim}{R}$. Let us choose an increasing sequence of positive integers $n_1, n_2, \ldots, n_j, \ldots$ such that $\|\Phi_n - \Phi_{n_j}\| < 1/2^j$ if $n > n_j$. Then $\|\Phi_{n_{j+1}} - \Phi_{n_j}\| < 1/2^j$ for $j = 1,2,\ldots$, and this implies that

$$\sum_{j=k}^{\infty} \|\Phi_{n_{j+1}} - \Phi_{n_j}\| < 1/2^{k-1}$$

for $k = 1,2,\ldots$. By making use of Lemma 3.2 we can conclude that

$$\Phi(s) - \Phi_{n_k}(s) = \sum_{j=k}^{\infty} [\Phi_{n_{j+1}}(s) - \Phi_{n_j}(s)] \varepsilon \underset{\sim}{R}$$

and

$$\|\Phi - \Phi_{n_k}\| \leqq \sum_{j=k}^{\infty} \|\Phi_{n_{j+1}} - \Phi_{n_j}\| < 1/2^{k-1}$$

for $k = 1,2,\ldots$. Since $\Phi_{n_k}(s) \varepsilon \underset{\sim}{R}$ and $\Phi(s) - \Phi_{n_k}(s) \varepsilon \underset{\sim}{R}$, it follows that $\Phi(s) \varepsilon \underset{\sim}{R}$.

If $n > n_k$, then we have

$$\|\Phi - \Phi_n\| \leq \|\Phi - \Phi_{n_k}\| + \|\Phi_{n_k} - \Phi_n\| < \frac{1}{2^{k-1}} + \frac{1}{2^k} = \frac{3}{2^k}$$

for $k = 1,2,\ldots$. This implies that $\lim_{n \to \infty} \|\Phi - \Phi_n\| = 0$. So we can conclude that the space $\underset{\sim}{R}$ is complete.

13.2. Let

$$a_n(s) = \sum_{k=-\infty}^{\infty} a_k^{(n)} s^k \in \underset{\sim}{A} \quad \text{and} \quad \|a_n\| = \sum_{k=-\infty}^{\infty} |a_k^{(n)}| < \infty$$

for $n = 1,2,\ldots$. By assumption, for every $\varepsilon > 0$ there exists an N such that

$$\|a_m - a_n\| = \sum_{k=-\infty}^{\infty} |a_k^{(m)} - a_k^{(n)}| < \varepsilon$$

if $m > N$ and $n > N$. This implies that for each k $(k = 0, \pm 1, \pm 2, \ldots)$ $|a_k^{(m)} - a_k^{(n)}| < \varepsilon$ if $m > N$ and $n > N$, that is, $\{a_k^{(n)} ; n = 1,2,\ldots\}$ is a Cauchy sequence. Thus the limit $\lim_{n \to \infty} a_k^{(n)} = a_k$ exists for each $k = 0, \pm 1, \pm 2, \ldots$. Now for any fixed K we have

$$\sum_{k=-K}^{K} |a_k^{(m)} - a_k^{(n)}| < \varepsilon$$

if $m > N$ and $n > N$. Let $m \to \infty$. Then we obtain

$$\sum_{k=-K}^{K} |a_k - a_k^{(n)}| \leq \varepsilon$$

for $n > N$ and for any K. Let $K \to \infty$. Then we obtain

$$\sum_{k=-\infty}^{\infty} |a_k - a_k^{(n)}| \leqq \varepsilon$$

for $n > N$. Since $|a_k| \leq |a_k - a_k^{(n)}| + |a_k^{(n)}|$, it follows that

$$\sum_{k=-\infty}^{\infty} |a_k| \leqq \varepsilon + \sum_{k=-\infty}^{\infty} |a_k^{(n)}| < \infty .$$

Accordingly, if

$$a(s) = \sum_{k=-\infty}^{\infty} a_k s^k ,$$

then $a(s) \in A$ and $\|a - a_n\| \leqq \varepsilon$ if $n > N$. This implies that $\lim_{n\to\infty} \|a - a_n\| = 0$. So we can conclude that the space $\underset{\sim}{A}$ is complete.

13.3. We observe that $\Phi(s) = \underset{\sim}{E}\{e^{-s\eta}\}$ where η has the density function $f(x) = e^{-|x|}/2$ for $-\infty < x < \infty$. Thus $\Phi(s) \in \underset{\sim}{R}$ and

$$\Phi^{+}(s) = \underset{\sim}{E}\{e^{-s\eta^{+}}\} = \frac{1}{2} + \frac{1}{2}\int_0^{\infty} e^{-sx-x}\, dx = \frac{1}{2}\left(1 + \frac{1}{1+s}\right)$$

for $\operatorname{Re}(s) > -1$.

In this case we can also apply (5.8) with $0 < \varepsilon < 1$ to obtain

$$\Phi^{+}(s) = \frac{s}{2\pi i} \int_{C_\varepsilon^{+}} \frac{\Phi(z)}{z(s-z)}\, dz = \frac{s}{2\pi i} \int_{C_\varepsilon^{+}} \frac{dz}{z(s-z)(1-z^2)}$$

for $\operatorname{Re}(s) > \varepsilon > 0$. In the right half-plane $\operatorname{Re}(z) > 0$, the integrand has two poles $z = s$ and $z = 1$, and by Cauchy's theorem of residues (see e.g. W. Osgood [23 p. 162]) we obtain that

$$\Phi^+(s) = \frac{1}{1-s^2} - \frac{s}{2(1-s)} = \frac{1}{2}\left(1 + \frac{1}{1+s}\right)$$

for $\text{Re}(s) > 0$.

If we apply formula (5.1), then we obtain that

$$\Phi^+(s) = \frac{1}{2} + \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{dz}{z(s-z)(1-z^2)} =$$

$$= \frac{1}{2} + \lim_{\varepsilon \to 0} \frac{s}{\pi} \int_\varepsilon^\infty \frac{dy}{(1+y^2)(s^2+y^2)} =$$

$$= \frac{1}{2} + \frac{s}{\pi(1-s^2)} \int_0^\infty \left[\frac{1}{s^2+y^2} - \frac{1}{1+y^2}\right]dy =$$

$$= \frac{1}{2} + \frac{s}{1-s^2}\left[\frac{1}{2s} - \frac{1}{2}\right] = \frac{1}{2}\left(1 + \frac{1}{1+s}\right)$$

for $\text{Re}(s) > 0$.

13.4. Since $\Phi(s) = \underset{\sim}{E}\{e^{-s\nu_m}\}$ where

$$\underset{\sim}{P}\{\nu_m = m - 2j\} = \binom{m}{j} p^j q^{m-j}$$

for $j = 0,1,\ldots, m$, it follows that $\Phi(s) \in \underset{\sim}{R}$. If we write

$$\Phi(s) = \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j} e^{-(m-2j)s}$$

and apply $\underset{\sim}{T}$ term by term, then we obtain that

$$\Phi^{+}(s) = \sum_{2j<m} \binom{m}{j} p^{j} q^{m-j} e^{-(m-2j)s} + \sum_{2j\geq m} \binom{m}{j} p^{j} q^{m-j} =$$

$$= 1 + \sum_{2j<m} \binom{m}{j} p^{j} q^{m-j} [e^{-(m-2j)s} - 1] .$$

The same result can be obtained by using formula (5.1). Accordingly, if $\mathrm{Re}(s) > 0$, then

$$\Phi^{+}(s) = \frac{1}{2}\Phi(0) + \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\Phi(z)}{z(s-z)} dz =$$

$$= \frac{1}{2}\Phi(0) + \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \left[\frac{\Phi(iy)}{s-iy} - \frac{\Phi(-iy)}{s+iy} \right] \frac{dy}{y} =$$

$$= \frac{1}{2}\Phi(0) + \lim_{\varepsilon \to 0} \frac{s}{\pi} \int_{\varepsilon}^{\infty} \frac{s\,\mathrm{Im}[\Phi(iy)] + y\,\mathrm{Re}[\Phi(iy)]}{(s^2+y^2)y} dy .$$

Thus we obtain that

$$\Phi^{+}(s) = \frac{1}{2} + \frac{s}{\pi} \sum_{j=0}^{m} \binom{m}{j} p^{j} q^{m-j} \int_{0}^{\infty} \frac{s \sin(2j-m)y + y \cos(2j-m)y}{(s^2+y^2)} dy$$

for $\mathrm{Re}(s) > 0$. If we take into consideration that

$$\int_{0}^{\infty} \frac{\cos ay}{s^2+y^2} dy = \frac{\pi e^{-as}}{2s} \quad \text{and} \quad \int_{0}^{\infty} \frac{\sin ay}{(s^2+y^2)y} dy = \frac{\pi(1-e^{-as})}{2s^2}$$

for $a \geq 0$ and $\mathrm{Re}(s) > 0$, then it follows that

$$\Phi^{+}(s) = \frac{1}{2} + \sum_{2j<m} \binom{m}{j} p^{j} q^{m-j} [e^{-(m-2j)s} - \frac{1}{2}] + \frac{1}{2} \sum_{2j\geq m} \binom{m}{j} p^{j} q^{m-j}$$

which is in agreement with the previous result.

13.5. Since $\Phi(s) = E\{e^{-s\eta}\}$ where

$$P\{\eta \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du ,$$

it follows that $\Phi(s) \in R$ and

$$\Phi^+(s) = E\{e^{-s\eta^+}\} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-sx-\frac{x^2}{2}} dx =$$

$$= \frac{1}{2} + \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(s+x)^2} dx = \frac{1}{2} + \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_s^\infty e^{-u^2/2} du .$$

If we introduce the function

$$w(s) = \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_0^s e^{-u^2/2} du$$

for any complex s, then we can write that

$$\Phi^+(s) = \frac{1 + e^{s^2/2}}{2} - w(s)$$

for any complex s. We note that the function $w(\sqrt{2}\, is)\sqrt{\pi}/i$ has been tabulated by K. A. Karpov [17].

13.6. Let ξ be a nonnegative random variable for which $E\{e^{-s\xi}\} = \phi(s)$ if $Re(s) \geq 0$. Let $P\{\theta \leq x\} = 1 - e^{-\lambda x}$ for $x \geq 0$, and $P\{\theta \leq x\} = 0$ for $x < 0$. Then

$$E\{e^{-s\theta}\} = \frac{\lambda}{\lambda + s}$$

for $\mathrm{Re}(s) > -\lambda$. If ξ and θ are independent, then

$$\underset{\sim}{E}\{e^{-s(\xi-\theta)}\} = \frac{\lambda\phi(s)}{\lambda - s}$$

for $0 \leq \mathrm{Re}(s) < \lambda$. Accordingly,

$$\underset{\sim}{T}\{\frac{\lambda\phi(s)}{\lambda - s}\} = \underset{\sim}{E}\{e^{-s[\xi-\theta]^+}\}$$

for $\mathrm{Re}(s) \geq 0$.

If $x \geq 0$, then

$$\underset{\sim}{E}\{e^{-s[\xi-\theta]^+} | \xi = x\} = \lambda \int_0^x e^{-s(x-u)-\lambda u}\, du + \lambda \int_x^\infty e^{-\lambda u}\, du =$$

$$= \begin{cases} \dfrac{\lambda e^{-sx} - s\, e^{-\lambda x}}{\lambda - s} & \text{for } s \neq \lambda , \\ \lambda x\, e^{-\lambda x} + e^{-\lambda x} & \text{for } s = \lambda . \end{cases}$$

Hence

$$\underset{\sim}{E}\{e^{-s[\xi-\theta]^+}\} = \begin{cases} \dfrac{\lambda \underset{\sim}{E}\{e^{-s\xi}\} - s\underset{\sim}{E}\{e^{-\lambda\xi}\}}{\lambda - s} & \text{if } s \neq \lambda , \\ \lambda \underset{\sim}{E}\{\xi e^{-\lambda\xi}\} + \underset{\sim}{E}\{e^{-\lambda\xi}\} & \text{if } s = \lambda , \end{cases}$$

and $\mathrm{Re}(s) \geq 0$.

Finally,

$$\underset{\sim}{T}\{\frac{\lambda\phi(s)}{\lambda - s}\} = \begin{cases} \dfrac{\lambda\phi(s) - s\phi(\lambda)}{\lambda - s} & \text{if } s \neq \lambda , \\ \phi(\lambda) - \lambda\phi'(\lambda) & \text{if } s = \lambda , \end{cases}$$

and $\mathrm{Re}(s) \geq 0$. The same result can be obtained by applying formula (5.8).

13.7. Let $q = \lambda + i\tau$ where λ and τ are real numbers. Since

$$\int_0^\infty e^{-qx+sx}\,dx = \int_0^\infty e^{-\lambda x}\,e^{-i\tau x+sx}\,dx = \frac{1}{q-s}$$

for $\mathrm{Re}(s) = 0$, we can write that

$$\frac{1}{\lambda}\,\underset{\sim}{E}\{-e^{-i\tau\eta-s(-\eta)}\} = \frac{1}{s-q}$$

for $\mathrm{Re}(s) = 0$ where η is a random variable with density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. This shows at once that $1/(s-q) \in \underset{\sim}{R}$. Thus by (5.1)

$$\underset{\sim}{T}\{\frac{\Phi(s)}{s-q}\} = -\frac{\Phi(0)}{2q} + \lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi(z)}{z(s-z)(z-q)}\,dz$$

for $\mathrm{Re}(s) > 0$. Since

$$\frac{1}{(s-z)(z-q)} = \frac{1}{(s-q)}\left[\frac{1}{s-z} - \frac{1}{q-z}\right]$$

if $s \neq q$ and $z \in L_\varepsilon$, it follows that

$$\underset{\sim}{T}\{\frac{\Phi(s)}{s-q}\} = -\frac{\Phi(0)}{2q} + \frac{1}{(s-q)}\left[\Phi^+(s) - \frac{1}{2}\Phi(0)\right] -$$

$$-\frac{s}{(s-q)q}\left[\Phi^+(q) - \frac{1}{2}\Phi(0)\right] = \frac{1}{(s-q)}\left[\Phi^+(s) - \frac{s}{q}\Phi^+(q)\right]$$

for $\mathrm{Re}(s) > 0$ and $s \neq q$. For $\mathrm{Re}(s) \geq 0$ we obtain the formula to be proved by continuity.

13.8. Let ξ be a nonnegative random variable for which $\underset{\sim}{E}\{e^{-s\xi}\} = \phi(s)$ if $\mathrm{Re}(s) \geq 0$. Let $\underset{\sim}{P}\{\theta \leq x\} = 1 - e^{-\lambda x}$ for $x \geq 0$ and $\underset{\sim}{P}\{\theta \leq x\} = 0$ for $x < 0$. Then

$$\underset{\sim}{E}\{e^{-s\theta}\} = \frac{\lambda}{\lambda + s}$$

for $\text{Re}(s) > -\lambda$. If ξ and θ are independent, then

$$\underset{\sim}{E}\{e^{-s(\theta-\xi)}\} = \frac{\lambda\phi(-s)}{\lambda + s}$$

for $-\lambda < \text{Re}(s) \leq 0$, and

$$\underset{\sim}{T}\{\frac{\lambda\phi(-s)}{\lambda + s}\} = \underset{\sim}{E}\{e^{-s[\theta-\xi]^{+}}\} = 1 - \frac{\phi(\lambda)s}{\lambda + s}$$

for $\text{Re}(s) > -\lambda$. For if $x \geq 0$, then

$$\underset{\sim}{E}\{e^{-s[\theta-\xi]^{+}} | \xi = x\} = \lambda\int_0^x e^{-\lambda u}du + \lambda\int_x^\infty e^{-s(u-x)-\lambda u}du =$$

$$= 1 - e^{-\lambda x} + e^{-\lambda x}\frac{\lambda}{\lambda + s},$$

and unconditionally we have

$$\underset{\sim}{E}\{e^{-s[\theta-\xi]^{+}}\} = 1 - \phi(\lambda) + \phi(\lambda)\frac{\lambda}{\lambda + s}$$

for $\text{Re}(s) > -\lambda$. The same result can also be obtained by using formula (5.9).

Note. If $\Phi(s) \in \underset{\sim}{R}$, and $\Phi^{+}(s) = \underset{\sim}{T}\{\Phi(s)\}$, then we can write that

$$\underset{\sim}{T}\{\Phi(-s)\} = \Phi(-s) - \Phi^{+}(-s) + \Phi(0)$$

for $\text{Re}(s) = 0$. This follows from the following identity

$$e^{-s[-x]^+} = e^{sx} - e^{s[x]^+} + 1$$

which holds for any s and real x.

Thus we can deduce the solution of Problem 13.8 from the solution of Problem 13.6 if $\text{Re}(s) = 0$ and by analytic continuation we can obtain the solution for $\text{Re}(s) \geq 0$ too.

13.9. Let ξ be a nonnegative random variable for which $\underset{\sim}{E}\{e^{-s\xi}\} = \phi(s)$ if $\text{Re}(s) \geq 0$. Let

$$\underset{\sim}{P}\{\theta \leq x\} = \begin{cases} 1 - \sum_{j=0}^{m-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$\underset{\sim}{E}\{e^{-s\theta}\} = \lambda^m/(\lambda+s)^m$$

for $\text{Re}(s) > -\lambda$. If ξ and θ are independent, then

$$\underset{\sim}{E}\{e^{-s(\xi-\theta)}\} = \lambda^m\phi(s)/(\lambda-s)^m$$

for $0 \leq \text{Re}(s) < \lambda$, and

$$\underset{\sim}{T}\left\{\frac{\lambda^m \phi(s)}{(\lambda-s)^m}\right\} = \underset{\sim}{E}\{e^{-s[\xi-\theta]^+}\}$$

for $\text{Re}(s) \geq 0$.

If $x \geq 0$, then

$$\underset{\sim}{E}\{e^{-s[\xi-\theta]^+} \mid \xi = x\} = \frac{\lambda^m}{(m-1)!}\int_0^x e^{-s(x-u)-\lambda u} u^{m-1}\, du + \sum_{j=0}^{m-1} e^{-\lambda x}\frac{(\lambda x)^j}{j!} =$$

$$\begin{cases} \dfrac{\lambda^m e^{-sx} - e^{-\lambda x}\sum_{j=0}^{m-1}\frac{x^j}{j!}[\lambda^m(\lambda-s)^j - \lambda^j(\lambda-s)^m]}{(\lambda-s)^m} & \text{for } s \neq \lambda , \\ e^{-\lambda x}\sum_{j=0}^{m}\dfrac{x^j}{j!} & \text{for } s = \lambda . \end{cases}$$

Hence

$$\underset{\sim}{E}\{e^{-s[\xi-\theta]^+}\} = \begin{cases} \dfrac{\lambda^m\phi(s) - \sum_{j=0}^{m-1}\frac{(-1)^j\phi^{(j)}(\lambda)}{j!}[\lambda^m(\lambda-s)^j - \lambda^j(\lambda-s)^m]}{(\lambda-s)^m} & \text{if } s \neq \lambda , \\ \sum_{j=0}^{m}\dfrac{(-1)^j\lambda^j\phi^{(j)}(\lambda)}{j!} & \text{if } s = \lambda , \end{cases}$$

and $\text{Re}(s) \geqq 0$. The same result can be obtained by using formula (5.8).

13.10. If we use the same notation as in the solution of Problem 13.9, then we can write that

$$\underset{\sim}{T}\{\frac{\lambda^m\phi(-s)}{(\lambda+s)^m}\} = \underset{\sim}{E}\{e^{-s[\theta-\xi]^+}\}$$

for $\text{Re}(s) \geqq 0$. If $x \geqq 0$, then we have

$$\underset{\sim}{E}\{e^{-s[\theta-\xi]^+}|\xi = x\} = \int_0^x e^{-\lambda u} \frac{(\lambda u)^{m-1}}{(m-1)!} \lambda du + \int_x^\infty e^{-s(u-x)-\lambda u} \frac{(\lambda u)^{m-1}}{(m-1)!} \lambda du$$

$$= 1 - \sum_{j=0}^{m-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!} + \sum_{j=0}^{m-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!} \left(\frac{\lambda}{\lambda + s}\right)^{m-j} .$$

Hence it follows that

$$\underset{\sim}{T}\left\{\frac{\lambda^m \phi(-s)}{(\lambda + s)^m}\right\} = 1 - \sum_{j=0}^{m-1} \frac{(-1)^j \lambda^j \phi^{(j)}(\lambda)}{j!} \left[1 - \left(\frac{\lambda}{\lambda + s}\right)^{m-j}\right]$$

for $\mathrm{Re}(s) \geq 0$.

13.11. In this case we can write that

$$\gamma(s) = \frac{\pi_{m-1}(s)}{\prod_{j=1}^{m} (s+\alpha_j)}$$

for $\mathrm{Re}(s) \geq 0$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leq m-1$. Since $|\gamma(s)| \leq 1$ for $\mathrm{Re}(s) \geq 0$, it follows that $\mathrm{Re}(\alpha_j) > 0$ for $j = 1,2,\ldots, m$. By formula (5.8) we have

$$\underset{\sim}{T}\{\phi(s)\gamma(-s)\} = \frac{s}{2\pi i} \int_{C_\varepsilon^+} \frac{\phi(z)\gamma(-z)}{z(s-z)} dz$$

for $\mathrm{Re}(s) > \varepsilon > 0$ where ε is a sufficiently small positive number. In the right half-plane $\mathrm{Re}(z) > 0$, the integrand has poles $z = s$ and $z = \alpha_j$ $(j = 1,2,\ldots, m)$. If $\alpha_1, \alpha_2,\ldots, \alpha_m$ are distinct and $s \neq \alpha_j$ $(j = 1,2,\ldots, m)$, then by Cauchy's theorem of residues we obtain that

$$\underset{\sim}{T}\{\phi(s)\gamma(-s)\} = \phi(s)\gamma(-s) + \sum_{j=1}^{m} \frac{s\phi(\alpha_j)\pi_{m-1}(-\alpha_j)}{(s-\alpha_j)\alpha_j} \frac{1}{\prod_{i\neq j}(\alpha_j - \alpha_i)}$$

for $Re(s) \geqq 0$ and $s \neq \alpha_j$ $(j = 1,2,\ldots, m)$.

If the numbers $\alpha_1, \alpha_2,\ldots, \alpha_m$ are not distinct, then we can also apply Cauchy's theorem of residues to obtain $\underset{\sim}{T}\{\phi(s)\gamma(-s)\}$.

13.12. As in the solution of Problem 13.11 we can write that

$$\gamma(s) = \frac{\pi_{m-1}(s)}{\prod_{j=1}^{m}(s+\alpha_j)}$$

for $Re(s) \geqq 0$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leqq m-1$ and $Re(\alpha_j) > 0$ for $j = 1,2,\ldots, m$.

By (5.1) we have

$$\underset{\sim}{T}\{\Phi(s)\gamma(-s)\} = \frac{\Phi(0)\gamma(0)}{2} + \lim_{\varepsilon\to 0} \frac{s}{2\pi i} \int_{L_\varepsilon} \frac{\Phi(z)\pi_{m-1}(-z)}{z(s-z)(\alpha_1 - z)\ldots(\alpha_m - z)} dz$$

for $Re(s) > 0$. If $\alpha_1, \alpha_2,\ldots, \alpha_m$ are distinct and if we use partial fraction expansion in the integrand and apply (5.1) repeatedly, then we obtain that

$$\underset{\sim}{T}\{\Phi(s)\gamma(-s)\} = \Phi^+(s)\gamma(-s) + \sum_{j=1}^{m} \frac{s\Phi^+(\alpha_j)\pi_{m-1}(-\alpha_j)}{(s-\alpha_j)\alpha_j} \frac{1}{\prod_{i\neq j}(\alpha_j - \alpha_i)}$$

for $Re(s) \geqq 0$ and $s \neq \alpha_j$ $(j = 1,2,\ldots, m)$ where $\Phi^+(s) = \underset{\sim}{T}\{\Phi(s)\}$.

In general we can write that

$$\underset{\sim}{T}\{\Phi(s)\gamma(-s)\} = \Phi^+(s)\gamma(-s) + \frac{s\, G_{m-1}(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_m)}$$

for $\text{Re}(s) \geq 0$ and $s \neq \alpha_j$ $(j = 1,2,\ldots, m)$ where $G_{m-1}(s)$ is a polynomial of degree $\leq m-1$. The polynomial $G_{m-1}(s)$ is uniquely determined by the requirement that

$$z\, G_{m-1}(z) - \phi^{+}(z)\pi_{m-1}(-z) = 0$$

whenever $z = \alpha_j$ $(j = 1,2,\ldots, m)$ and if the number α_j occurs r times among $\alpha_1, \alpha_2,\ldots, \alpha_m$, then $z = \alpha_j$ is a root of multiplicity r of the above equation.

13.13. As in the solution of Problem 13.11 we can write that

$$\gamma(s) = \frac{\pi_{m-1}(s)}{\prod_{j=1}^{m} (s+ \alpha_j)}$$

for $\text{Re}(s) \geq 0$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leq m-1$ and $\text{Re}(\alpha_j) > 0$ for $j = 1,2,\ldots, m$.

By (5.9) we have

$$\underset{\sim}{T}\{\gamma(s)\phi(-s)\} = 1 + \frac{s}{2\pi i} \int_{C_\varepsilon^-} \frac{\gamma(z)\phi(-z)}{z(s-z)}\, dz$$

for $\text{Re}(s) \geq 0$ where ε is a sufficiently small positive number. In the left half-plane $\text{Re}(z) < 0$, the integrand has poles $z = -\alpha_j$ for $j = 1,2,\ldots, m$. If $\alpha_1, \alpha_2,\ldots, \alpha_m$ are distinct, then by Cauchy's theorem of residues we obtain that

$$\underset{\sim}{T}\{\gamma(s)\phi(-s)\} = 1 - \sum_{j=1}^{m} \frac{s\phi(\alpha_j)\pi_{m-1}(-\alpha_j)}{\alpha_j(s+\alpha_j)} \frac{1}{\prod_{i\neq j} (\alpha_j - \alpha_i)}$$

for $\operatorname{Re}(s) \geq 0$.

If the numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ are not distinct, then we can also apply Cauchy's theorem of residues to obtain $\underset{\sim}{T}\{\gamma(s)\phi(-s)\}$.

13.14. Let $\underset{\sim}{P}\{\nu = j\} = pq^j$ for $j = 0,1,2,\ldots$. Then $\underset{\sim}{E}\{s^\nu\} = p/(1-qs)$ for $|s| < 1/q$. If ξ and ν are independent, then

$$\underset{\sim}{E}\{s^{\xi-\nu}\} = \frac{p\ s\ g(s)}{s - q}$$

for $q < |s| \leq 1$. Accordingly, we have

$$\underset{\sim}{\Pi}\{\frac{p\ s\ g(s)}{s-q}\} = \underset{\sim}{E}\{s^{[\xi-\nu]^+}\}$$

for $|s| \leq 1$. If $k = 0,1,2,\ldots$, then

$$\underset{\sim}{E}\{s^{[\xi-\nu]^+} | \xi = k\} = p \sum_{j=0}^{k} q^j s^{k-j} + q^{k+1} =$$

$$= \begin{cases} \dfrac{ps^{k+1} - (1-s)q^{k+1}}{s - q} & \text{for } s \neq q , \\ (1 + kp)q^k & \text{for } s = q . \end{cases}$$

If we multiply the above equation by $\underset{\sim}{P}\{\xi = k\}$ and add for $k = 0,1,2,\ldots,$ then we obtain that

$$\underset{\sim}{\Pi}\{\frac{p\ s\ g(s)}{s-q}\} = \begin{cases} \dfrac{p\ s\ g(s) - (1-s)q\ g(s)}{s-q} & \text{if } s \neq q , \\ g(q) + p\ q\ g'(q) & \text{if } s = q , \end{cases}$$

and $|s| \leq 1$. The same result can be obtained by using (11.10).

13.15. If we use the same notation as in the solution of Problem 13.14, then we can write that

$$\mathbf{E}\{s^{\nu-\xi}\} = \frac{p\, g(1/s)}{1-qs}$$

for $1 \leq |s| < 1/q$. Accordingly, we have

$$\mathbf{\Pi}\{\frac{p\, g(1/s)}{1-qs}\} = \mathbf{E}\{s^{[\nu-\xi]^+}\} = 1 - \frac{q\, g(q)(1-s)}{1-qs}$$

for $|s| < 1/q$. For

$$\mathbf{E}\{s^{[\nu-\xi]^+} | \xi = k\} = p \sum_{j=0}^{k} q^j + p \sum_{j=k+1}^{\infty} q^j s^{j-k} =$$

$$= 1 - q^{k+1} + \frac{p\, q^{k+1} s}{1-qs}$$

whenever $k = 0,1,2,\ldots$ and $|s| < 1/q$. If we multiply this equation by $\mathbf{P}\{\xi = k\}$ and add for $k = 0,1,2,\ldots$, then we obtain the above formula. The same result can also be obtained by using formula (11.12).

Note. If $a(s) \in \mathbf{A}$ and $a^+(s) = \mathbf{\Pi}\{a(s)\}$, then we can write that

$$\mathbf{\Pi}\{a(\tfrac{1}{s})\} = a(\tfrac{1}{s}) - a^+(\tfrac{1}{s}) + a(1)$$

for $|s| = 1$. This follows easily from the following identity

$$s^{[-k]^+} = s^{-k} - s^{-[k]^+} + 1$$

which holds for any s and $k = 0, \pm 1, \pm 2, \ldots$.

Thus we can deduce the solution of Problem 13.15 from the solution of Problem 13.14 if $|s| = 1$ and by analytic continuation we can obtain the solution for $|s| \leq 1$ too.

13.16. Let

$$\underset{\sim}{P}\{\nu = j\} = \binom{m+j-1}{m-1} p^m q^j$$

for $j = 0,1,2,\ldots$. Then $\underset{\sim}{E}\{s^\nu\} = p^m/(1-qs)^m$ for $|s| < 1/q$. If ξ and ν are independent random variables, then

$$\underset{\sim}{E}\{s^{\xi-\nu}\} = \frac{p^m s^m g(s)}{(s-q)^m}$$

for $q < |s| \leq 1$ and

$$\underset{\sim}{\Pi}\{\frac{p^m s^m g(s)}{(s-q)^m}\} = \underset{\sim}{E}\{s^{[\xi-\nu]^+}\}$$

for $|s| \leq 1$. If $k = 0,1,2,\ldots$, then

$$\underset{\sim}{E}\{s^{[\xi-\nu]^+} | \xi = k\} = p^m \sum_{j=0}^{k} \binom{m+j-1}{m-1} q^j s^{k-j} + 1 - p^m \sum_{j=0}^{k} \binom{m+j-1}{m-1} q^j =$$

$$= \begin{cases} \dfrac{p^m s^{m+k} - \sum_{j=0}^{m-1} \binom{m+k}{j} q^{m+k-j} [p^m (s-q)^j - p^j (s-q)^m]}{(s-q)^m} & \text{for } s \neq q, \\ \sum_{j=0}^{m} \binom{m+k}{j} p^j q^{m+k-j} & \text{for } s = q. \end{cases}$$

If we multiply this equation by $\underset{\sim}{P}\{\xi = k\}$ and add for $k = 0,1,2,\ldots$, then we obtain that

$$\underset{\sim}{\Pi}\{\frac{p^m s^m g(s)}{(s-q)^m}\} = \begin{cases} \dfrac{p^m s^m g(s)}{(s-q)^m} - \sum_{j=0}^{m-1} \dfrac{1}{j!} \left(\dfrac{d^j q^m g(q)}{dq^j}\right) \left[\dfrac{p^m (s-q)^j - p^j (s-q)^m}{(s-q)^m}\right] & \text{for } s \neq q, \\ \sum_{j=0}^{m} \dfrac{p^j}{j!} \left(\dfrac{d^j q^m g(q)}{dq^j}\right) & \text{for } s = q, \end{cases}$$

and $|s| \leq 1$.

Note. In the above proof we used the following identity

$$\sum_{j=0}^{k} \binom{a+j}{b} q^j s^{k-j} = \frac{s^{k+1} \sum_{j=0}^{b} \binom{a}{j}(s-q)^j q^{b-j} - q^{k+1} \sum_{j=0}^{b} \binom{a+k+1}{j}(s-q)^j q^{b-j}}{(s-q)^{b+1}}$$

which holds if $s \neq q$ and a and b are nonnegative integers. This follows from the relation

$$\frac{1}{b!} \frac{d^b}{dz^b} \left(z^a \sum_{j=0}^{k} (qz)^j s^{k-j}\right)_{z=1} = \frac{1}{b!} \frac{d^b}{dz^b} \left(z^a \frac{s^{k+1} - (qz)^{k+1}}{s - qz}\right)_{z=1} .$$

13.17. If we use the same notation as in the solution of Problem 13.16, then we can write that

$$\mathbf{E}\{s^{\nu-\xi}\} = p^m g(1/s)/(1-qs)^m$$

for $1 \leq |s| < 1/q$ and

$$\Pi\left\{\frac{p^m g(1/s)}{(1-qs)^m}\right\} = \mathbf{E}\{s^{[\nu-\xi]^+}\}$$

for $|s| \leq 1$. If $k = 0,1,2,\ldots$, then

$$\mathbf{E}\{s^{[\nu-\xi]^+} | \xi = k\} = p^m \sum_{j=0}^{k} \binom{m+j-1}{m-1} q^j + p^m \sum_{j=k+1}^{\infty} \binom{m+j-1}{m-1} q^j s^{j-k} =$$

$$= 1 - \sum_{j=0}^{m-1} \binom{m+k}{j} p^j q^{m+k-j} + \sum_{j=0}^{m-1} \binom{m+k}{j} p^j q^{m+k-j} \left(\frac{ps}{1-qs}\right)^{m-j}$$

for $|s| < 1/q$ and hence it follows that

$$\underset{\sim}{\Pi}\left\{\frac{p^m g(1/s)}{(1-qs)^m}\right\} = 1 - \sum_{j=0}^{m-1} \frac{p^j}{j!}\left(\frac{d^j q^m g(q)}{dq^j}\right)\left[1 - \left(\frac{qs}{1-qs}\right)^{m-j}\right]$$

for $|s| < 1/q$. The same result can also be obtained by using formula (11.12).

Note. in the above proof we used the relations

$$\sum_{j=0}^{\infty} \binom{m-j-1}{m-1} q^j s^j = \frac{1}{(1-qs)^m} ,$$

$$\sum_{j=0}^{k} \binom{m-j-1}{m-1} q^j s^j = \frac{1}{(1-qs)^m} - \sum_{j=0}^{m-1} \binom{m+k}{j} \frac{(qs)^{m+k-j}}{(1-qs)^{m-j}}$$

and

$$\sum_{j=k+1}^{\infty} \binom{m-j-1}{m-1} q^j s^j = \sum_{j=0}^{m-1} \binom{m+k}{j} \frac{(qs)^{m+k-j}}{(1-qs)^{m-j}}$$

which hold for $|s| < 1/q$.

13.18. We can write that

$$b(s) = \frac{\pi_{m-1}(s)}{\prod_{j=1}^{m} (1-\beta_j s)}$$

for $|s| \leqq 1$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leqq m-1$. Since $|b(s)| \leqq 1$ for $|s| \leqq 1$, it follows that $|\beta_j| < 1$ for $j = 1,2,\ldots, m$. By formula (11.10) we have

$$\underset{\sim}{\Pi}\{a(s)b(\tfrac{1}{s})\} = \frac{1-s}{2\pi i} \int_{C_\varepsilon^+} \frac{g(z)b(\frac{1}{z})}{(1-z)(s-z)}\, dz$$

for $|s| < 1-\varepsilon$ where ε is a sufficiently small positive number. In the unit circle $|z| < 1$ the integrand has poles at $z = s$ and $z = \beta_j$ for $j = 1,2,\ldots, m$. If $\beta_1, \beta_2,\ldots, \beta_m$ are distinct and $s \neq \beta_j$ $(j = 1,2,\ldots, m)$, then by Chaucy's theorem of residues we obtain that

$$\underset{\sim}{\Pi}\{a(s)b(\tfrac{1}{s})\} = a(s)b(\tfrac{1}{s}) - \sum_{j=1}^{m} \frac{(1-s)\beta_j^m\, a(\beta_j)\, \pi_{m-1}(1/\beta_j)}{(1-\beta_j)(s-\beta_j)} \frac{1}{\prod_{i\neq j} (\beta_j - \beta_i)}$$

for $|s| \leqq 1$ and $s \neq \beta_j$ $(j = 1,2,\ldots, m)$. If $\beta_1, \beta_2,\ldots, \beta_m$ are not distinct, then we can obtain $\underset{\sim}{\Pi}\{a(s)b(\frac{1}{s})\}$ in a similar way.

13.19. As in the solution of Problem 13.18 we can write that

$$b(s) = \frac{\pi_{m-1}(s)}{\prod_{j=1}^{m} (1-\beta_j s)}$$

for $|s| \leqq 1$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leqq m-1$ and $|\beta_j| < 1$ for $j = 1,2,\ldots, m$. By (11.12) we have

$$\underset{\sim}{\Pi}\{a(\tfrac{1}{s})b(s)\} = 1 + \frac{1-s}{2\pi i} \int_{C_\varepsilon^-} \frac{a(\frac{1}{z})\, b(z)}{(1-z)(s-z)}\, dz$$

for $|s| \leqq 1$ where ε is a sufficiently small positive number. In the domain $|z| > 1$ the integrand has poles at $z = 1/\beta_j$ for $j = 1,2,\ldots, m$. If $\beta_1, \beta_2,\ldots, \beta_m$ are distinct, then by Cauchy's theorem of residues we obtain that

$$\underset{\sim}{\Pi}\{a(\tfrac{1}{s})b(s)\} = 1 - \sum_{j=1}^{m} \frac{(1-s)\beta_j^m\, a(\beta_j)\pi_{m-1}(1/\beta_j)}{(1-\beta_j)(1-s\beta_j)} \frac{1}{\prod_{i\neq j}(\beta_j-\beta_i)}$$

for $|s| \leq 1$. If $\beta_1, \beta_2, \ldots, \beta_m$ are not distinct, then we can obtain $\underset{\sim}{\Pi}\{a(\frac{1}{s})b(s)\}$ in a similar way.

13.20. Let $\{\nu_n\}$ be a sequence of mutually independent random variables for which $\underset{\sim}{P}\{\nu_n = j\} = h_j$ for $j = 0,1,2,\ldots$ and $n = 1,2,\ldots$. Define a sequence of random variables ξ_n $(n = 0,1,2,\ldots)$ by the recurrence formula

$$\xi_n = [\xi_{n-1} + 1 - \nu_n]^+$$

where $n = 1,2,\ldots$ and ξ_0 is a random variable which takes on only non-negative integers and which is independent of $\{\nu_n\}$. It can easily be seen that $\{\xi_n\}$ is a homogeneous Markov chain with state space $I = \{0,1,2,\ldots\}$ and transition probability matrix π . Accordingly, we can use the aforementioned representation of $\{\xi_n\}$ in finding the distribution of ξ_n for $n = 1,2,\ldots$. Let us introduce the notation

$$U_n(s) = \underset{\sim}{E}\{s^{\xi_n}\}$$

for $n = 0,1,2,\ldots$ and $|s| \leq 1$ and

$$h(s) = \sum_{j=0}^{\infty} h_j s^j$$

for $|s| \leq 1$. Then we can write that

$$U_n(s) = \underset{\sim}{\Pi}\{U_{n-1}(s)sh(\tfrac{1}{s})\}$$

for $n = 1,2,\ldots$. By Theorem 10.1 we obtain that

$$\sum_{n=0}^{\infty} U_n(s)\rho^n = e^{-\Pi\{\log[1-\rho sh(\frac{1}{s})]\}} \Pi\{U_0(s) \frac{e^{\Pi\{\log[1-\rho sh(\frac{1}{s})]\}}}{1-\rho sh(\frac{1}{s})}\}$$

for $|s| \leq 1$ and $|\rho| < 1$. If, in particular, $P\{\xi_0 = 0\} = 1$, that is, $U_0(s) \equiv 1$, then

$$\sum_{n=0}^{\infty} U_n(s)\rho^n = e^{-\Pi\{\log[1-\rho sh(\frac{1}{s})]\}}$$

for $|s| \leq 1$ and $|\rho| < 1$.

We observe that if $|\rho| < 1$, then the equation

$$\rho h(z) = z$$

has exactly one root $z = \delta(\rho)$ in the unit circle $|z| < 1$. If we use the notation $N_n = \nu_1 + \nu_2 + \ldots + \nu_n$ for $n = 1,2,\ldots,$ and $N_0 = 0$, then by Lagrange's expansion we obtain that

$$[\delta(\rho)]^k = \sum_{n=k}^{\infty} \frac{k}{n} P\{N_n = n-k\}\rho^n$$

for $k = 1,2,\ldots$ and $|\rho| < 1$.

Thus by (12.2) we can write that

$$1 - \rho sh(\frac{1}{s}) = g^+(s, \rho)g^-(s, \rho)$$

for $|s| = 1$ and $|\rho| < 1$ where

$$g^+(s, \rho) = 1 - s\delta(\rho)$$

for $|s| \leq 1$ and

$$g^{-}(s, \rho) = \frac{1 - \rho s h(\frac{1}{s})}{1 - s\delta(\rho)}$$

for $|s| \geq 1$. Hence by (12.13) we have

$$\sum_{n=0}^{\infty} U_n(s)\rho^n = \frac{1}{1-s\delta(\rho)} \Pi\{ \frac{U_0(s)[1-s\delta(\rho)]}{1 - \rho s h(\frac{1}{s})} \}$$

for $|s| \leq 1$ and $|\rho| < 1$. If, in particular, $P\{\xi_0 = 0\} = 1$, then by (12.14) we have

$$(1-\rho) \sum_{n=0}^{\infty} U_n(s)\rho^n = \frac{1 - \delta(\rho)}{1 - s\delta(\rho)}$$

for $|s| \leq 1$ and $|\rho| < 1$, that is,

$$(1-\rho) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P\{\xi_n = k | \xi_0 = 0\} s^k \rho^n = \frac{1 - \delta(\rho)}{1 - s\delta(\rho)} .$$

Hence

$$(1-\rho) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P\{\xi_n \geq k | \xi_0 = 0\} s^k \rho^n = \frac{1}{1 - s\delta(\rho)}$$

and

$$(1-\rho) \sum_{n=0}^{\infty} P\{\xi_n \geq k | \xi_0 = 0\} \rho^n = [\delta(\rho)]^k$$

for $k = 0,1,2,\ldots$ and $|\rho| < 1$. From this formula we can conclude that if $k = 1,2,\ldots$, then

$$\underset{\sim}{P}\{\xi_n \geqq k | \xi_0 = 0\} = \sum_{j=k}^{n} \frac{k}{j} \underset{\sim}{P}\{N_j = j - k\}$$

for $n = k, k+1, \ldots$.

If $\underset{\sim}{P}\{\xi_0 = i\} = 1$ where $i = 0,1,2,\ldots,$ then

$$[1-s\delta(\rho)] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \underset{\sim}{P}\{\xi_n = k | \xi_0 = i\} s^k \rho^n = \underset{\sim}{\Pi}\{ \frac{s^i[1 - s\delta(\rho)]}{1 - \rho s h(\frac{1}{s})} \}$$

for $|s| \leqq 1$ and $|\rho| < 1$. If we multiply this equation by w^i and add for $i = 0,1,2,\ldots,$ then we obtain that

$$[1-s\delta(\rho)] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \underset{\sim}{P}\{\xi_n = k | \xi_0 = i\} s^k \rho^n w^i = \underset{\sim}{\Pi}\{\frac{1 - s\delta(\rho)}{(1-sw)[1-\rho s h(\frac{1}{s})]} \}$$

for $|s| \leqq 1$, $|\rho| < 1$ and $|w| < 1$. Hence it follows that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \underset{\sim}{P}\{\xi_n \geqq k | \xi_0 = i\} s^k \rho^n w^i = \frac{1}{(1-s)(1-\rho)(1-w)} -$$

$$- \frac{s}{(1-s)[1-s\delta(\rho)]} \underset{\sim}{\Pi}\{\frac{1 - s\delta(\rho)}{(1-sw)[1-\rho s h(\frac{1}{s})]} \}$$

for $|s| \leqq 1$, $|\rho| < 1$ and $|w| < 1$. By (11.10) we can prove that

$$\underset{\sim}{\Pi}\{\frac{1 - s\delta(\rho)}{(1-sw)[1-\rho s h(\frac{1}{s})]}\} = \frac{1 - \delta(\rho)}{(1-w)(1-\rho)} + \frac{w(1-s)\ [w-\delta(\rho)]}{(1-w)(1-ws)[w-\rho h(w)]}$$

The above formulas make it possible to find $\underset{\sim}{P}\{\xi_n \geqq k | \xi_0 = i\}$ explicitly. If $k = 1,2,\ldots$ and $i = 0,1,\ldots,$ then we have

$$\underset{\sim}{P}\{\xi_n \geq k | \xi_0 = i\} = \underset{\sim}{P}\{N_n \leq n+i-k\} + \sum_{j=k}^{n} \underset{\sim}{P}\{N_j = j-k\}\underset{\sim}{P}\{N_{n-j} > n+i-j\}$$

for $n = 1,2,\ldots$.

If $h_0 > 0$ and $h_0 + h_1 < 1$, then $\{\xi_n\}$ is an irreducible and aperiodic Markov chain with state space $I = \{0, 1,2,\ldots\}$. Thus $\lim_{n\to\infty} \underset{\sim}{P}\{\xi_n = k\} = P_k$ exists for $k = 0,1,2,\ldots$ and is independent of the initial distribution. There are two possibilities: either $P_k > 0$ for $k = 0,1,2,\ldots$ and $\sum_{k=0}^{\infty} P_k = 1$, or $P_k = 0$ for $k = 0,1,2,\ldots$. In finding $\{P_k\}$ we may assume without loss of generality that $\underset{\sim}{P}\{\xi_0 = 0\} = 1$. Then by Abel's theorem we obtain that

$$\sum_{k=0}^{\infty} P_k s^k = \lim_{\rho\to+1} (1-\rho) \sum_{n=0}^{\infty} U_n(s)\rho^n = \frac{1-\delta}{1-s\delta}$$

where $\delta = \lim_{\rho\to+1} \delta(\rho)$. Accordingly, $P_k = (1-\delta)\delta^k$ for $k = 0,1,2,\ldots$.

We can easily prove that $\delta = 0$ if $\alpha \leq 1$, whereas $0 < \delta < 1$ if $\alpha > 1$.

CHAPTER II

21.1. Denote by $\tau_k = \rho_1 + \rho_2 + \ldots + \rho_k$ $(k = 1,2,\ldots)$ the k-th ladder index for $k = 1,2,\ldots$ and let $\tau_0 = 0$. Then $\rho_1, \rho_2,\ldots, \rho_k,\ldots$ are mutually independent and identically distributed random variables. Since $\underset{\sim}{P}\{\zeta_n > 0\} = 1/2$ for $n = 1,2,\ldots$, by Theorem 19.3 we obtain that

$$\underset{\sim}{E}\{z^{\rho_k}\} = \pi(z) = 1 - e^{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{z^n}{n}} = 1 - \sqrt{1-z}$$

for $|z| < 1$. Hence

$$\underset{\sim}{E}\{z^{\tau_k}\} = (1 - \sqrt{1-z})^k = \sum_{j=k}^{\infty} \frac{k}{2j-k} \binom{2j-k}{j} \frac{z^j}{2^{2j-k}}$$

for $|z| < 1$, and consequently

$$\underset{\sim}{P}\{\tau_k = j\} = \frac{k}{2j-k} \binom{2j-k}{j} \frac{1}{2^{2j-k}} = \left[\binom{2j-k-1}{j-1} - \binom{2j-k-1}{j}\right] \frac{1}{2^{2j-k}}$$

for $1 \leqq k \leqq j$.

Obviously we have

$$\underset{\sim}{P}\{\nu_n \geqq k\} = \underset{\sim}{P}\{\tau_k \leqq n\}$$

for $k = 0,1,2,\ldots$ and $n = 1,2,\ldots$. This implies that

$$\underset{\sim}{P}\{\nu_n = k\} = \underset{\sim}{P}\{\tau_k \leqq n\} - \underset{\sim}{P}\{\tau_{k+1} \leqq n\} = \binom{2n-k}{n} \frac{1}{2^{2n-k}}$$

for $0 \leqq k \leqq n$.

<u>Note.</u> The power series expansion of $[\pi(z)]^k$ can be proved either by mathematical induction if we take into consideration that

$$[\pi(z)]^k = 2[\pi(z)]^{k-1} - z[\pi(z)]^{k-2}$$

for $k = 2,3,\ldots$, or by Lagrange's expansion if we take into consideration that $w = \pi(z)$ is the only root of $w^2 - 2w + z = 0$ in the unit circle $|w| < 1$ whenever $|z| < 1$. The Lagrange's expansion of $[\pi(z)]^k$ is as follows:

$$[\pi(z)]^k = \frac{z^k}{2^k} + \sum_{n=1}^{\infty} \frac{k!}{2^n n!} \left(\frac{d^{n-1}\, a^{k-1}\, a^{2n}}{da^{n-1}}\right)_{a=z/2}$$

for $|z| < 1$.

21.2. In this case the sequence $\{\zeta_n\ ;\ n = 0,1,2,\dots\}$ describes the path of a one-dimensional random walk on the x-axis and $\tau_k = k+2m$ $(m = 0,1,\dots)$ if and only if the particle reaches the point $x = k$ for the first time at the $(k+2m)$-th step. By Lemma 20.3 we have

$$\mathbf{P}\{\tau_k = k+2m\} = \left[\binom{k+2m-1}{m} - \binom{k+2m-1}{m-1}\right]p^{k+m} q^m =$$

$$= \frac{k}{k+2m}\binom{k+2m}{m}p^{k+m} q^m$$

for $k = 1,2,\dots$ and $m = 0,1,2,\dots$. The same result can also be obtained by using the reflection principle. See formula (36.49).

We note that by the solution of Problem 21.1 we can write that

$$\mathbf{E}\{z^{\tau_k}\} = z^k\sum_{m=0}^{\infty} \frac{kp^k}{k+2m}\binom{k+2m}{m}(pqz^2)^m = \left[\frac{1-\sqrt{1-4pqz^2}}{2qz}\right]^k$$

for $k = 1,2,\dots$ and $|z| < 1$. This formula can be proved directly as follows. Since

$$\mathbf{P}\{\tau_1 = 2m+1\} = \frac{p}{2m+1}\binom{2m+1}{m}(pq)^m = \frac{p}{m+1}\binom{2m}{m}(pq)^m = (-1)^m 2p\binom{\frac{1}{2}}{m+1}(4pq)^m$$

for $m = 0,1,2,\dots$, therefore

$$\mathbf{E}\{z^{\tau_1}\} = 2p\, z\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m+1}(-4pqz^2)^m = 2p\,\frac{1-\sqrt{1-4pqz^2}}{4pqz}$$

for $|z| < 1$,and the relation $\mathbf{E}\{z^{\tau_k}\} = [\mathbf{E}\{z^{\tau_1}\}]^k$ proves the desired result.

Finally, we note that

$$\underset{\sim}{P}\{\tau_k < \infty\} = \lim_{z \to 1-0} \underset{\sim}{E}\{z^{\tau_k}\} = \left(\frac{1 - |p-q|}{2q}\right)^k = \begin{cases} 1 & \text{if } p \geq q, \\ (p/q)^k & \text{if } p < q. \end{cases}$$

21.3. Let $u_n(s) = \underset{\sim}{E}\{s^{\eta_n}\}$ for $n = 0,1,2,\ldots$ and $|s| \leq 1$, and $\gamma(s) = \underset{\sim}{E}\{s^{\xi_n}\} = ps + qs^{-1}$ for $s \neq 0$. We have $u_0(s) = 1$ and

$$u_n(s) = \underset{\sim}{\Pi}\{\gamma(s)\, u_{n-1}(s)\}$$

for $n = 1,2,\ldots$ and $|s| \leq 1$ where $\underset{\sim}{\Pi}$ is defined in Section 9. If $|s| = 1$ and $|z| < 1$, then we can write that

$$1 - z\,\gamma(s) = g^+(s,z)\, g^-(s,z)$$

where

$$g^+(s,z) = s - \frac{1 + \sqrt{1 - 4pqz^2}}{2pz} = s - \frac{2qz}{1 - \sqrt{1 - 4pqz^2}}$$

and

$$g^-(s,z) = \frac{1 - \sqrt{1 - 4pqz^2}}{2s} - pz$$

satisfy the conditions (a_1), (a_2), (b_1), (b_2), (b_3) in Section 12. By Theorem 12.2 we have

$$(1-z) \sum_{n=0}^{\infty} u_n(s) z^n = \frac{g^+(1,z)}{g^+(s,z)} = \frac{1 - 2qz - \sqrt{1 - 4pqz^2}}{s - 2qz - s\sqrt{1 - 4pqz^2}}$$

for $|s| \leq 1$ and $|z| < 1$. Hence

$$(1-z) \sum_{n=0}^{\infty} \underset{\sim}{P}\{\eta_n = k\} z^k = \left[\frac{1 - \sqrt{1 - 4pqz^2}}{2qz}\right]^k - \left[\frac{1 - \sqrt{1 - 4pqz^2}}{2qz}\right]^{k+1}$$

or

$$(1-z) \sum_{n=0}^{\infty} \underset{\sim}{P}\{\eta_n \geq k\}\, z^k = \left[\frac{1 - \sqrt{1 - 4pqz^2}}{2qz}\right]^k$$

for $k = 0,1,2,\ldots$ and $|z| < 1$. Hence by the solution of Problem 21.2 we get

$$\mathbf{P}\{\eta_n \geq k\} = \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{k}{k+2m} \binom{k+2m}{m} p^{k+m} q^m$$

for $k = 0,1,2,\ldots$.

We note that if τ_k $(k = 1,2,\ldots)$ denotes the k-th ladder index for the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ and $\tau_0 = 0$ then we have the obvious relation $\mathbf{P}\{\eta_n \geq k\} = \mathbf{P}\{\tau_k \leq n\}$ for $n \geq 0$ and $k \geq 0$. Thus $\mathbf{P}\{\eta_n \geq k\}$ can also be obtained immediately by the solution of Problem 21.2.

21.4. For $k = 1,2,\ldots$ we can write that $\tau_k = \rho_1 + \rho_2 + \ldots + \rho_k$ where $\rho_1, \rho_2, \ldots, \rho_k, \ldots$ are mutually independent and identically distributed random variables. Since

$$\underset{\sim}{P}\{\zeta_n > 0\} = \underset{\sim}{P}\{\frac{\zeta_n}{n^{1/\alpha}} > 0\} = \underset{\sim}{P}\{\xi_1 > 0\} = 1 - R_\alpha(0) = q$$

is independent of n, by Theorem 19.3 we obtain that

$$\underset{\sim}{E}\{z^{\rho_k}\} = \pi(z) = 1 - e^{-q \sum_{n=1}^{\infty} \frac{z^n}{n}} = 1 - (1-z)^q$$

for $|z| < 1$. Hence we obtain that

$$\underset{\sim}{P}\{\rho_k = j\} = (-1)^{j-1} \binom{q}{j}$$

for $j = 1,2,\ldots$. Since

$$\underset{\sim}{E}\{z^{\tau_k}\} = [1 - (1-z)^q]^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (1-z)^{rq} ,$$

it follows that

$$\underset{\sim}{P}\{\tau_k = j\} = (-1)^j \sum_{r=1}^{k} (-1)^r \binom{k}{r} \binom{rq}{j}$$

for $j = 1,2,\ldots$. Obviously $\underset{\sim}{P}\{\tau_k = j\} = 0$ for $j < k$. Accordingly,

$$\underset{\sim}{P}\{\tau_k \leq n\} = (-1)^n \sum_{r=1}^{k} (-1)^r \binom{k}{r} \binom{rq-1}{n} = \sum_{r=1}^{k} (-1)^r \binom{k}{r} \binom{n-rq}{n}$$

for $1 \leq k \leq n$.

We note that by (42.192) we have

$$q = \frac{1}{2} + \frac{1}{\alpha\pi} \text{arc tan}(\beta \tan \frac{\alpha\pi}{2}) .$$

See also Problem 46. 7.

21.5. Since $\underset{\sim}{P}\{\max_{1 \leq r \leq n} (N_r - r) < k\} = 0$ if $k < 0$, we can write that

$$\underset{\sim}{E}\{\max_{0 \leq r \leq n} (N_r - r)\} = \sum_{k=1}^{\infty} [1 - \underset{\sim}{P}\{\max_{1 \leq r \leq n} (N_r - r) < k\}] =$$

$$= \sum_{k=1}^{\infty} [1 - \underset{\sim}{P}\{\max_{1 \leq r \leq n} (N_r - r) < k\}] - \sum_{k=-\infty}^{0} \underset{\sim}{P}\{\max_{1 \leq r \leq n} (N_r - r) < k\} + \underset{\sim}{P}\{\max_{1 \leq r \leq n} (N_r - r) < 0\}$$

and the probabilities in question can be obtained by (20.8) and (20.13). Accordingly, we have

$$\underset{\sim}{E}\{\max_{0 \leq r \leq n} (N_r - r)\} = \underset{\sim}{E}\{[N_n - n]^+\} + \sum_{j=1}^{n-1} \sum_{\ell=0}^{n-j} (1 - \frac{\ell}{n-j}) \underset{\sim}{P}\{N_n - N_j = \ell, N_j > j\} -$$

$$- \underset{\sim}{E}\{[n - N_n]^+\} + \sum_{j=1}^{n-1} \sum_{\ell=0}^{n-j} (1 - \frac{\ell}{n-j}) \underset{\sim}{P}\{N_n - N_j = \ell, N_j \leq j\} + \frac{1}{n} \underset{\sim}{E}\{[n - N_n]^+\} =$$

$$= \sum_{j=1}^{n-1} \underset{\sim}{E}\{[1 - \frac{N_n - N_j}{n-j}]^+\} + \underset{\sim}{E}\{N_n - n\} - \frac{(n-1)}{n} \underset{\sim}{E}\{[n - N_n]^+\} =$$

$$= \sum_{j=0}^{n-1} \underset{\sim}{E}\{[1 - \frac{N_n - N_j}{n-j}]^+\} + \underset{\sim}{E}\{N_n - n\} = \sum_{j=1}^{n} \frac{1}{j} \underset{\sim}{E}\{[j - N_j]^+\} + \underset{\sim}{E}\{N_n - n\} =$$

$$= \sum_{j=1}^{n} \frac{1}{j} \underset{\sim}{E}[N_j - j]^+\}$$

because if $E\{\nu_j\} = \gamma < \infty$, then $E\{[j-N_j]^+\} = E\{j-N_j\} + E\{[N_j-j]^+\} = j(1-\gamma) + E\{[N_j-j]^+\}$ for $j = 1,2,\ldots, n$. If $\gamma = \infty$, then both sides of the equation to be proved are infinite.

21.6. Now $\max_{0\leq r\leq n} (r-N_r)$ is a discrete random variable which may take on the integers $0, 1,\ldots, n$ only. Thus by (20.17) we have

$$E\{\max_{0\leq r\leq n} (r-N_r)\} = \sum_{k=1}^{n} P\{\max_{1\leq r\leq n} (r-N_r) \geq k\} =$$

$$= \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{k}{j} P\{N_j = j-k\} = \sum_{j=1}^{n} \frac{1}{j} E\{[j-N_j]^+\} .$$

21.7. We have

$$E\{\eta_n\} = \sum_{j=1}^{n} \frac{1}{j} E\{\zeta_j^+\} .$$

If $E\{\xi_n^+\} = \infty$, then both sides of the above equation are infinite. Let us suppose that $E\{\xi_n^+\} < \infty$. Then $E\{\eta_n\} < \infty$ for $n = 1,2,\ldots$ because, obviously, $E\{\eta_n\} \leq n\, E\{\xi_1^+\}$. Since by (15.1)

$$\sum_{n=0}^{\infty} E\{e^{-s\eta_n}\}\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} E\{e^{-s\zeta_k^+}\}\}$$

for $Re(s) \geq 0$ and $|\rho| < 1$, it follows that

$$\sum_{n=1}^{\infty} E\{\eta_n\}\rho^n = \frac{1}{1-\rho} \sum_{k=1}^{\infty} \frac{\rho^k}{k} E\{\zeta_k^+\}$$

for $|\rho|< 1$. If we form the coefficient of ρ^n on the right-hand side, then we obtain $E\{\eta_n\}$ which was to be determined.

We note that in a similar way we can express $E\{\eta_n^r\}$ for $r = 1,2,\ldots$ with the aid of the moments $E\{[\zeta_j^+]^s\}$ $(s = 1,2,\ldots,r$ and $j = 1,2,\ldots, n)$.

21.8. Let us introduce the following notation: $P\{\chi_n \leq x\} = H(x)$, $P\{\chi_1+\ldots+ \chi_n \leq x\} = H_n(x)$, $E\{e^{-s\chi_n}\} = \psi(s)$, $E\{e^{-s\eta_n}\} = \Phi_n(s)$ and let

$$a_k(\lambda) = \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^k}{k!} [\int_0^u (1-\frac{x}{u})dH_k(u)]\lambda du =$$

$$= \frac{(-1)^{k-1}\lambda^{k+1}}{k!} \frac{d^{k-1}}{d\lambda^{k-1}} (\frac{[\psi(\lambda)]^k}{\lambda^2})$$

for $k = 1,2,\ldots$.

By Theorem 15.3 we have

$$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = e^{-T\{\log[1-\frac{\lambda\rho\psi(s)}{\lambda -s}]\}}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho| < 1$. By the first example in Section 18 we can also write that

$$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\lambda[\gamma(\rho) - s]}{\gamma(\rho)[\lambda-s-\lambda\rho\psi(s)]}$$

where $s = \gamma(\rho)$ is the only root of the equation

$$\lambda - s - \lambda\rho\psi(s) = 0$$

in the domain $\mathrm{Re}(s) \geq 0$ whenever $|\rho| < 1$.

By Lagrange's expansion we obtain that

$$\frac{1}{\gamma(\rho)} = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{k=1}^{\infty} \rho^k a_k(\lambda)$$

for $|\rho| < 1$, and consequently

$$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = [1- \frac{s}{\lambda-s} \sum_{k=1}^{\infty} \rho^k a_k(\lambda)] \sum_{j=0}^{\infty} (\frac{\lambda\rho\psi(s)}{\lambda - s})^j$$

for $|\lambda\rho\psi(s)| < |s-\lambda|$ and $\mathrm{Re}(s) \geq 0$. Hence

$$\Phi_n(s) = (\frac{\lambda\psi(s)}{\lambda - s})^n - \frac{s}{\lambda-s} \sum_{k=1}^{n} a_k(\lambda)(\frac{\lambda\psi(s)}{\lambda - s})^{n-k}$$

for $n = 1,2,\ldots$, $\mathrm{Re}(s) \geq 0$ and $s \neq \lambda$. If we write $s = \lambda - (\lambda-s)$ in front of the sum, then by inversion we obtain that

$$\mathop{P}_{\sim}\{\eta_n \leq x\} = K_n(x) - \sum_{k=1}^{n} a_k(\lambda)[K^*_{n-k}(x) - K_{n-k}(x)]$$

for any x where

$$K_n(x) = \frac{\lambda^n}{(n-1)!} \int_0^{\infty} H_n(u+x)e^{-\lambda u} u^{n-1}\, du \quad \text{and} \quad K^*_n(x) = \frac{\lambda^{n+1}}{n!} \int_0^{\infty} H_n(u+x)e^{-\lambda u}u^n du$$

for $n = 1,2,\ldots$, $K_0(x) = 1$ for $x \geq 0$, $K_0(x) = 0$ for $x < 0$, and $K^*_0(x) = 1$ for $x \geq 0$, $K^*_0(x) = e^{\lambda x}$ for $x < 0$. Here we took into consideration that

$$\int_{-\infty}^{\infty} e^{-sx} dK_n(x) = (\frac{\lambda\psi(s)}{\lambda - s})^n \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-sx} dK^*_n(x) = \frac{\lambda^{n+1}[\psi(s)]^n}{(\lambda-s)^{n+1}}$$

for $n = 0,1,2,\ldots$ and $0 \leq \mathrm{Re}(s) < \lambda$.

If $x < 0$, then obviously $\mathbf{P}\{\eta_n \leq x\} = 0$. Furthermore, $\mathbf{P}\{\eta_n = 0\} = \lim_{s \to \infty} \Phi_n(s) = a_n(\lambda)$.

We can also write down that

$$\mathbf{P}\{\eta_n \leq x\} = (-1)^n I_n(x) + \sum_{k=1}^{n} (-1)^{n-k} a_k(\lambda) [I^*_{n-k}(x) + I_{n-k}(x)]$$

for $x \geq 0$ where

$$I_n(x) = \frac{\lambda^{n-1} e^{\lambda x}}{(n-1)!} \int_0^x e^{-\lambda y}(x-y)^{n-1} dH_n(y) \quad \text{and} \quad I^*_n(x) = \frac{\lambda^n e^{\lambda x}}{n!} \int_0^x e^{-\lambda y}(x-y)^n dH_n(y)$$

for $x \geq 0$ and $n = 1,2,\ldots$, $I_0(x) = 1$ for $x \geq 0$ and $I^*_0(x) = (e^{\lambda x}-1)/\lambda$ for $x \geq 0$. Here we used that $\mathbf{P}\{\eta_n = 0\} = a_n(\lambda)$,

$$\int_0^\infty e^{-sx}\, dI_n(x) = \left(\frac{\lambda\psi(s)}{s-\lambda}\right)^n \quad \text{and} \quad \int_0^\infty e^{-sx}\, dI^*_n(x) = \frac{\lambda^{n+1}[\psi(s)]^n}{(s-\lambda)^{n+1}}$$

for $\mathrm{Re}(s) > \lambda$, and

$$\int_0^\infty e^{-sx+\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda dx = \left(\frac{\lambda}{s-\lambda}\right)^n$$

for $n = 1,2,\ldots$ and $\mathrm{Re}(s) > \lambda$.

21.9. Let us introduce the following notation: $\mathbf{P}\{\chi_n \leq x\} = H(x)$, $\mathbf{P}\{\chi_1+\ldots+\chi_n \leq x\} = H_n(x)$, $H_0(x) = 1$ for $x \geq 0$, $H_0(x) = 0$ for $x < 0$ and

$$F_n(x) = \begin{cases} 1 - \sum_{j=0}^{n-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!} & \text{for } x \geqq 0 , \\ 0 & \text{For } x < 0 . \end{cases}$$

Furhtermore, let $\underset{\sim}{E}\{e^{-s\chi_n}\} = \psi(s)$ and $\underset{\sim}{E}\{e^{-s\eta_n}\} = \Phi_n(s)$ for $Re(s) \geqq 0$. Since in this case

$$\underset{\sim}{E}\{e^{-s\theta_n}\} = \left(\frac{\lambda}{\lambda + s}\right)^m$$

for $Re(s) > -\lambda$ and $n = 1,2,\ldots,$ we can apply the solution of the second example in Section 18 to obtain

$$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\lambda^m}{(\lambda-s)^m - \lambda^m \rho\psi(s)} \prod_{i=1}^{m} \left(1 - \frac{s}{\gamma_i(\rho)}\right)$$

for $Re(s) \geqq 0$ and $|\rho| < 1$ where $s = \gamma_i(\rho)$ $(i = 1,2,\ldots, m)$ are the m roots of the equation

$$(\lambda - s)^m - \lambda^m \rho\psi(s) = 0$$

in the domain $Re(s) \geqq 0$ whenever $|\rho| < 1$.

By forming the Lagrange expansion of the root $\gamma_i(\rho)$ for $i = 1,2,\ldots, m$ and $|\rho| < 1$, we can write that

$$\prod_{i=1}^{m} \left(1 - \frac{s}{\gamma_i(\rho)}\right) = \left(1- \frac{s}{\lambda}\right)^m + \sum_{r=0}^{m} \left(1- \frac{s}{\lambda}\right)^r \sum_{k=1}^{\infty} a_{k,r}(\lambda)\rho^k$$

for any s where $a_{k,r}(\lambda)$ $(k = 1,2,\ldots; r = 0,1,\ldots, m)$ are appropriate functions of λ .

If $|\lambda^m \rho\psi(s)| < |(\lambda-s)^m|$ and $\mathrm{Re}(s) \geq 0$, then obviously

$$\frac{\lambda^m}{(\lambda-s)^m - \lambda^m \rho\psi(s)} = \left(\frac{\lambda}{\lambda - s}\right)^m \sum_{j=0}^{\infty} \left(\frac{\lambda^m \rho\psi(s)}{(\lambda-s)^m}\right)^j .$$

By using these expansions we can conclude that

$$\Phi_n(s) = \left(\frac{\lambda^m \psi(s)}{(\lambda-s)^m}\right)^n + \sum_{r=0}^{m} \sum_{k=1}^{n} a_{k,r}(\lambda)\left(\frac{\lambda}{\lambda-s}\right)^{(n-k+1)m-r}[\psi(s)]^{n-k}$$

for $n = 1,2,\ldots$, $\mathrm{Re}(s) \geq 0$ and $s \neq \lambda$. Hence it follows by inversion that

$$P\{\eta_n \leq x\} = K_{n,mn}(x) + \sum_{r=0}^{m} \sum_{k=1}^{n} a_{k,r}(\lambda) K_{n-k,(n-k+1)m-r}(x)$$

for any x where

$$K_{n,j}(x) = \frac{\lambda^j}{(j-1)!} \int_0^\infty H_n(u+x) e^{-\lambda u} u^{j-1}\, du$$

for $n = 0,1,2,\ldots$ and $j = 1,2,\ldots$ and $K_{n,0}(x) = H_n(x)$ for $n = 0,1,2,\ldots$.
Here we used that

$$\int_{-\infty}^{\infty} e^{-sx}\, dK_{n,j}(x) = \left(\frac{\lambda}{\lambda-s}\right)^j [\psi(s)]^n$$

for $0 \leq \mathrm{Re}(s) < \lambda$ and $n = 0,1,2,\ldots;\ j = 0,1,2,\ldots$.

If $x < 0$, then obviously $P\{\eta_n \leq x\} = 0$. Furthermore, $P\{\eta_n = 0\} = \lim_{s\to\infty} \Phi_n(s) = a_{n,m}(\lambda)$.

We can also write down that

$$\mathbf{P}\{\eta_n \leq x\} = (-1)^{mn} I_{n,mn}(x) + \sum_{r=0}^{m} \sum_{k=1}^{n} (-1)^{(n-k+1)m-r} a_{k,r}(\lambda) I_{n-k,(n-k+1)m-r}(x)$$

for $x \geq 0$

$$I_{n,j}(x) = \frac{\lambda^{j-1} e^{\lambda x}}{(j-1)!} \int_0^x e^{-\lambda y} (x-y)^{j-1} dH_n(y)$$

for $x \geq 0$, $n \geq 0$, $j \geq 1$ and $I_{n,0}(x) = H_n(x)$ for $n = 0,1,2,\ldots$.

Here we used that

$$\int_0^\infty e^{-sx} dI_{n,j}(x) = \left(\frac{\lambda}{\lambda-s}\right)^j [\psi(s)]^n$$

for $Re(s) > \lambda$ and $n \geq 0$, $j \geq 0$, and that $\mathbf{P}\{\eta_n = 0\} = a_{n,m}(\lambda)$.

We note that

$$\sum_{r=0}^{m} a_{k,r}(\lambda) = 0$$

for $k = 1,2,\ldots$ and

$$\sum_{k=1}^{\infty} a_{k,0}(\lambda)\rho^k = \prod_{i=1}^{m} \left(1- \frac{\lambda}{\gamma_i(\rho)}\right) .$$

21.10. We shall prove that the probability in question depends only on n and k and thus we can denote this probability by $P(n, k) = S(n, k)/n!$ where $S(n, k)$ is the number of favorable cases. We shall prove that

$$P(n, k) = \begin{cases} 1 - \frac{k}{n} & \text{if } 0 \leq k < n , \\ 0 & \text{if } k \geq n . \end{cases}$$

If $n = 1$, then $P(1, 0) = 1$ and $P(1, k) = 0$ for $k \geq 1$. Let us suppose that the above formula is true for every k if n is replaced by $n-1$ $(n = 2,3,\dots)$. We shall prove that it is true for every k and n . Thus by mathematical induction we can conclude that it is true for $n = 1,2,\dots$ and $k = 0,1,2,\dots$. If $k \geq n$,then obviously $P(n, k) = 0$. Let $0 \leq k < n$. Since the last number drawn may be k_i $(i = 1,2,\dots,n)$, we have

$$S(n, k) = \sum_{i=1}^{n} S(n-1, k-k_i) ,$$

or in other words,

$$P(n, k) = \frac{1}{n} \sum_{i=1}^{n} P(n-1, k-k_i) .$$

If $0 \leq k < n$, then by the induction hypothesis the right-hand side becomes

$$P(n, k) = \frac{1}{n} \sum_{i=1}^{n} (1- \frac{k-k_i}{n-1}) = 1 - \frac{k}{n-1} + \frac{k}{n(n-1)} = 1 - \frac{k}{n} .$$

This proves that $P(n, k)$ depends only on n and k and that the aformentioned formula is true.

CHAPTER III

27.1. By Theorem 23.1 we have

$$\mathbf{P}\{\Delta_n = j\} = \mathbf{P}\{\Delta_j = j\}\,\mathbf{P}\{\Delta_{n-j} = 0\}$$

for $0 \leqq j \leqq n$ and by Theorem 24.1 we have

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = n\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n > 0\}\}$$

and

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = 0\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n \leqq 0\}\}$$

for $|\rho| < 1$. Since in our case $\mathbf{P}\{\zeta_n > 0\} = \frac{1}{2}$ for $n = 1,2,\ldots,$ it follows that

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = n\}\rho^n = \sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = 0\}\rho^n = (1-\rho)^{-1/2}$$

for $|\rho| < 1$. Thus

$$\mathbf{P}\{\Delta_n = n\} = \mathbf{P}\{\Delta_n = 0\} = \binom{2n}{n}\frac{1}{2^{2n}}$$

and

$$\mathbf{P}\{\Delta_n = j\} = \binom{2j}{j}\binom{2n-2j}{n-j}\frac{1}{2^{2n}}$$

for $0 \leqq j \leqq n$.

We note that if Δ_n^* denotes the number of nonnegative elements in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, then obviously $\mathbf{P}\{\Delta_n^* = j\} = \mathbf{P}\{\Delta_n = j\}$ for $0 \leqq j \leqq n$.

<u>Remark</u>. We can also obtain $P\{\Delta_n = j\}$ for $j = 0,1,\ldots,n$ in a simpler way. First, we observe that $P\{\Delta_n = n\} = P\{\Delta_n = 0\}$ for $n = 0,1,2,\ldots$. Since $P\{\xi_r \leq x\} = P\{-\xi_r \leq x\}$ for $r = 1,2,\ldots,n$, we have $P\{\Delta_n = n\} = P\{\Delta_n^* = 0\}$ and since $P\{\zeta_r = 0\} = 0$ for $r = 1,2,\ldots,n$, we have $P\{\Delta_n^* = 0\} = P\{\Delta_n = 0\}$. Thus we can write that

$$P\{\Delta_n = j\} = P\{\Delta_j = 0\} P\{\Delta_{n-j} = 0\}$$

for $0 \leq j \leq n$. Hence

$$(*) \qquad \sum_{j=0}^{n} P\{\Delta_j = 0\} P\{\Delta_{n-j} = 0\} = 1$$

for $n = 0,1,2,\ldots$. From this equation we obtain step by step that

$$P\{\Delta_n = 0\} = \binom{2n}{n} \frac{1}{2^{2n}} = (-1)^n \binom{-\frac{1}{2}}{n}$$

for $n = 0,1,2,\ldots$.

If we multiply (*) by z^n and add for $n = 0,1,2,\ldots,$ then we obtain that

$$\sum_{n=0}^{\infty} P\{\Delta_n = 0\} z^n = \frac{1}{\sqrt{1-z}}$$

for $|z| < 1$, and this also yields the above result.

27.2. Since in this case $\mathbf{P}\{\zeta_n > 0\} = q$ for $n = 1,2,\ldots$ where

$$q = \frac{1}{2} + \frac{1}{\alpha\pi} \text{ arc } \tan(\beta \tan \frac{\alpha\pi}{2}) ,$$

in exactly the same way as in the solution of Problem 27.1 we obtain that

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = n\}\rho^n = (1-\rho)^{-q}$$

and

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = 0\}\rho^n = (1-\rho)^{q-1}$$

for $|\rho| < 1$. Thus

$$\mathbf{P}\{\Delta_n = j\} = \mathbf{P}\{\Delta_j = j\}\mathbf{P}\{\Delta_{n-j} = 0\} = (-1)^n\binom{-q}{j}\binom{q-1}{n-j}$$

for $0 \leq j \leq n$.

We note that if Δ_n^* denotes the number of nonnegative elements in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, then we have $\mathbf{P}\{\Delta_n^* = j\} = \mathbf{P}\{\Delta_n = j\}$ for $0 \leq j \leq n$ because the random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ have a continuous distribution function.

27.3 By Theorem 22.1 we can write that

$$\mathbf{P}\{\Delta_n = k\} = \mathbf{P}\{\Delta_k = k\}\,\mathbf{P}\{\Delta_{n-k} = 0\}$$

for $0 \leq k \leq n$.

Define $\mathbf{P}\{\Delta_n = 0\} = a_n(p)$ for $n = 0,1,2,\ldots$. Since

$$P\{\Delta_n = n\} = P\{\xi_1 = 1\} \, P\{\zeta_i - \zeta_1 \geq 0 \quad \text{for} \quad 1 \leq i \leq n\} =$$

$$= p \, P\{-\zeta_r \leq 0 \quad \text{for} \quad 0 \leq r \leq n-1\} = p \, a_{n-1}(q)$$

for $n = 1,2,\ldots,$ we can write that

$$P\{\Delta_n = k\} = p \, a_{k-1}(q) \, a_{n-k}(p)$$

for $k = 1,2,\ldots,n$. Thus it remains only to determine $a_n(p)$ for $n = 0,1,2,\ldots$ and $0 < p < 1$.

By the solution of Problem 21.3 we have

$$a_n(p) = P\{\eta_n = 0\} = 1 - p \sum_{m=0}^{[\frac{n-1}{2}]} \binom{2m}{m} \frac{(pq)^m}{m+1}$$

for $n = 1,2,\ldots,$ and $a_0(p) = 1$.

<u>Remark</u>. We can also determine $a_n(p)$ for $n = 0,1,2,\ldots$ and $0 < p < 1$ as follows. Since

$$\sum_{k=0}^{n} P\{\Delta_n = k\} = 1 ,$$

we get

$$p \sum_{k=1}^{n} a_{k-1}(q) \, a_{n-k}(p) = 1 - a_n(p)$$

for $n = 1,2,\ldots$ and $a_0(p) = 1$. If we introduce the generating function

$$A(z,p) = \sum_{n=0}^{\infty} a_n(p) \, z^n$$

for $|z| < 1$ and $0 < p < 1$, then we get

$$pz(1-z) \, A(z,p) \, A(z,q) + (1-z) \, A(z,p) - 1 = 0 \quad .$$

If we interchange p and q in the above equation, then we obtain that

$$qz(1-z)\,A(z,p)\,A(z,q) + (1-z)\,A(z,q) - 1 = 0 .$$

Consequently, we have

$$q\,A(z,p) - \frac{q}{1-z} = p\,A(z,q) - \frac{p}{1-z} .$$

This implies that

$$q\,a_n(p) - q = p\,a_n(q) - p$$

for $n = 0,1,2,\ldots,$ and

$$q\,z(1-z)[A(z,p)]^2 + (1-2qz)\,A(z,p) - 1 = 0 .$$

Accordingly,

$$A(z,p) = \frac{\sqrt{1-4pqz^2} - (1-2qz)}{2qz(1-z)}$$

for $0 < p < 1$ and $|z| < 1$. Finally, we obtain that

$$a_n(p) = 1 - \frac{1}{2q} + \frac{1}{2q} \sum_{j=0}^{[\frac{n+1}{2}]} (-1)^j \binom{\frac{1}{2}}{j} (4pq)^j$$

for $n = 0,1,2,\ldots$ and $0 < p < 1$. This is in agreement with the previous result.

27.4. The random variables $\nu_1, \nu_2, \ldots, \nu_n$ are interchangeable random variables taking on nonnegative integers and having sum $\nu_1 + \ldots + \nu_n = n$. By Theorem 26.3 we have

$$P\{\Delta_n^* = j\} = \begin{cases} \displaystyle\sum_{i=n-j}^{n-1} \frac{1}{i(n-i)} P\{N_i = i+1\} & \text{for } 1 \leq j \leq n-1, \\ \displaystyle 1 - \sum_{i=1}^{n-1} \frac{1}{(n-i)} P\{N_i = i+1\} & \text{for } j = n, \end{cases}$$

and evidently

$$\mathop{P}_{\sim}\{N_i = i+1\} = \binom{n}{i+1}\left(\frac{i}{n}\right)^{i+1}\left(1-\frac{i}{n}\right)^{n-i-1}$$

for $i = 1,2,\ldots, n-1$.

Thus we can write that

$$\mathop{P}_{\sim}\{\Delta_n^* = j\} = \frac{1}{n}\sum_{r=1}^{j}\frac{1}{r}\binom{n}{r-1}\left(\frac{r}{n}\right)^{r-1}\left(1-\frac{r}{n}\right)^{n-r}$$

for $1 \leq j \leq n$. For $1 \leq j \leq n-1$ this is obvious. For $j = n$ we used that

$$\sum_{r=1}^{n}\frac{1}{r}\binom{n-1}{r-1}\left(\frac{r}{n}\right)^{r-1}\left(1-\frac{r}{n}\right)^{n-r} = 1$$

for $n = 1,2,\ldots$. We note that

$$\mathop{P}_{\sim}\{\Delta_n^* = n\} = \frac{(n+1)^{n-1}}{n^n}$$

for $n = 1,2,\ldots$.

27.5. If we apply Theorem 22.2 to the random variables $\xi_i = 1-\nu_i$ $(i = 1,2,\ldots, n)$, then we obtain that

$$\mathop{P}_{\sim}\{\Delta_n^{(c)} = j\} = \mathop{P}_{\sim}\{N_r < r-c \text{ for } j \text{ subscripts } r = 1,2,\ldots, n\} =$$

$$= \mathop{P}_{\sim}\{j-N_j > r-N_r - c \text{ for } 0 \leq r < j \text{ and } j-N_j \geq r-N_r-c \text{ for } j \leq r \leq n\}.$$

If $\Delta_n^{(c)} = j$ and $j \geq 1$, then there is an r such that $N_r = r-c$. Hence $N_j \leq j$ necessarily holds. Consequently, we can write that

$$P\{\Delta_n^{(c)} = j\} = \sum_{\ell=0}^{j} P\{N_j - N_r < j-r+c \ \text{ for } \ 0 \leq r < j,\ N_j = \ell,\ N_r - N_j \geq r-j-c \ \text{ for } \ j \leq r \leq n\} =$$

$$= \sum_{\ell=0}^{j} [P\{N_j = \ell\} - \sum_{i=1}^{j-1} (1 - \frac{\ell-i-c}{j-i}) P\{N_i = i+c\ ,\ N_j = \ell\}] \cdot$$

$$[1 - \sum_{r=c+1+j}^{n} \frac{c+1}{r-j} P\{N_r - N_j = r-c-1 | N_j = \ell\}]\ .$$

In the sum the first factor can be obtained by (20.13) and the second factor by (20.17) . Accordingly, we have

$$P\{\Delta_n^{(c)} = j\} = \sum_{\ell=0}^{j} [P\{N_j = \ell\} - \sum_{r=c+1+j}^{n} \frac{c+1}{r-j} P\{N_j = \ell,\ N_r - N_j = r-c-1\}] -$$

$$- \sum_{\ell=0}^{j} \sum_{i=1}^{j-1} (1 - \frac{\ell-i-c}{j-i}) [P\{N_i = i+c,\ N_j = \ell\} - \sum_{r=c+1+j}^{n} \frac{c+1}{r-j} P\{N_i = i+c,\ N_j = \ell,$$

$$N_r - N_j = r-c-1\}]$$

for $c = 0,1,\ldots, n-1$ and $j = 1,2,\ldots, n-c$.

CHAPTER IV

34.1. Denote by Δ_n the number of positive partial sums in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, and by Δ_n^* the number of nonnegative partial sums in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$. By Theorem 29.1 we have

$$\mathbf{P}\{\alpha_{nk} = j\} = \sum_{\max(0,j+k-n)\leq r\leq \min(j,k)} \mathbf{P}\{\Delta_j^* = r\}\mathbf{P}\{\Delta_{n-j} = k-r\} .$$

By Theorem 23.1 we have

$$\mathbf{P}\{\Delta_n = k\} = \mathbf{P}\{\Delta_k = k\}\mathbf{P}\{\Delta_{n-k} = 0\}$$

and

$$\mathbf{P}\{\Delta_n^* = k\} = \mathbf{P}\{\Delta_k^* = k\}\mathbf{P}\{\Delta_{n-k}^* = 0\}$$

for $0 \leq k \leq n$. Furthermore, by Theorem 24.1 we have

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = n\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n > 0\}\} ,$$

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n = 0\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n \leq 0\}\} ,$$

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n^* = n\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n \geq 0\}\} ,$$

and

$$\sum_{n=0}^{\infty} \mathbf{P}\{\Delta_n^* = 0\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n < 0\}\}$$

for $|\rho| < 1$. Accordingly, $\mathbf{P}\{\alpha_{nk} = j\}$ is completely determined by the probabilities $\mathbf{P}\{\zeta_r > 0\}$ and $\mathbf{P}\{\zeta_r < 0\}$ for $r = 1,2,\ldots, n$.

34.2. It follows easily from (31.9) that

$$(1-\rho)\sum_{n=k}^{\infty} \Phi_{nk}(s)\rho^n = \frac{\sum_{n=k}^{\infty} \Phi_{nn}(s)\rho^n}{\sum_{n=0}^{\infty} \Phi_{nn}(s)\rho^n}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$.

In our particular case, $\Phi_{nn}(s) = \Phi_n(s)$ is given explicitly for $\mathrm{Re}(s) \geqq 0$ and $n = 0,1,2,\ldots$ in the solution of Problem 21.7. Thus by the above formula we can also determine explicitly $\Phi_{nk}(s)$ for $0 \leqq k \leqq n$.

We note that by Theorem 31.2 and by the first example in Section 18 we obtain that

$$(1-\omega)(1-\rho)\sum_{n=0}^{\infty}\sum_{k=0}^{n} \Phi_{nk}(s)\rho^n\omega^k =$$

$$= 1-\omega\,\frac{\gamma(\rho)[\gamma(\rho\omega)-s]\,[\lambda-s-\lambda\rho\psi(s)]}{\gamma(\rho\omega)[\gamma(\rho)-s]\,[\lambda-s-\lambda\rho\omega\psi(s)]}$$

for $\mathrm{Re}(s) \geqq 0$, $|\rho| < 1$, $|\rho\omega| < 1$ where $s = \gamma(\rho)$ is the only root of the equation

$$\lambda-s-\lambda\rho\psi(s) = 0$$

in the domain $\mathrm{Re}(s) \geqq 0$ whenever $|\rho| < 1$.

CHAPTER V

40.1. We can write that

$$p_n(a,b) = pP^*(n-1, a-1)$$

for $n = 1,2,\ldots$ where

$$P^*(n,j) = pP^*(n-1, j-1) + qP^*(n-1, j+1)$$

for $n = 1,2,\ldots$ and $-b < j < a$, $P^*(n,a) = P^*(n,-b) = 0$ for $n = 1,2,\ldots,$ $P^*(0,0) = 1$, and $P^*(0,j) = 0$ for $j \neq 0$. See (37.29). Let

$$U_j(z) = \sum_{n=0}^{\infty} P^*(n,j)z^j$$

for $-b \leqq j \leqq a$. Then $U_a(z) \equiv U_{-b}(z) \equiv 0$ and

$$U_j(z) = pzU_{j-1}(z) + qzU_{j+1}(z) + P^*(0,j)$$

for $-b < j < a$. Since the equation $qz\omega^2 - \omega + pz = 0$ has two roots

$$\omega_1 = \frac{1+\sqrt{1-4pqz^2}}{2qz} \quad \text{and} \quad \omega_2 = \frac{1-\sqrt{1-4pqz^2}}{2qz}$$

for $z \neq 0$ and $|4pqz^2| < 1$, the general solution of the above difference equation can be expressed as

$$U_j(z) = A\omega_1^j - B\omega_2^j - \delta(j)\,\frac{\omega_1^j - \omega_2^j}{qz(\omega_1 - \omega_2)}$$

where A and B are arbitrary constants and $\delta(j) = 0$ for $j \leq 0$ and $\delta(j) = 1$ for $j \geqq 1$. The requirements $U_a(z) = U_{-b}(z) = 0$ yield that

$$A = \frac{\omega_1^b(\omega_1^a - \omega_2^a)}{qz(\omega_1 - \omega_2)(\omega_1^{a+b} - \omega_2^{a+b})} \quad \text{and} \quad B = \frac{\omega_2^b(\omega_1^a - \omega_2^a)}{qz(\omega_1 - \omega_2)(\omega_1^{a+b} - \omega_2^{a+b})} .$$

Accordingly we have

$$\sum_{n=1}^{\infty} p_n(a,b)z^n = pz\, U_{a-1}(z) = \frac{p(\omega_1\omega_2)^{a-1}(\omega_1^b - \omega_2^b)}{q(\omega_1^{a+b} - \omega_2^{a+b})} =$$

$$= (2pz)^a \left\{ \frac{[1+\sqrt{1-4pqz^2}]^b - [1-\sqrt{1-4pqz^2}]^b}{[1+\sqrt{1-4pqz^2}]^{a+b} - [1-\sqrt{1-4pqz^2}]^{a+b}} \right\}$$

for $|4pqz^2| < 1$. We can obtain $p_n(a,b)$ explicitly either by (37.24) or by (37.25).

40.2. In exactly the same way as in the solution of Problem 40.1 we obtain that

$$\sum_{n=1}^{\infty} P\{\underset{\sim}{\rho} = n\}z^n = pz\, U_{a-1}(z)$$

for $|4pqz^2| < 1$ where

$$U_j(z) = pz\, U_{j-1}(z) + qz\, U_{j+1}(z) + P^*(0,j)$$

for $-\infty < j < a$, $U_a(z) \equiv 0$, $P^*(0,0) = 1$ and $P^*(0,j) = 0$ for $j \neq 0$. Since $|U_j(z)| \leq 1/(1-|z|)$ for all $j < a$ and $|z| < 1$, it follows that in the general solution $B = 0$ and A is determined by the condition $U_a(z) \equiv 0$. Thus we obtain that

$$U_j(z) = \frac{(\omega_1^a - \omega_2^a)\omega_1^j - \delta(j)(\omega_1^j - \omega_2^j)\omega_1^a}{qz(\omega_1 - \omega_2)\omega_1^a} .$$

for $j < a$ and $|4pqz^2| < 1$. Finally

$$\sum_{n=1}^{\infty} \underset{\sim}{P}\{\rho = n\}z^n = \omega_2^a = [\frac{1 - \sqrt{1-4pqz^2}}{2qz}]^a$$

for $|4pqz^2| < 1$. The probability $\underset{\sim}{P}\{\rho = a + 2m\}$ for $m = 0,1,2,\ldots$ is given explicitly by (36.42).

40.3. Let us consider a one-dimensional symmetric random walk. Denote by η_{2n} the position of the particle at the $2n$-th step. Then η_{2n} has the characteristic function

$$\underset{\sim}{E}\{e^{it\eta_{2n}}\} = (\cos t)^{2n}$$

and

$$Q_{2n} = \underset{\sim}{P}\{\eta_{2n} = 0\} = \binom{2n}{n} \frac{1}{2^{2n}} = \frac{1}{2\pi} \int_0^{2\pi} (\cos t)^{2n} dt = \frac{2}{\pi} \int_0^{\pi/2} (\cos t)^{2n} dt .$$

This relation can also be proved directly. Let us define

$$I_k = \int_0^{\pi/2} (\cos t)^k dt$$

for $k = 0,1,2,\ldots$. Then $I_0 = \frac{\pi}{2}$, $I_1 = 1$ and by integrating by parts we obtain that

$$I_k = \frac{(k-1)}{k} I_{k-2}$$

for $k = 2,3,\ldots$. Hence

$$I_{2n} = \frac{\pi}{2}\frac{1.3\ \ldots\ (2n-1)}{2.4\ \ldots\ 2n} = \frac{\pi}{2}\binom{2n}{n}\frac{1}{2^{2n}} = \frac{\pi}{2}Q_{2n}$$

which is in agreement with the prece ding formula.

Since $0 < \cos t < 1$ for $0 < t < \frac{\pi}{2}$, therefore

$$I_{2n+1} < I_{2n} < I_{2n-1}$$

or

$$\frac{2n}{2n+1} < \frac{I_{2n}}{I_{2n-1}} < 1$$

for $n = 1,2,\ldots$. If we take into consideration that

$$I_{2n-1} = \frac{2.4\ \ldots\ (2n-2)}{3.5\ \ldots\ (2n-1)} = \frac{\pi}{4n\ I_{2n}} = \frac{1}{2n\ Q_{2n}},$$

then it follows that

$$\frac{2n}{2n+1} < n\pi\ Q_{2n}^2 < 1$$

which implies the inequalities to be proved.

From the last inequalities it follows that

$$\frac{4}{\pi} = \lim_{n\to\infty} 4n\ Q_{2n}^2 = \lim_{n\to\infty}\frac{3^2.5^2\ \ldots\ (2n-3)^2(2n-1)^2}{2.4^2.6^2\ldots\ (2n-2)^2 2n} = \frac{3.3.5.5.7.7\ \ldots}{2.4.4.6.6.8\ \ldots}$$

This product representation of $4/\pi$ was found in 1665 by J. Wallis [66].

40.4. We have

$$(x-\omega_1)(x-\omega_2)\ldots(x-\omega_n) = a_0x^n - a_1x^{n-1} + \ldots + (-1)^n a_n$$

or

$$(1-x\omega_1)(1-x\omega_2) \ldots (1-x\omega_n) = a_0 - a_1 x + \ldots + (-1)^n a_n x^n .$$

If $|x|$ is sufficiently small, then we can write that

$$a_0 - a_1 x + \ldots + (-1)^n a_n x^n = e^{\sum_{i=1}^{n} \log(1-x\omega_i)} = e^{-s_1 x - \frac{s_2}{2} x^2 - \frac{s_3}{3} x^3 - \ldots} .$$

Hence it follows easily the relation to be proved.

We note that by the relation

$$s_1 x + \frac{s_2}{2} x^2 + \frac{s_3}{3} x^3 + \ldots = -\log(a_0 - a_1 x + \ldots + (-1)^n a_n x^n) =$$

$$= \sum_{r=1}^{\infty} \frac{(a_1 x - a_2 x^2 + \ldots + (-1)^{n-1} a_n x^n)^r}{r}$$

we can also express s_k with the aid of $a_1, a_2, \ldots, a_n$.

40.5. Denote by $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ the variables $\xi_1, \xi_2, \ldots, \xi_m$ arranged in increasing order of magnitude. In general, we have

$$\delta_m^+ = \sup_{-\infty<x<\infty} [F_m(x) - F(x)] = \max_{1 \leqq r \leqq m} [F_m(\xi_r^*) - F(\xi_r^*)]$$

and

$$\delta_m^- = \sup_{-\infty<x<\infty} [F(x) - F_m(x)] = \max_{1 \leqq r \leqq m} [F(\xi_r^*) - F_m(\xi_r^* - 0)] .$$

If $F(x)$ is a continuous distribution function, then in finding the distributions of δ_m^+ and δ_m^- we may assume without loss of generality that $F(x) = x$ for $0 \leqq x \leqq 1$. Then $F(\xi_r^*) = \xi_r^*$ and $F_m(\xi_r^*) = \frac{r}{m}$ with probability 1. In this case

$$\delta_m^+ = \max_{1 \leq r \leq m} [\frac{r}{m} - \xi_r^*] \quad \text{and} \quad \delta_m^- = \max_{1 \leq r \leq m} [\xi_r^* - \frac{r-1}{m}] \ .$$

If in δ_m^+ we replace ξ_r^* by $1 - \xi_{m+1-r}^*$ for $r = 1,2,\ldots, m$, then we obtain a new random variable which has exactly the same distribution as δ_m^+. This new random variable

$$\max_{1 \leq r \leq m} [\xi_{m+1-r}^* - \frac{m-r}{m}] = \max_{1 \leq i \leq m} [\xi_i^* - \frac{i-1}{m}] \ ,$$

is evidently δ_m^- .

40.6. Denote by $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ the random variables $\xi_1, \xi_2, \ldots, \xi_m$ arranged in increasing order of magnitude and by $\eta_1^*, \eta_2^*, \ldots, \eta_n^*$ the random variables $\eta_1, \eta_2, \ldots, \eta_n$ arranged in increasing order of magnitude . In general we have

$$\delta_{m,n}^+ = \sup_{-\infty<x<\infty} [F_m(x) - G_n(x)] = \max_{1 \leq r \leq n} [F_m(\eta_r^*) - G_n(\eta_r^* - 0)]$$

$$= \max_{1 \leq r \leq m} [F_m(\xi_r^*) - G_n(\xi_r^*)]$$

and

$$\delta_{m,n}^- = \sup_{-\infty<x<\infty} [G_n(x) - F_m(x)] = \max_{1 \leq r \leq n} [G_n(\eta_r^*) - F_m(\eta_r^*)] =$$

$$= \max_{1 \leq r \leq m} [G_n(\xi_r^*) - F_m(\xi_r^* - 0)] \ .$$

Let us define ν_r $(r = 1,2,\ldots, n+1)$ as the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ falling in the interval $(\eta_{r-1}^*, \eta_r^*]$ where $\eta_0^* = -\infty$ and $\eta_{n+1}^* = \infty$. Let $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1,2,\ldots, n+1$. Clearly,

$N_{n+1} = m$. Then $F_m(\eta_r^*) = N_r/m$, $G_n(\eta_r^*) = r/n$ and $G_n(\eta_r^* - 0) = (r-1)/n$ with probability 1 , and we can write that

$$\delta_{m,n}^{+} = \max_{1 \leq r \leq n} \left[\frac{N_r}{m} - \frac{r-1}{n}\right] \text{ and } \delta_{m,n}^{-} = \max_{1 \leq r \leq n} \left[\frac{r}{n} - \frac{N_r}{m}\right] .$$

If $F(x) \equiv G(x)$ is a continuous distribution function, then $\nu_1, \nu_2, \ldots, \nu_{n+1}$ are interchangeable random variables. If in $\delta_{m,n}^{+}$ we replace ν_r by ν_{n+2-r} for $r = 1,2,\ldots, n$, then we obtain a new random variable which has exactly the same distribution as $\delta_{m,n}^{+}$. This new random variable,

$$\max_{1 \leq r \leq n} \left[\frac{n-r+1}{n} - \frac{N_{n+1-r}}{m}\right] = \max_{1 \leq i \leq n} \left[\frac{i}{n} - \frac{N_i}{m}\right] ,$$

is evidently $\delta_{m,n}^{-}$.

40.7. The random variables $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ are the coordinates arranged in increasing order of m points distributed uniformly and independently on the interval $(0,1)$. The random variables $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ have a joint density function $f(x_1, x_2, \ldots, x_m) = 1/m!$ for $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_m \leqq 1$ and $f(x_1, x_2, \ldots, x_m) = 0$ otherwise. We have

$$P\{\xi_j^* \leqq x\} = \frac{m!}{(j-1)!(m-j)!} \int_0^x u^{j-1}(1-u)^{m-j} du = \sum_{k=j}^{m} \binom{m}{k} x^k (1-x)^{m-k}$$

for $0 \leqq x \leqq 1$ and $j = 1,2,\ldots, m$, and

$$E\{(\xi_j^*)^r\} = \frac{j(j+1)\ldots(j+r-1)}{(m+1)(m+2)\ldots(m+r)}$$

for $r = 1,2,\ldots$. Hence $E\{\xi_j^*\} = j/(m+1)$ and $Var\{\xi_j^*\} = j(m+1-j)/(m+1)^2(m+2)$.

Furthermore, we have $\underset{\sim}{\mathrm{Cov}}\{\xi_i^*, \xi_j^*\} = i(m+1-j)/(m+1)^2(m+2)$ for $1 \leq i \leq j \leq n$. This last result can easily be proved if we take into consideration that $\xi_1^*, \xi_2^* - \xi_1^*, \ldots, \xi_m^* - \xi_{m-1}^*, 1-\xi_m^*$ are interchangeable random variables with sum 1. For by this property

$$\underset{\sim}{\mathrm{Cov}}\{\xi_i^*, \xi_j^*\} = \frac{i(m+1-j)}{m} \underset{\sim}{\mathrm{Var}}\{\xi_1^*\}$$

if $1 \leq i \leq j \leq m$.

40.8. The random variables $N_1, N_2, \ldots, N_n$ can be interpreted in the following way. We arrange m white balls and n black balls in a row in such a way that all the $\binom{m+n}{m}$ possible arrangements are equally probable. Denote by N_i $(i = 1,2,\ldots, n)$ the number of white balls preceding the i-th black ball. We have

$$\underset{\sim}{P}\{N_i = j_i \text{ for } i = 1,2,\ldots, n\} = \frac{1}{\binom{m+n}{m}}$$

for $0 \leq j_1 \leq j_2 \leq \cdots \leq j_n \leq m$. Hence it follows that

$$\underset{\sim}{P}\{N_i = s\} = \frac{\binom{i+s-1}{s}\binom{m+n-i-s}{m-s}}{\binom{m+n}{n}}$$

for $0 \leq s \leq m$ and $1 \leq i \leq n$, and

$$\underset{\sim}{E}\{\binom{N_i}{r}\} = \frac{\binom{i+r-1}{r}\binom{m+n}{n+r}}{\binom{m+n}{m}}$$

for $1 \leq r \leq m$. In particular, we have $\underset{\sim}{E}\{N_i\} = im/(n+1)$ and

$$\underset{\sim\sim}{\text{Var}}\{N_i\} = \frac{i(n+1-i)m(m+n+1)}{(n+1)^2(n+2)}$$

for $1 \leq i \leq n$. Furthermore, we have

$$\underset{\sim\sim}{\text{Cov}}\{N_i, N_j\} = \frac{i(n+1-j)m(m+n+1)}{(n+1)^2(n+2)}$$

for $1 \leq i \leq j \leq n$. This last result can easily be proved if we take into consideration that $N_1, N_2 - N_1, \ldots, N_n - N_{n-1}, m-N_n$ are interchangeable random variables with sum m . For this property implies that

$$\underset{\sim\sim}{\text{Cov}}\{N_i, N_j\} = \frac{i(n+1-j)}{n} \underset{\sim\sim}{\text{Var}}\{N_1\}$$

for $1 \leq i \leq j \leq n$.

40.9. In finding the joint distribution of δ_m^+ and δ_m^- we may assume without loss of generality that $F(x) = x$ for $0 \leq x \leq 1$. Then by the solution of Problem 40.5 we have

$$\delta_m^+ = \max_{1 \leq r \leq m} \left[\frac{r}{m} - \xi_r^*\right] \quad \text{and} \quad \delta_m^- = \max_{1 \leq r \leq m} \left[\xi_r^* - \frac{r-1}{m}\right]$$

with probability 1 and consequently

$$\underset{\sim}{P}\{\delta_m^+ \leq x,\ \delta_m^- \leq y\} = \underset{\sim}{P}\left\{\frac{r}{m} - x \leq \xi_r^* \leq \frac{r-1}{m} + y \quad \text{for } r = 1,2,\ldots, m\right\} .$$

Let

$$a_r = \max\left(0, \frac{r}{m} - x\right) \quad \text{and} \quad b_r = \min\left(\frac{r-1}{m} + y, 1\right)$$

for $r = 1,2,\ldots, m$. Then $0 \leq a_1 \leq \ldots \leq a_m \leq 1$, $0 \leq b_1 \leq \ldots \leq b_m \leq 1$

and $a_r \leqq b_r$ if $x+y \geqq \frac{1}{m}$, $x \geqq 0$ and $y \geqq 0$.

In general, if $0 \leqq a_1 \leqq \dots \leqq a_m \leqq 1$, $0 \leqq b_1 \leqq \dots \leqq b_m \leqq 1$ and $a_r \leqq b_r$ for $r = 1,2,\dots, m$, then we have

$$\underset{\sim}{P}\{a_r \leqq \xi_r^* \leqq b_r \text{ for } r = 1,2,\dots, m\} =$$

$$= m! \sum_{\nu=0}^{m-1} (-1)^{m-\nu-1} \sum_{\substack{0=k_0<k_1<\dots<k_{\nu+1}=m \\ a_{k_{i+1}} \leqq b_{k_i+1} (i=1,\dots,\nu)}} \prod_{i=0}^{\nu} \frac{(b_{k_i+1} - a_{k_{i+1}})^{k_{i+1}-k_i}}{(k_{i+1} - k_i)!} =$$

$$= m! \quad \mathrm{Det}\left|\delta(i,j)\right|_{\substack{1\leqq i\leqq m \\ 1\leqq j\leqq m}}$$

where

$$\delta(i,j) = \begin{cases} \dfrac{([b_i - a_j]^+)^{j-i+1}}{(j-i+1)!} & \text{if } i \leqq j+1, \\ 0 & \text{if } i > j+1. \end{cases}$$

For we have

$\underset{\sim}{P}\{a_r \leqq \xi_r^* \leqq b_r$ for $r = 1,\dots, m\} = m!\underset{\sim}{P}\{a_r \leqq \xi_r \leqq b_r$ for $r = 1,\dots, m$ and $\xi_1 \leqq \xi_2 \leqq \dots \leqq \xi_m\} = m!\underset{\sim}{P}\{a_r \leqq \xi_r \leqq b_r$ for $r = 1,\dots, m$ and none of the events $\xi_i > \xi_{i+1}$ $(i = 1,\dots, m-1)$ occurs$\}$ $= m! \sum_{\nu=0}^{m-1} (-1)^{m-1-\nu}$ $\sum_{0=k_0<k_1<\dots<k_{\nu+1}=m} \underset{\sim}{P}\{a_r \leqq \xi_r \leqq b_r$ for $r = 1,\dots, m$ and $\xi_r > \xi_{r+1}$ for $r \neq k_1,\dots, k_\nu\}$,

and here

$$\underset{\sim}{P}\{a_r \leq \xi_r \leq b_r \text{ for } r = 1,\ldots,m \text{ and } \xi_r > \xi_{r+1} \text{ for } r \neq k_1,\ldots,k_\nu\} =$$

$$= \underset{\sim}{P}\{a_r \leq \xi_r \leq b_r \text{ for } k_i < r \leq k_{i+1},\ \xi_r > \xi_{r+1} \text{ for } k_i < r < k_{i+1} \text{ and } 0 \leq i \leq \nu\}$$

$$= \underset{\sim}{P}\{a_{k_{i+1}} \leq \xi_{k_{i+1}} \leq \cdots \leq \xi_{k_i+1} \leq b_{k_i+1} \text{ for } 0 \leq i \leq \nu\} =$$

$$= \prod_{i=0}^{\nu} \frac{([b_{k_i+1} - a_{k_{i+1}}]^+)^{k_{i+1}-k_i}}{(k_{i+1} - k_i)!} .$$

40.10. By using the same notation as in the solution of Problem 40.6 we can write that

$$\delta^+_{m,n} = \max_{1 \leq r \leq n} \left[\frac{N_r}{m} - \frac{r-1}{n}\right] \text{ and } \delta^-_{m,n} = \max_{1 \leq r \leq n} \left[\frac{r}{n} - \frac{N_r}{m}\right]$$

with probability 1 . Thus we have

$$\underset{\sim}{P}\{\delta^+_{m,n} \leq x,\ \delta^-_{m,n} \leq y\} = \underset{\sim}{P}\{\frac{mr}{n} - my \leq N_r \leq \frac{m(r-1)}{n} + mx \text{ for } 1 \leq r \leq n\}$$

$$= \underset{\sim}{P}\{a_r \leq N_r \leq b_r \text{ for } 1 \leq r \leq n\}$$

where a_r is the smallest integer $\geq \max(0, \frac{mr}{n} - my)$ and b_r is the largest integer $\leq \min(m, \frac{m}{n}(r-1) + mx)$. We have $0 \leq a_1 \leq \cdots \leq a_n \leq m$ and $0 \leq b_1 \leq \cdots \leq b_n \leq m$, and $a_r \leq b_r$ $(r = 1,\ldots, n)$ whenever $x+y \geq \frac{1}{n}$, $x \geq 0$, $y \geq 0$. For any such $\{a_r\}$ and $\{b_r\}$ we have

$$\underset{\sim}{P}\{a_r \leq N_r \leq b_r \text{ for } 1 \leq r \leq n\} =$$

$$= \frac{1}{\binom{m+n}{m}} \sum_{\nu=0}^{n-1} (-1)^{n-\nu-1} \sum_{\substack{0=j_0<j_1<\dots<j_{\nu+1}=n \\ a_{j_{i+1}} \le b_{j_i}+1 \ (i=1,\dots,\nu)}} \prod_{i=0}^{\nu} \binom{b_{j_i+1} - a_{j_{i+1}} + 1}{j_{i+1} - j_i} =$$

$$= \frac{1}{\binom{m+n}{m}} \operatorname{Det} \left| d(i,j) \right|_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

where

$$d(i,j) = \begin{cases} \binom{[b_i - a_j + 1]^+}{j-i+1} & \text{if } i \le j+1 , \\ 0 & \text{if } i > j+1 . \end{cases}$$

If we take into consideration that

$$P\{N_i = j_i \text{ for } i = 1,2,\dots, n\} = \frac{1}{\binom{m+n}{m}}$$

for nonnegative integers $0 \le j_1 \le j_2 \le \dots \le j_n \le m$, then the above result can be proved in a similar way as the corresponding result in Problem 40.9.

40.11. Let us define the random variables $N_1, N_2, \dots, N_n$ in the same way as in the solution of Problem 40.6. Define

$$\eta_{m,n}(u) = \sqrt{\frac{mn}{m+n}} \left[\frac{N_{[nu]}}{m} - \frac{[nu]}{n} \right]$$

for $0 \le u \le 1$ and $m \ge 1$, $n \ge 1$. It is sufficient to prove that if $m \to \infty$ and $n \to \infty$, then the finite dimensional distribution functions of the process $\{\eta_{m,n}(u), 0 < u < 1\}$ converge to the finite dimensional distribution functions of the Gaussian process $\{\eta(u) , 0 < u < 1\}$ for which $E\{\eta(u)\} = 0$ and $\operatorname{Cov}\{\eta(u) , \eta(v)\} = u(1-v)$ for $0 < u \le v < 1$. Then (39.79) follows by a theorem of M. D. Donsker [245].

Now

$$E\{\eta_{m,n}(u)\} = \sqrt{\frac{mn}{m+n}}\left[\frac{[nu]}{n+1} - \frac{[nu]}{n}\right] \to 0$$

as $m \to \infty$ and $n \to \infty$, and if $0 < u \leq v < 1$, then

$$\text{Cov}\{\eta_{m,n}(u), \eta_{m,n}(v)\} = \frac{n(n+m-1)[nu](n+1-[nv])}{(m+n)(n+1)^2(n+2)} \to u(1-v)$$

as $m \to \infty$ and $n \to \infty$. Hence we can easily conclude that if $m \to \infty$ and $n \to \infty$, then the joint distribution function of the random variables $\eta_{m,n}(t_1)$, $\eta_{m,n}(t_2), \dots, \eta_{m,n}(t_k)$ where $0 < t_1 < t_2 < \dots < t_k < 1$ converges to a k-dimensional normal distribution function of type (39.21). This completes the proof of the statement.

40.12. First, we shall prove that

$$\lim_{\substack{m \to \infty \\ n \to \infty}} P\{\sqrt{\frac{mn}{m+n}}\,\delta^+_{m,n}(0,\alpha) \leq x\} = P\{\sup_{0 \leq t \leq \alpha} \eta(t) \leq x\} =$$

$$= \int_{-\infty}^{x} (1 - e^{-2x(x-u)/\alpha})\,dP\{\eta(\alpha) \leq u\}$$

for $x \geq 0$ where $\{\eta(t), 0 \leq t \leq 1\}$ is a separable Gaussian process for which $E\{\eta(t)\} = 0$ if $0 \leq t \leq 1$ and $E\{\eta(s)\eta(t)\} = s(1-t)$ for $0 \leq s \leq t \leq 1$. The first equality follows from a theorem of M. D. Donsker [245]. To prove the second equality let us calculate the limit in the particular case when $m = n$ and $n \to \infty$. By using the same notation as in the solution of Problem 40.6, the above limit can be expressed as $\lim_{n \to \infty} P\{N_r < r + a \text{ for } 1 \leq r \leq j\}$ where $a = [x\sqrt{2n}\,]$ and $j = [n\alpha]$. Since in this case

$$P\{N_r < r+a \quad \text{for} \quad 1 \leqq r \leqq j \mid N_j = j+s\} = 1 - \frac{\binom{2j+s-1}{a+j}}{\binom{2j+s-1}{j-1}}$$

for $0 \leqq j + s < j + a$,

$$\lim_{j \to \infty} P\{ \frac{N_j - j}{\sqrt{2n}} \leqq u\} = P\{\eta(\alpha) \leqq u\}$$

and

$$\lim_{n \to \infty} \frac{\binom{2j+s-1}{a+j}}{\binom{2j+s-1}{j-1}} = e^{-2x(x-u)/\alpha}$$

whenever $j = [n\alpha]$, $a = [x\sqrt{2n}]$ and $s = [u\sqrt{2n}]$, the aforementioned limit theorem follows easily.

By the repeated application of the above limit theorem we can easily prove that

$$\lim_{\substack{m \to \infty \\ n \to \infty}} P\{\sqrt{\frac{mn}{m+n}}\, \delta^+_{m,n}(0,\alpha) \leqq x , \sqrt{\frac{mn}{m+n}}\, \delta^+_{m,n}(\alpha,\beta) \leqq y , \sqrt{\frac{mn}{m+n}}\, \delta^+_{m,n}(\beta,1) \leqq z\} =$$

$$= P\{ \sup_{0 \leqq t \leqq \alpha} \eta(t) \leqq x , \sup_{\alpha \leqq t \leqq \beta} \eta(t) \leqq y , \sup_{\beta \leqq t \leqq 1} \eta(t) \leqq z\} =$$

$$= \iint_{\substack{-\infty < u \leqq \min(x,y) \\ -\infty < v \leqq \min(y,z)}} (1-e^{-2x(x-u)/\alpha})(1-e^{-2(y-u)(y-v)/(\beta-\alpha)})(1-e^{-2z(z-v)/(1-\beta)}) \cdot$$

$$\cdot\ d_u\, d_v\, P\{\eta(\alpha) \leqq u , \eta(\beta) \leqq v\}$$

for $0 < \alpha < \beta < 1$ and $x \geqq 0$, $y \geqq 0$.

40.13. By the solution of Problem 40.12 we have

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \underset{\sim}{P}\{\delta^+_{m,n}(\alpha,\beta) \leq 0\} = \int_{-\infty}^{0}\int_{-\infty}^{0} (1-e^{-2uv/(\beta-\alpha)})d_u d_v \underset{\sim}{P}\{\eta(\alpha) \leq u \ , \ \eta(\beta) \leq v\} =$$

$$= \frac{1}{\pi} \text{arc sin} \sqrt{\frac{\alpha(1-\beta)}{\beta(1-\alpha)}} \ .$$

40.14. We have

$$\lim_{m\to\infty} \underset{\sim}{P}\{\sqrt{m}\ \delta^+_m\ (\alpha,\beta) \leq y\} = \underset{\sim}{P}\{ \sup_{\alpha\leq t\leq \beta} \eta(t) \leq y\}$$

where $\{\eta(t) \ , \ 0 \leq t \leq 1\}$ is a separable Gaussian process for which $\underset{\sim}{E}\{\eta(t)\} = 0$ if $0 \leq t \leq 1$ and $\underset{\sim}{E}\{\eta(s)\eta(t)\} = s(1-t)$ if $0 \leq s \leq t \leq 1$. This probability can be obtained by the solution of Problem 40.12 .

For the case of $\beta = 1$ we deduce another formula. By (39.119) we have

$$\underset{\sim}{P}\{\delta^+_m(\alpha,1) > x\} = \sum_{m(x+\alpha)\leq j\leq m} \frac{mx}{mx+m-j} \underset{\sim}{P}\{\chi_m\ (\frac{j-mx}{m}) = \frac{j}{m}\}$$

for $x > 0$ where

$$\underset{\sim}{P}\{\chi_m(u) = \frac{j}{m}\} = \binom{m}{j} u^j (1-u)^{m-j}$$

for $0 \leq j \leq m$ and $0 \leq u \leq 1$. If we put $x = y/\sqrt{m}$ and $j = mu$ in the above formula and let $m \to \infty$, then we obtain that

$$\lim_{m\to\infty} \underset{\sim}{P}\{\sqrt{m}\ \delta^+_m(\alpha,1) > y\} = \frac{y}{\sqrt{2\pi}} \int_{\alpha}^{1} \frac{e^{-\frac{y^2}{2u(1-u)}}}{u^{1/2}(1-u)^{3/2}} du$$

for $y > 0$.

40.15. By (39.123) we have

$$\mathbf{P}\{\mu_m^+(\alpha,1) > x\} = \sum_{m(x+1)\alpha \leq j \leq m} \frac{mx}{mx+m-j} \mathbf{P}\{\chi_m(\frac{j}{m(x+1)}) = \frac{j}{m}\}$$

for $x > 0$ where $m\chi_m(u)$ has a Bernoulli distribution with parameters m and u. If we put $x = y/\sqrt{m}$ and $j = mu$ in the above formula and let $m \to \infty$, then we obtain that

$$\lim_{m \to \infty} \mathbf{P}\{\sqrt{m}\, \mu_m^+(\alpha,1) > y\} = \frac{y}{\sqrt{2\pi}} \int_\alpha^1 \frac{e^{-\frac{u^2y^2}{2u(1-u)}}}{u^{1/2}(1-u)^{3/2}} du =$$

$$= 2[1 - \Phi(y\sqrt{\frac{\alpha}{1-\alpha}})]$$

for $y > 0$ where $\Phi(x)$ is the normal distriburion function.

46.1. First, we shall prove that $\psi(0) = 1$. Since $\psi(0) > 0$, this follows from

$$[\psi(0)]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2/2} r\,dr = 1 .$$

Here in the second integral we made the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

We can write that

$$\psi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-sx-\frac{x^2}{2}}\,dx = \frac{e^{s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+s)^2/2}dx = \frac{e^{s^2/2}}{\sqrt{2\pi}}\int_{L_s} e^{-z^2/2}dz$$

where $L_s = \{z : z = x+s \text{ and } -\infty < x < \infty\}$. If we integrate $e^{-z^2/2}$ along the rectangle $(R,0)$, $(R, i\mathrm{Im}(s))$, $(-R, i\mathrm{Im}(s))$, $(-R,0)$ and let $R \to \infty$, then by Cauchy's formula it follows that the integral in the last formula does not depend on s . Thus $\psi(s) = e^{s^2/2}\psi(0) = e^{s^2/2}$ for any s .

46.2. In this case

$$\psi(u) = \int_{-\infty}^{\infty} e^{iux} f(x)dx = e^{-|u|^{1/2}(1-i\beta\frac{u}{|u|})}$$

for $-\infty < u < \infty$ and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\psi(u)du = \mathrm{Re}\{\frac{1}{\pi}\int_0^{\infty} e^{-iux}\psi(u)du\} .$$

By the substitution $v = z - (1-i)\sqrt{ux/2}$ where $z = [(1+\beta)+i(1-\beta)]/\sqrt{8x}$ we obtain that

$$\int_0^{\infty} e^{-iux}\psi(u)du = \frac{1}{ix} + \frac{ze^{-z^2}}{x}(\sqrt{\pi} + 2i\int_0^z e^{v^2}dv)$$

for $x > 0$. Thus

$$f(x) = \mathrm{Re}\{ \frac{z}{\pi x} [\sqrt{\pi}\, e^{-z^2} + 2iw(z)]\}$$

for $x > 0$ where

$$w(z) = e^{-z^2} \int_0^z e^{v^2} dv$$

and $z = [(1+\beta) + i(1-\beta)]/\sqrt{8x}$.

46.3 By integrating by parts we obtain that

$$\int_{-a}^{a} |x|^{\delta} dF(x) = \delta \int_0^a x^{\delta-1}[F(a)-F(x)+F(-x)-F(-a)]dx =$$

$$= \delta \int_0^a x^{\delta-1} [1-F(x)+F(-x)]dx - a^{\delta}[1-F(a)+F(-a)]$$

for $a \geq 0$. If $\int_{-\infty}^{\infty} |x|^{\delta} dF(x) < \infty$, then

$$0 \leq a^{\delta}[1-F(a)+F(-a)] \leq \int_{|x| \geq a} |x|^{\delta} dF(x) \to 0 \quad \text{as} \quad a \to \infty ,$$

and thus the statement is true. If $\int_{-\infty}^{\infty} |x|^{\delta} dF(x) = \infty$,

$$\int_0^{\infty} x^{\delta-1} [1-F(x) + F(-x)]dx = \infty$$

necessarily holds.

46.4. Let us consider the complex plane cut along the positive real axis and define a path of integration C as follows: We integrate along a straight line from $z = i\varepsilon$ to $z = R + i\varepsilon$ where $0 < \varepsilon < 1$ and $R > 1$, then from $z = R + i\varepsilon$ to $z = R - i\varepsilon$ along the circle $z^2 = R^2 + \varepsilon^2$ in the positive direction, then along a straight line from $z = R - i\varepsilon$ to $z = -i\varepsilon$, and finally from $z = -i\varepsilon$ to $z = i\varepsilon$ along the circle $|z| = \varepsilon$ in the negative direction. If we interpret $z^\delta = e^{\delta \log z}$ where $\log z = \log|z| + i\arg z$ and $0 \leq \arg z \leq 2\pi$, then $z^\delta/(1+z^2)$ is a one-valued function in the region bounded by C and regular except at the poles $z = i$ and $z = -i$. By the theorem of residues we obtain that

$$\frac{2}{\pi} \int_C \frac{z^\delta}{1+z^2}\, dz = 2[e^{\frac{i\delta\pi}{2}} - e^{\frac{3i\delta\pi}{2}}] = -4ie^{i\delta\pi} \sin\frac{\delta\pi}{2} .$$

If $\varepsilon \to 0$ and $R \to \infty$, then the integral on the left-hand side tends to

$$\frac{2}{\pi}(1 - e^{2i\delta\pi}) \int_0^\infty \frac{x^\delta}{1+x^2}\, dx = -4ie^{i\delta\pi} \sin\frac{\delta\pi}{2} .$$

Hence it follows that

$$E\{|\xi|^\delta\} = \frac{2}{\pi} \int_0^\infty \frac{x^\delta}{1+x^2}\, dx = \frac{1}{\cos\frac{\delta\pi}{2}}$$

for $-1 < \delta < 1$. See D. Bierens de Haan [11 p.42 and p. 50].

46.5 By using Cauchy's integral theorem we can express $I_\alpha(s)$ by known real integrals which can be found for example in the book of D. Bierens de Haan [11].

First, let us suppose that $0 < \alpha < 1$. If we use the solution of Problem 46.4 and if we introduce a new variable $z = sx$ in the integral, then we obtain that

$$I_\alpha(s) = s^\alpha J_\alpha(s) + \frac{s\alpha\pi}{2\cos\frac{\alpha\pi}{2}}$$

where

$$J_\alpha(s) = \int_{L_s} (e^{-z} - 1) \frac{\alpha dz}{z^{\alpha+1}}$$

and $L_s = \{z : z = sx , 0 \leq x < \infty\}$. The integrand in $J_\alpha(s)$ is a regular function of z in the region bounded by the lines L_1 and L_s and the arcs $|z| = \varepsilon$ and $|z| = R$ where $0 < \varepsilon < R$. If we integrate along the boundary of this region and let $\varepsilon \to 0$ and $R \to \infty$, then we obtain that

$$J_\alpha(s) = J_\alpha(1) = \int_0^\infty (e^{-x}-1) \frac{\alpha dx}{x^{\alpha+1}} = -\Gamma(1-\alpha)$$

where $\Gamma(1-\alpha)$ is the gamma function. See [11 p. 132].

Thus

$$I_\alpha(s) = -\Gamma(1-\alpha)s^\alpha + \frac{s\alpha\pi}{2\cos\frac{\alpha\pi}{2}}$$

for $0 < \alpha < 1$ if $\mathrm{Re}(s) \geq 0$.

Now we shall prove that

$$I_1(s) = s \log s - s(1-C)$$

if $\mathrm{Re}(s) > 0$ where $C = 0.5772157...$ is Euler's constant. By [11 p. 135] we have $I_1(1) = C-1$. If we introduce a new variable $z = sx$ in $I_1(s)$, then we can write that

$$I_1(s) = s \int_{L_s} (e^{-z}-1+ \frac{z}{1+z^2})dz + s \int_0^\infty [\frac{1}{1+x^2} - \frac{1}{1+s^2x^2}] \frac{dx}{x}$$

where $L_s = \{z : z = sx , 0 \leq x < \infty\}$. By using Cauchy's integral theorem we can prove that the first integral on the right-hand side of the above equation does not depend on s . Thus the first term becomes $sI_1(1) = s(C-1)$. The integral in the second term on the right-hand side of the above formula can be calculated as follows:

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \left[\frac{1}{1+x^2} - \frac{1}{1+s^2x^2}\right] \frac{dx}{x} = \lim_{\varepsilon \to 0} \left[\int_{\varepsilon}^{\infty} \frac{dx}{x(1+x^2)} - \int_{L_s(\varepsilon)} \frac{dz}{z(1+z^2)}\right]$$

where $L_s(\varepsilon) = \{z : z = sx , \varepsilon \leq x < \infty\}$. Since the function $1/z(1+z^2)$ is regular in the domain $\mathrm{Re}(z) > 0$, the last term can be expressed as

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon s} \frac{dz}{z(1+z^2)} = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon s} \frac{dz}{z} = \log s .$$

This proves the formula for $I_1(s)$ in the case when $\mathrm{Re}(s) > 0$. If $\mathrm{Re}(s) \geq 0$, then $I_1(s)$ can be obtained by continuity.

If $s = -i\omega$ where ω is real, we have

$$I_1(-i\omega) = -i\omega\log \omega - \frac{\omega\pi}{2} + i\omega(1-C)$$

for $\omega > 0$, $I_1(0) = 0$ and $I_1(i\omega) = \overline{I_1(-i\omega)}$. This can be proved directly as follows. If $\omega > 0$, then

$$I_1(-i\omega) = \int_0^{\infty} (e^{i\omega x}-1 - \frac{i\omega x}{1+x^2}) \frac{dx}{x} = \int_0^{\infty} \frac{\cos\omega x-1}{x^2} dx +$$

$$+ i \int_0^{\infty} (\sin\omega x - \frac{\omega x}{1+\omega^2x^2}) \frac{dx}{x^2} - i\omega \int_0^{\infty} (\frac{x}{1+x^2} - \frac{x}{1+\omega^2x^2}) \frac{dx}{x^2} =$$

$$= \omega \int_0^{\infty} \frac{\cos u-1}{u^2} du + i\omega \int_0^{\infty} (\frac{\sin u}{u} - \frac{1}{1+u^2}) du - i\omega \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon\omega} \frac{du}{u(1+u^2)} =$$

$$= -\frac{\omega\pi}{2} + i\omega(1-C) - i\omega\log \omega .$$

For by [11 p. 220]

$$\int_0^\infty \frac{1-\cos u}{u^2}\, du = \frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\sin u - u\cos u}{u^2}\, du = 1$$

and by [11 p. 256]

$$\int_0^\infty \left(\frac{1}{1+u^2} - \cos u\right) \frac{du}{u} = C .$$

Finally let us suppose that $1 < \alpha < 2$. Then we can write that

$$I_\alpha(s) = \int_0^\infty (e^{-sx}-1+sx)\, \frac{\alpha dx}{x^{\alpha+1}} + s\alpha \int_0^\infty \frac{x^{1-\alpha}}{1+x^2}\, dx .$$

Thus by the solution of Problem 46.4 we have

$$I_\alpha(s) = s^\alpha J_\alpha(s) + \frac{s\alpha\pi}{2\cos\frac{\alpha\pi}{2}}$$

where

$$J_\alpha(s) = \int_{L_s} (e^{-z}-1+z)\, \frac{\alpha dz}{z^{\alpha+1}}$$

and $L_s = \{z : z = sx , 0 \leq x < \infty\}$. By using Cauchy's theorem we can prove that $J_\alpha(s)$ does not depend on s , and thus by [11 p. 132] we have

$$J_\alpha(s) = J_\alpha(1) = \int_0^\infty (e^{-x}-1+x)\, \frac{\alpha dx}{x^{\alpha+1}} = -\Gamma(1-\alpha) = \frac{\pi}{\Gamma(\alpha)\sin \alpha\pi} .$$

Finally,

$$I_\alpha(s) = -\Gamma(1-\alpha)s^\alpha + \frac{s\alpha\pi}{2\cos\frac{\alpha\pi}{2}} = \frac{-s^\alpha\pi}{\Gamma(\alpha)\sin\alpha\pi} + \frac{s\alpha\pi}{2\cos\frac{\alpha\pi}{2}}$$

for $1 < \alpha < 2$ and $\mathrm{Re}(s) \geq 0$.

46.6 Let

$$\gamma = \frac{2}{\pi} \arctan\left(\beta \tan \frac{\alpha\pi}{2}\right)$$

where $-1 < \gamma < 1$. By (42.128) and (42.130) we have

$$E\{|\xi|^{\delta}\} = \int_{-\infty}^{\infty} |x|^{\delta} f(x;\alpha,\beta,c,0)dx = \left(\frac{c}{\cos\frac{\gamma\pi}{2}}\right)^{\frac{\delta}{\alpha}} \int_{-\infty}^{\infty} |x|^{\delta} h(x;\alpha,\gamma)dx =$$

$$= \left(\frac{c}{\cos\frac{\gamma\pi}{2}}\right)^{\frac{\delta}{\alpha}} \left[\int_0^{\infty} x^{\delta} h(x;\alpha,\gamma)dx + \int_0^{\infty} x^{\delta} h(x;\alpha,-\gamma)dx\right] .$$

The case $\delta = 0$ is obvious. Let $\delta \neq 0$ and $-1 < \delta < \alpha$. By (42.131) we have

$$\int_0^{\infty} x^{\delta} h(x;\alpha,\gamma)dx = \frac{1}{\pi} \int_0^{\infty} x^{\delta} \mathrm{Re}\{\int_0^{\infty} e^{-ixu-u^{\alpha}e^{-\frac{\gamma\pi i}{2}}} du\}dx =$$

$$= \frac{\Gamma(1+\delta)}{\pi} \mathrm{Re}\{e^{-\frac{(1+\delta)\pi i}{2}} \int_0^{\infty} e^{-u^{\alpha}e^{-\frac{\gamma\pi i}{2}}} u^{-\delta-1} du\} =$$

$$= \frac{\Gamma(1+\delta)}{\pi\alpha} \mathrm{Re}\{e^{-\frac{(1+\delta)\pi i}{2} - \frac{\delta\gamma\pi i}{2\alpha}} \int_L e^{-z} z^{-\frac{\delta}{\alpha}-1} dz\} =$$

$$= \frac{\Gamma(1+\delta)\Gamma(-\frac{\delta}{\alpha})}{\pi\alpha} \mathrm{Re}\{e^{-\frac{(1+\delta)\pi i}{2} - \frac{\delta\gamma\pi i}{2}}\} =$$

$$= -\frac{\Gamma(1+\delta)\Gamma(1-\frac{\delta}{\alpha})}{\pi\delta} \cos\left(\frac{(1+\delta)\pi}{2} + \frac{\delta\gamma\pi}{2}\right) = \frac{\Gamma(1+\delta)\Gamma(1-\frac{\delta}{\alpha})}{\pi\delta} \sin\left(1+\frac{\gamma}{\alpha}\right)\frac{\delta\pi}{2}$$

where $L = \{z : z = e^{-\frac{\gamma\pi i}{2}} u^{\alpha}, 0 \leq u < \infty\}$.

Thus finally,

$$E\{|\xi|^{\delta}\} = \left(\frac{c}{\cos\frac{\gamma\pi}{2}}\right)^{\frac{\delta}{\alpha}} \frac{2\Gamma(1+\delta)\Gamma(1-\frac{\delta}{\alpha})}{\pi\delta} \sin\frac{\delta\pi}{2} \cos\frac{\gamma\delta\pi}{2\alpha} =$$

$$= \left(\frac{c}{\cos\frac{\gamma\pi}{2}}\right)^{\frac{\delta}{\alpha}} \frac{\Gamma(1-\frac{\delta}{\alpha})\cos\frac{\gamma\delta\pi}{2\alpha}}{\Gamma(1-\delta)\cos\frac{\delta\pi}{2}}$$

for $-1 < \delta < \alpha$. This result is in agreement with (42.198).

46.7. By the solution of Problem 46.6 we obtain that

$$P\{\xi \geq 0\} = \int_0^{\infty} f(x;\alpha,\beta,c,0)dx = \int_0^{\infty} h(x;\alpha,\gamma)dx =$$

$$= \lim_{\delta \to 0} \int_0^{\infty} x^{\delta} h(x;\alpha,\gamma)dx = \lim_{\delta \to 0} \frac{\Gamma(1+\delta)\Gamma(1-\frac{\delta}{\alpha})}{\pi\delta} \sin(1+\frac{\gamma}{\alpha})\frac{\delta\pi}{2} =$$

$$= \frac{1}{2} + \frac{\gamma}{2\alpha}$$

where

$$\gamma = \frac{2}{\pi} \text{arc tan}(\beta \tan\frac{\alpha\pi}{2})$$

and $-1 < \gamma < 1$. This implies (42.192).

46.8. If $0 < \alpha < 1$ and $\beta = 1$, then $R(0) = 0$ and therefore $T\{\psi(s)\} = \psi(s)$ for $Re(s) \geq 0$. If $0 < \alpha < 1$ and $\beta = -1$, then $R(0) = 1$ and $T\{\psi(s)\} = 1$ for $Re(s) \geq 0$. In the remaining cases we have

$$\psi^+(s) = \frac{1}{2} - \frac{\gamma}{2\alpha} + \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi}\int_0^\infty \frac{e^{-cx^\alpha s^\alpha/\cos\frac{\gamma\pi}{2}}}{1-2x\sin\frac{\gamma\pi}{2\alpha}+x^2}\,dx$$

for $\operatorname{Re}(s) \geqq 0$ where

$$\gamma = \frac{2}{\pi}\operatorname{arc\,tan}(\beta\tan\frac{\alpha\pi}{2})$$

and $-1 < \gamma < 1$. We note that $R(0) = \frac{1}{2} - \frac{\gamma}{2\alpha}$.

It is sufficient to prove the above formula for $\operatorname{Re}(s) > 0$ and for some particular $c > 0$. For $\operatorname{Re}(s) \geqq 0$ we obtain $\psi^+(s)$ by continuity. We shall prove the above formula for $c = \cos\frac{\gamma\pi}{2}$ and by replacing s by $s(c/\cos\frac{\gamma\pi}{2})^{1/\alpha}$ we obtain $\psi^+(s)$ for a general $c > 0$.

Thus let us assume that $c = \cos\frac{\gamma\pi}{2}$. Then we have

$$\psi(iy) = \begin{cases} e^{-y^\alpha e^{i\gamma\pi/2}} & \text{for } y \geqq 0\,, \\ e^{-(-y)^\alpha e^{-i\gamma\pi/2}} & \text{for } y \leqq 0\,. \end{cases}$$

By Theorem 5.1 we have

$$\psi^+(s) = \frac{1}{2} + \lim_{\varepsilon\to 0}\frac{s}{2\pi i}\left[\int_\varepsilon^\infty \frac{\psi(iy)}{y(s-iy)}\,dy - \int_\varepsilon^\infty \frac{\psi(-iy)}{y(s+iy)}\,dy\right]$$

for $\operatorname{Re}(s) > 0$. If we substitute $y = e^{-i\gamma\pi/2\alpha}sz$ in the first integral and $y = e^{i\gamma\pi/2\alpha}sz$ in the second integral then we obtain that

$$\psi^+(s) = \frac{1}{2} + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\left[\int_{L_1(\varepsilon)} \frac{e^{-z^\alpha s^\alpha}}{z(1-ize^{-i\gamma\pi/2\alpha})}\,dz - \int_{L_2(\varepsilon)} \frac{e^{-z^\alpha s^\alpha}}{z(1+ize^{i\gamma\pi/2\alpha})}\,dz\right]$$

for $Re(s) > 0$ where $L_1(\varepsilon) = \{z : z = e^{i\gamma\pi/2\alpha} y/s$ and $\varepsilon \leqq y < \infty\}$ and $L_2(\varepsilon) = \{z : z = e^{-i\gamma\pi/2\alpha} y/s$ and $\varepsilon \leqq y < \infty\}$. Denote by $C_1(\varepsilon)$ the path which varies from $z = e^{i\gamma\pi/2\alpha}\varepsilon/s$ to $z = \varepsilon/|s|$ along the arc $|z| = \varepsilon/|s|$ and from $z = \varepsilon/|s|$ to ∞ along the real axis. Denote by $C_2(\varepsilon)$ the path which varies from $z = e^{-i\gamma\pi/2\alpha}\varepsilon/s$ to $z = \varepsilon/|s|$ along the arc $|z| = \varepsilon/|s|$ and from $z = \varepsilon/|s|$ to ∞ along the real axis. If we replace $L_1(\varepsilon)$ by $C_1(\varepsilon)$ in the first integral and $L_2(\varepsilon)$ by $C_2(\varepsilon)$ in the second integral, then by Cauchy's integral theorem both integrals remain unchanged. If $\varepsilon \to 0$, then the difference of the two integrals taken along the arcs tend to

$$\lim_{\varepsilon \to 0} [- \int_{\varepsilon/|s|}^{\varepsilon e^{i\gamma\pi/2\alpha}/s} \frac{dz}{z} + \int_{\varepsilon/|s|}^{\varepsilon e^{-i\gamma\pi/2\alpha}/s}] = \log e^{-\frac{i\gamma\pi}{2\alpha}} \frac{s}{|s|} - \log e^{\frac{i\gamma\pi}{2\alpha}} \frac{s}{|s|} = -\frac{i\gamma\pi}{\alpha}$$

and consequently,

$$\psi^+(s) = \frac{1}{2} - \frac{\gamma}{2\alpha} + \frac{1}{2\pi i} \int_0^\infty [\frac{e^{-x^\alpha s^\alpha}}{x(1-ixe^{-i\gamma\pi/2\alpha})} - \frac{e^{-x^\alpha s^\alpha}}{x(1+ixe^{i\gamma\pi/2\alpha}}]dx =$$

$$= \frac{1}{2} - \frac{\gamma}{2\alpha} + \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{e^{-x^\alpha s^\alpha}}{1-2x \sin\frac{\gamma\pi}{2\alpha} + x^2} dx$$

for $Re(s) > 0$ and $c = \cos\frac{\gamma\pi}{2\alpha}$. Since

$$\psi^+(0) = \frac{1}{2} - \frac{\gamma}{2\alpha} + \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{1}{1-2x \sin\frac{\gamma\pi}{2\alpha} + x^2} dx = 1$$

we can also write that

$$\psi^+(s) = 1 - \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{1 - e^{-x^\alpha s^\alpha}}{1-2x \sin\frac{\gamma\pi}{2\alpha} + x^2} dx$$

for $\mathrm{Re}(s) \geqq 0$ and $c = \cos \frac{\gamma\pi}{2\alpha}$.

46.9. By (42.115) we have

$$P\{\eta \leq x\} = \begin{cases} 2[1-\Phi(\frac{c}{\sqrt{x}})] & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

whence the statement follows. We note that $E\{e^{-s\eta}\} = e^{-c\sqrt{2s}}$ for $\mathrm{Re}(s) \geqq 0$ or $E\{e^{-s\eta}\} = e^{-c|s|^{1/2}(1+\frac{s}{|s|})}$ for $\mathrm{Re}(s) = 0$.

46.10. Since $F(x)$ belongs to the domain of attraction of itself, Theorem 44.8 is applicable. By (42.104) and (42.105) we can choose $B_n = n^{1/\alpha}$ in (46.247). Thus by (46.244) and (46.247) we obtain that

$$\lim_{n \to \infty} n\, F(-n^{1/\alpha}x) = \frac{c_1}{x^\alpha} \quad \text{and} \quad \lim_{n \to \infty} n[1-F(n^{1/\alpha}x)] = \frac{c_2}{x^\alpha}$$

for $x > 0$. Hence the assertions follow.

46.11. Let us denote by a the expectation of $F(x)$ if it exists, and by σ^2 the variance of $F(x)$ if it exists.

If $\sigma^2 = 0$, then $F(x)$ is degenerate, and $c = 0$ and $m = a$. (α and β are immaterial.)

If $0 < \sigma^2 < \infty$, then $F(x)$ is a normal distribution, and $\alpha = 2$, $c = \sigma^2/2$, $m = a$. (β is immaterial.)

If $\sigma^2 = \infty$, then $F(x)$ belongs to the domain of attraction of itself and thus by (46.245) we have

$$\lim_{x \to \infty} \frac{1-F(x) + F(-x)}{1-F(\rho x) + F(-\rho x)} = \rho^{\alpha}$$

for any $\rho > 0$. This determines α . The constants β and c are determined by the solution of Problem 46.10. It remains to find m . If $1 \leq \alpha < 2$, then m can be determined in the following way. On the one hand in Theorem 46.8

$$A_n = n \int_{|x|<\tau n^{1/\alpha}} x \, d\, F(x) - (\mu(\tau) + \varepsilon_n) n^{1/\alpha}$$

where $\tau > 0$, $\mu(\tau)$ is defined by (46.243) and $\lim_{n \to \infty} \varepsilon_n = 0$. On the other hand by (42.104) and (42.105)

$$A_n = \begin{cases} m(n-n^{1/\alpha}) & \text{for } \alpha \neq 1 \\ \dfrac{2\beta c n \log n}{\pi} & \text{for } \alpha = 1 \end{cases}$$

is a possible choice in Theorem 46.8.

A comparison of the above two formulas show that if $1 < \alpha < 2$, then

$$m = a = \int_{-\infty}^{\infty} x dF(x) ,$$

and if $\alpha = 1$, then

$$m = \lim_{n \to \infty} \left[\int_{|x|<\tau n} x dF(x) - \frac{2\beta c \log n}{\pi} \right] - \frac{2\beta c}{\pi} [\log \tau - (1-C)]$$

where $\tau > 0$ and C is Euler's constant. It is convenient to choose $\tau = e^{1-C}$ in the last formula.

<u>Note.</u> If $\alpha = 1$, then by the last formula we obtain that

$$\lim_{y \to \infty} \frac{1}{\log y} \int_{|x|<y} x dF(x) = \frac{2\beta c}{\pi} .$$

If $0 < \alpha < 1$ and if we compare the aforementioned two formulas for A_n, then we obtain that

$$\lim_{y \to \infty} \frac{1}{y^{1-\alpha}} \int_{|x|<y} x dF(x) = \frac{2\beta c \Gamma(1+\alpha) \sin \frac{\alpha\pi}{2}}{(1-\alpha)\pi} .$$

These formulas can also be proved directly by the solution of Problem 46.10 .

46.12. For any $k = 1,2,\ldots$ we have

$$\lim_{n \to \infty} P\{ \frac{\xi_1 + \xi_2 + \ldots + \xi_{nk} - A_{nk}}{B_{nk}} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$. If we write

$$\frac{\xi_1+\ldots+\xi_{nk} - kA_n}{B_n} = \frac{\xi_1+\ldots+\xi_n - A_n}{B_n} + \frac{\xi_{n+1}+\ldots+\xi_{2n} - A_n}{B_n} + \ldots + \frac{\xi_{(k-1)n+1}+\ldots+\xi_{kn} - A_n}{B_n} ,$$

then we can easily see that

$$\lim_{n \to \infty} P\{\frac{\xi_1 + \xi_2 + \ldots + \xi_{nk} - kA_n}{B_n} \leq x\} = R_k(x)$$

where $R_k(x)$ is the k-th iterated convolution of $R(x)$ with itself. By (42.103) $R_k(x)$ is a stable distribution function of type $S(\alpha,\beta,kc,km)$. Thus by (42.104) and (42.105)

$$\frac{\xi_1+\xi_2+\ldots+\xi_{nk}-kA_n}{k^{1/\alpha}B_n} - \begin{cases} \frac{2\beta c \log k}{\pi} & \text{for } \alpha = 1 , \\ m(k^{1-\frac{1}{\alpha}}-1) & \text{for } \alpha \neq 1 \end{cases}$$

has the limiting distribution $R(x)$ as $n \to \infty$. Hence by Lemma 44.2 it follows that necessarily

$$\lim_{n\to\infty} \frac{B_{nk}}{k^{1/\alpha}B_n} = 1$$

and

$$\lim_{n\to\infty} \frac{A_{nk}-kA_n}{B_{nk}} = \begin{cases} \frac{2\beta c}{\pi}\log k & \text{for } \alpha = 1 , \\ m(k^{1-\frac{1}{\alpha}}-1) & \text{for } \alpha \neq 1 . \end{cases}$$

Let us define

$$B(t) = B_n + (t-n)(B_{n+1}-B_n)$$

for $n < t \leqq n+1$ and $n = 0,1,2,\ldots$ and

$$a(t) = \frac{A_n}{B_n} + (t-n)\left[\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}\right]$$

for $n < t \leqq n+1$ and $n = 0,1,2,\ldots$. Since by (44.118) $B_n \to \infty$ and $B_{n+1}/B_n \to 1$ as $n \to \infty$, and by (44.125) $(A_{n+1}-A_n)/B_n \to 0$ as $n \to \infty$, we can conclude that

$$\lim_{t\to\infty} \frac{B(kt)}{k^{1/\alpha}B(t)} = 1$$

and

$$\lim_{t \to \infty} \left[a(kt) - \frac{k}{k^{1/\alpha}} a(t)\right] = \begin{cases} \frac{2\beta c}{\pi} \log k & \text{if } \alpha = 1 , \\ m(k^{1-\frac{1}{\alpha}} - 1) & \text{if } \alpha \neq 1 , \end{cases}$$

and $k = 1,2,\ldots$. The functions $B(t)$ and $a(t)$ are continuous functions of t , and therefore the above relations are also valid if k is replaced by ω where ω is any positive real number. If we write

$$B(t) = t^{1/\alpha} \rho(t)$$

for $t > 0$, then we have

$$\lim_{t \to \infty} \frac{\rho(\omega t)}{\rho(t)} = 1$$

for $\omega > 0$, and if we write

$$a(t) = \frac{h(t)}{t^{\frac{1}{\alpha}-1}} + \begin{cases} \frac{2\beta c}{\pi} \log t & \text{for } \alpha = 1 , \\ -m & \text{for } \alpha \neq 1 , \end{cases}$$

for $t > 0$, then we have

$$\lim_{t \to \infty} \frac{h(\omega t) - h(t)}{t^{\frac{1}{\alpha}-1}} = 0$$

for $\omega > 0$.

46.13. By (42.171) we have

$$\int_0^\infty e^{-sx}\, dR(x) = e^{-\Gamma(1-\alpha)s^\alpha}$$

for $R(s) \geqq 0$. Let $\phi(s) = \int_0^\infty e^{-sx}\, dF(x)$. Then

$$\phi(s) = 1-s\int_0^\infty [1-F(x)]e^{-sx}\,dx = 1-s\int_0^\infty h(x)x^{-\alpha}e^{-sx}dx =$$

$$= 1-\Gamma(1-\alpha)s^\alpha h(\tfrac{1}{s}) + o(s^\alpha)$$

as $s \to 0$. If $s > 0$, then

$$\lim_{n\to\infty} [\phi(\frac{s}{n^{1/\alpha}\,\rho(n)})]^n = e^{-\Gamma(1-\alpha)s^\alpha}$$

if and only if

$$\lim_{n\to\infty} \frac{h(n^{1/\alpha}\,\rho(n))}{(\rho(n))^\alpha} = 1 ,$$

and this proves the statement.

46.14. By (42.173) we have

$$\int_0^\infty e^{-sx}\,dR(x) = e^{-\Gamma(1-\alpha)s^\alpha}$$

for $\operatorname{Re}(s) \geqq 0$. Let $\phi(s) = \int_0^\infty e^{-sx}\,dF(x)$. Then

$$\phi(s) = 1-as+s\int_0^\infty [1-F(x)](1-e^{-sx})dx = 1-as +$$

$$+ s\int_0^\infty h(x)x^{-\alpha}(1-e^{-sx})dx = 1-as-\Gamma(1-\alpha)s^\alpha h(\tfrac{1}{s}) + o(s^\alpha)$$

and

$$\phi(s)e^{as} = 1-\Gamma(1-\alpha)s^\alpha h(\tfrac{1}{s}) + o(s^\alpha)$$

as $s \to 0$. If $s > 0$, then

$$\lim_{n\to\infty}\left[\phi(\frac{s}{n^{1/\alpha}\,\rho(n)})\, e^{\frac{as}{n^{1/\alpha}\rho(n)}}\right]^n = e^{-\Gamma(1-\alpha)s^\alpha}$$

if and only if

$$\lim_{n \to \infty} \frac{h(n^{1/\alpha}\,\rho(n))}{(\rho(n))^{\alpha}} = 1 ,$$

and this proves the statement.

46.15. Since

$$\sum_{j=0}^{k} (-1)^j \binom{a}{j} = (-1)^k \binom{a-1}{k}$$

for any a, we have

$$\underset{\sim}{P}\{\xi_n > 2k\} = 1 - \sum_{j=1}^{k} \frac{1}{2j-1} \binom{2j}{j} \frac{1}{2^{2j}} = 1 - \sum_{j=1}^{k} (-1)^{j-1} \binom{\frac{1}{2}}{j} =$$

$$= (-1)^{k-1} \binom{-\frac{1}{2}}{k} = \binom{2k}{k} \frac{1}{2^{2k}}$$

for $k = 1,2,\ldots$. By using the inequality

$$\frac{1}{\sqrt{\pi(k+\frac{1}{2})}} < \binom{2k}{k} \frac{1}{2^{2k}} < \frac{1}{\sqrt{\pi k}} ,$$

we get

$$\lim_{k \to \infty} \sqrt{2k}\, \underset{\sim}{P}\{\xi_n > 2k\} = \sqrt{\frac{2}{\pi}} .$$

Thus $\underset{\sim}{P}\{\xi_n \leq x\}$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\frac{1}{2},1,c,0)$ where $c > 0$. If we choose $A_n = 0$ and $B_n = (nb)^2$ where

$$b = \frac{\sqrt{\frac{2}{\pi}}}{\frac{2c}{\pi}\, \Gamma(\frac{1}{2}) \sin \frac{\pi}{4}} = \frac{1}{c} ,$$

then we have

$$\lim_{n\to\infty} \underset{\sim}{P}\{\frac{c^2(\xi_1+\ldots+\xi_n)}{n^2} \leqq x\} = R(x) = 2[1-\Phi(\frac{c}{\sqrt{x}})]$$

for $x \geqq 0$ where $\Phi(x)$ is the normal distribution function.

<u>Note.</u> Since

$$\underset{\sim}{P}\{\xi_1+\ldots+\xi_n = 2j\} = \frac{n}{2j-n}\binom{2j-n}{j}\frac{1}{2^{2j-n}}$$

for $j = n, n+1, \ldots,$ we can prove directly that

$$\lim_{n\to\infty} \underset{\sim}{P}\{\frac{\xi_1+\ldots+\xi_n}{n^2} > x\} = \frac{1}{\sqrt{2\pi}}\int_x^\infty y^{-\frac{3}{2}} e^{-\frac{1}{2y}}\,dy = 2\Phi(\frac{1}{\sqrt{x}}) - 1$$

for $x > 0$.

46.16. Since $\Gamma(a+1) \sim \sqrt{2\pi a}\,(\frac{a}{e})^a$ as $a \to \infty$, we have

$$k^q \underset{\sim}{P}\{\xi_n > k\} = k^q\binom{k-q}{k} = \frac{k^q\Gamma(k-q+1)}{\Gamma(1-q)\Gamma(k+1)} \to \frac{1}{\Gamma(1-q)} \quad \text{as } k \to \infty$$

and thus $\underset{\sim}{P}\{\xi_n \leqq x\}$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(q,1,c,0)$ where $c > 0$. If we choose $A_n = 0$ and $B_n = (nb)^{1/q}$ where

$$b = \frac{\pi}{\Gamma(1-q)2c\Gamma(q)\sin\frac{q\pi}{2}} = \frac{\sin q\pi}{2c\sin\frac{q\pi}{2}} = \frac{\cos\frac{q\pi}{2}}{c},$$

then

$$\lim_{n\to\infty} \underset{\sim}{P}\left\{\frac{\xi_1+\xi_2+\ldots+\xi_n}{(bn)^{1/q}} \leqq x\right\} = R(x) .$$

If $b = 1$, then $R(x)$ is of type $S(q,1,\cos \frac{q\pi}{2},0)$.

46.17. Since

$$\lim_{x \to \infty} \frac{1-F(x)}{1-F(\rho x)} = \rho$$

for $\rho > 0$, by Theorem 44.8 it follows that $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(1,1,c,m)$ where $c > 0$ and m is a real number. We have

$$\lim_{n \to \infty} P\{ \frac{\xi_1 + \ldots + \xi_n - A_n}{B_n} \leq x\} = R(x)$$

if we choose B_n in such a way that

$$\lim_{n \to \infty} \frac{n}{B_n (\log B_n)^2} = \frac{2c}{\pi}$$

and if

$$A_n = n \int_0^{\tau B_n} x dF(x) - mB_n - \frac{2c}{\pi} B_n[\log \tau - (1-C)] + \varepsilon_n B_n$$

where C is Euler's constant, $\tau > 0$, and $\varepsilon_n \to 0$ as $n \to \infty$.

If

$$B_n = \frac{n\pi}{2c(\log n)^2}$$

for $n > 1$, then the requirements are satisfied.

In our case

$$\int_0^\infty x dF(x) = \int_0^\infty [1-F(x)]dx = e + \int_e^\infty \frac{dx}{x(\log x)^2} = e + 1$$

and if $\tau B_n > e$, then

$$\int_0^{\tau B_n} x dF(x) = e-1 + \int_e^{\tau B_n} \left[\frac{1}{x(\log x)^2} + \frac{2}{x(\log x)^3}\right]dx = e + 1 - \frac{1}{\log \tau B_n} - \frac{1}{(\log \tau B_n)^2} .$$

Thus

$$A_n = n(e+1) - \frac{n}{\log n} + \frac{n}{(\log n)^2}\left[\log \frac{\pi}{2c} - C - \frac{m\pi}{2c}\right] - \frac{2n \log\log n}{(\log n)^2}$$

for $n > e$ is a possible choice.

If $c = \pi/2$ and $m = -C$, then we have

$$\lim_{n \to \infty} P\left\{ \frac{\xi_1 + \ldots + \xi_n - n(e+1)}{n(\log n)^{-2}} + \log n + 2\log\log n \leq x\right\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(1,1, \frac{\pi}{2}, -C)$.

46.18. In this case

$$1-F(x) + F(-x) = \frac{5}{6x(\log x)^2}$$

for $x \geq e$ and

$$\lim_{x \to \infty} \frac{F(-x)}{1-F(x)} = \frac{2}{3} .$$

Thus $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(1, \frac{1}{5}, c, m)$. If $c = 5\pi/12$, then we can choose $B_n = n/(\log n)^2$ for $n > 1$ and if $m = -C/6$ where C is Euler's constant, then

$$A_n = n(e-\frac{5}{6}) - \frac{n}{6\log n} - \frac{2n \log\log n}{6(\log n)^2}$$

for $n > e$ is a possible choice. Thus

$$\lim_{n\to\infty} P\{\frac{\xi_1+\ldots+\xi_n - A_n}{B_n} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(1, \frac{1}{5}, \frac{5\pi}{12}, -\frac{C}{6})$.

46.19. Since $\lim_{x\to\infty} x[1-F(x)] = 1$, it follows from Theorem 44.8 that $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(1,1,c,m)$. By (44.247) we can choose $B_n = n\pi/2c$ and by (44.248)

$$A_n = n\int_1^{\tau n\pi/2c} \frac{dx}{x} - \frac{n\pi}{2c}\{m + \frac{2c}{\pi}[\log\tau - (1-C)]\}$$

where $\tau > 0$ and C is Euler's constant. Thus

$$A_n = -m\frac{n\pi}{2c} + n\log\frac{n\pi}{2c} + n(1-C) .$$

If $c = \pi/2$ and $m = 1-C$, then we have

$$\lim_{n\to\infty} P\{\frac{\xi_1+\ldots+\xi_n}{n} - \log n \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(1,1,\frac{\pi}{2}, 1-C)$. The Laplace-Stieltjes transform of $R(x)$ is

$$\psi(s) = e^{-(1-C)s-|s|\frac{\pi}{2} + s\log|s|}$$

for $\mathrm{Re}(s) = 0$ or

$$\psi(s) = e^{-(1-C)s+s\log s}$$

for $\mathrm{Re}(s) \geqq 0$.

Note. We can prove the above result directly. The Laplace-Stieltjes transform of $F(x)$ can be expressed as

$$\phi(s) = \int_1^\infty \frac{e^{-sx}}{x^2}\,dx = s\int_s^\infty e^{-z}\frac{dz}{z^2} = s[C-1] + s\log s + 1 - s^2 + o(s^2)$$

for $\mathrm{Re}(s) \geqq 0$ where $o(s^2)/s^2 \to 0$ as $|s| \to 0$. (See N. Nielsen [142 p.5].) Thus

$$\lim_{n\to\infty} [\phi(\tfrac{s}{n})]^n e^{s\log n} = e^{s(C-1)+s\log s}$$

for $\mathrm{Re}(s) \geqq 0$. Accordingly,

$$\lim_{n\to\infty} \underset{\sim}{P}\{\frac{\xi_1+\ldots+\xi_n}{n} - \log n \leqq x\} = R(x)$$

where

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx}\,dR(x) = e^{-(1-C)s+s\log s}$$

for $\mathrm{Re}(s) \geqq 0$. If $\mathrm{Re}(s) = 0$, then $s\log s = s\log|s| - \frac{|s|\pi}{2}$

46.20. In this case $F(-x) = 1-F(x)$ and

$$1-F(x) + F(-x) = \frac{1+\log x}{x}$$

for $x \geqq 1$. Thus $F(x)$ belongs to the domain of attraction of a stable distribution function of type $S(1,0,c,0)$ where $c > 0$. If we choose $A_n = 0$ and $B_n = (\pi n\log n)/2c$, then

$$\lim_{n\to\infty} \mathbf{P}\{ \frac{2c(\xi_1+\ldots+\xi_n)}{\pi\, n \log n} \leq x\} = \frac{1}{2} + \frac{1}{\pi} \text{arc tan} \frac{x}{c} .$$

46.21. Let us suppose that $\mathbf{E}\{e^{-s\xi}\} = e^{-s^\alpha}$ for $\text{Re}(s) \geq 0$ where $0 < \alpha < 1$, $\mathbf{P}\{\eta \leq x\} = 1-e^{-x}$ for $x \geq 0$, and ξ and η are independent random variables. Then

$$\mathbf{P}\{\xi\eta^{-1} \leq x\} = \int_0^\infty \mathbf{P}\{\xi \leq xy\}e^{-y}dy = e^{-(1/x)^\alpha}$$

for $x > 0$. Hence

$$\mathbf{P}\{\eta^\alpha\ \xi^{-\alpha} \leq x\} = 1 - e^{-x}$$

for $x \geq 0$, and

$$\mathbf{E}\{e^{-s\eta^\alpha \xi^{-\alpha}}\} = \frac{1}{1+s}$$

for $\text{Re}(s) > -1$, or

$$\mathbf{E}\{e^{-s^\alpha \eta^\alpha \xi^{-\alpha}}\} = \int_0^\infty \mathbf{E}\{e^{-(su)^\alpha \xi^{-\alpha}}\}e^{-u}\, du = \frac{1}{1+s^\alpha}$$

for $\text{Re}(s) \geq 0$. On the other hand by (42.180) we have

$$\int_0^\infty E_\alpha(-s^\alpha u^\alpha)e^{-u}du = \sum_{k=0}^\infty \frac{(-1)^k\, s^{k\alpha}}{\Gamma(k\alpha+1)} \int_0^\infty e^{-u}\, u^{k\alpha}du = \frac{1}{1+s^\alpha}$$

for $|s| < 1$. Accordingly, we have

$$\int_0^\infty \mathbf{E}\{e^{-(su)^\alpha \xi^{-\alpha}}\}e^{-u}\, du = \int_0^\infty E_\alpha(-s^\alpha u^\alpha)e^{-u}\, du$$

for $|s| < 1$, and this implies that

$$\mathbf{E}\{e^{-w\xi^{-\alpha}}\} = E_\alpha(-w)$$

for every w. This proves (42.181).

46.22. In this case $\psi(s) = \underset{\sim}{E}\{e^{-s\xi}\} = \underset{\sim}{E}\{e^{-s\eta}\} = e^{-s^\alpha}$ for $\text{Re}(s) \geq 0$. Let us suppose that $\xi, \eta, \theta_1, \theta_2$ are mutually independent random variables and $\underset{\sim}{P}\{\theta_1 \leq x\} = \underset{\sim}{P}\{\theta_2 \leq x\} = 1-e^{-x}$ for $x \geq 0$. Then

$$\underset{\sim}{P}\{\xi\theta_1^{-1} \leq x\} = \underset{\sim}{P}\{\eta\theta_2^{-1} \leq x\} = \psi(\frac{1}{x})$$

for $x > 0$ and

$$G(x) = \underset{\sim}{P}\{\xi\eta^{-1}\theta_1^{-1}\theta_2 \leq x\} = \int\int_{uv^{-1}\leq x} \psi(\frac{1}{u})\psi(\frac{1}{v})du\, dv = \frac{x^\alpha}{1+x^\alpha}$$

for $x \geq 0$. Since

$$\underset{\sim}{E}\{\theta_1^{-s}\theta_2^{s}\} = \Gamma(1-s)\Gamma(1+s) = \frac{\pi s}{\sin \pi s}$$

for $|\text{Re}(s)| < 1$, it follows that

$$\int_0^\infty x^s dH(x) = \frac{\sin \pi s}{\pi s}\int_0^\infty x^s dG(x)$$

for $-1 < \text{Re}(s) < \alpha$. If we extend the definition of $G(x)$ by analytical continuation to the complex plane cut along the negative real axis from the origin to infinity, then we can write that

$$\frac{dH(x)}{dx} = \frac{G(xe^{\pi i}) - G(xe^{-\pi i})}{2\pi i x} = \frac{x^\alpha \sin \alpha\pi}{\pi x(1+2x^\alpha \cos \alpha\pi + x^{2\alpha})}$$

for $x > 0$. If, in particular, $\alpha = 1/2$, then

$$H(x) = \frac{2}{\pi} \text{arc tan} \sqrt{x}$$

for $x \geq 0$.

CHAPTER VII

53.1. If τ_k $(k = 0,1,2,\ldots)$ is defined as in Section 49, then by Theorem 43.3 we have

$$P\{\lim_{k \to \infty} \frac{\tau_k}{k} = a\} = 1 .$$

Hence if $0 < \varepsilon < a$, then

$$(a-\varepsilon)k \leqq \tau_k \leqq (a+\varepsilon)k$$

for sufficiently large k with probability 1. Since $\tau_{\nu(t)} \leqq t < \tau_{\nu(t)+1}$, therefore

$$\frac{1}{a+\varepsilon} - \frac{1}{t} \leqq \frac{\nu(t)}{t} \leqq \frac{1}{a-\varepsilon}$$

for sufficiently large t with probability 1. This implies that

$$P\{\lim_{t \to \infty} \frac{\nu(t)}{t} = \frac{1}{a}\} = 1 .$$

This result is also valid for $a = \infty$, if we define $1/a$ as 0 for $a = \infty$. This can be obtained from the previous result by truncating the recurrence times at m and letting $m \to \infty$.

53.2. Both ξ_1 and ξ_2 are necessarily discrete random variables, and there is a constant c such that $\xi_1 + c$ and $\xi_2 - c$ take on nonnegative integers only. Let $P\{\xi_1 + c = j\} = p_j$ and $P\{\xi_2 - c = j\} = q_j$ for $j = 0,1,2,\ldots$. Then we have

$$\sum_{j=0}^{k} p_j \, q_{k-j} = e^{-a} \frac{a^k}{k!}$$

for $k = 0,1,2,\ldots$. Hence $p_0 > 0$, $q_0 > 0$ and

$$p_k \leq \frac{e^{-a}}{q_0}\frac{a^k}{k!} \quad \text{and} \quad q_k \leq \frac{e^{-a}}{p_0}\frac{a^k}{k!}$$

for $k = 0,1,2,\ldots$. Let

$$g(z) = \sum_{j=0}^{\infty} p_j z^j \quad \text{and} \quad h(z) = \sum_{j=0}^{\infty} q_j z^j .$$

The function $g(z)$ is regular on the whole complex plane,

$$|g(z)| \leq \frac{1}{q_0} \sum_{k=0}^{\infty} e^{-a} \frac{a^k}{k!} |z|^k = \frac{1}{q_0} e^{-a(1-|z|)}$$

and $g(z)$ never vanishes. For $g(z)h(z) = e^{-a(1-z)}$ and $e^{-a(1-z)}$ has no zeros. Thus

$$\lim_{|z| \to \infty} \frac{\log g(z)}{z^2} = 0$$

and by Theorem 10.3 in the Appendix, it follows that $\log g(z) = a_1(z-1)$. Here we used that $g(1) = 1$. Accordingly, $g(z) = e^{-a_1(1-z)}$, and in a similar way we get $h(z) = e^{-a_2(1-z)}$. Obviously $a_1 \geq 0$, $a_2 \geq 0$ and $a_1 + a_2 = a$. If $a_1 = 0$ or $a_2 = 0$, then the random variable ξ_1 or ξ_2 has a degenerate distribution. If $a_1 > 0$ and $a_2 > 0$ then both $\xi_1 + c$ and $\xi_2 - c$ have a nondegenerate Poisson distribution.

53.3. Let

$$q_k(n) = \mathbf{P}\{\nu(i) = i \text{ for } k \text{ values } i = 1,2,\ldots, n \mid \nu(n) = n\}$$

for $1 \leq k \leq n$. It is easy to see that $q_1(1) = 1$ and

$$q_1(n) = 1 - \sum_{j=1}^{n-1} \mathbf{P}\{\nu(j) = j \mid \nu(n) = n\} q_1(j)$$

for $n > 1$. Furthermore,

$$q_k(n) = \sum_{j=1}^{n-k+1} \underset{\sim}{P}\{\nu(j) = j \mid \nu(n) = n\} q_1(j) q_{k-1}(n-j)$$

for $2 \leq k \leq n$. Define

$$Q_k(n) = \frac{n^n}{n!} q_k(n)$$

for $1 \leq k \leq n$. Then we have $Q_1(1) = 1$ and

$$Q_1(n) = \frac{n^n}{n!} - \sum_{j=1}^{n-1} Q_1(j) \frac{(n-j)^{n-j}}{(n-j)!}$$

for $n > 1$. Furthermore,

$$Q_k(n) = \sum_{j=1}^{n-k+1} Q_1(j) Q_{k-1}(n-j)$$

for $2 \leq k \leq n$. Hence

$$\sum_{n=1}^{\infty} Q_1(n) z^n = \frac{\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n}{1 + \sum_{n=1}^{\infty} \frac{n^n}{n!} z^n} = \frac{\frac{\rho(z)}{1-\rho(z)}}{\frac{1}{1-\rho(z)}} = \rho(z)$$

for $|z| < 1/e$ where $w = \rho(z)$ is the only root of $w e^{-w} = z$ in the circle $|w| < 1$,and

$$\sum_{n=k}^{\infty} Q_k(n) z^n = \left(\sum_{n=1}^{\infty} Q_1(n) z^n \right)^k = [\rho(z)]^k$$

for $k = 1, 2, \ldots$. By Lagrange's expansion we obtain that

$$Q_k(n) = \frac{k}{(n-k)!n}$$

for $1 \leq k \leq n$. (See (39.148) and (39.149).)

53.4. If τ_u $(0 \leq u < \infty)$ is defined by (49.24), then by the solution of Problem 46.17 we have

$$\lim_{u \to \infty} P\left\{ \frac{\tau_u - u(e+1)}{u(\log u)^{-2}} + \log u + 2\log\log u \leq x \right\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(1, 1, \frac{\pi}{2}, -C)$ where C is Euler's constant.

If

$$t = u(e+1) - \frac{u}{\log u} + x\frac{u}{(\log u)^2} - \frac{2u\log\log u}{(\log u)^2}$$

for $u > e$, then

$$\lim_{t \to \infty}\left[\frac{u - \frac{t}{e+1}}{t(e+1)^{-2}(\log t)^{-2}} - \log t - 2\log\log t\right] = \frac{1}{e+1} + \log(e+1) - x$$

and thus

$$\lim_{u \to \infty} P\{\tau_u \leq t\} = \lim_{t \to \infty} P\{\nu(t) \geq u\} = R(x)$$

implies that

$$\lim_{t \to \infty} P\left\{\frac{\nu(t) - \frac{t}{e+1}}{t(e+1)^{-2}(\log t)^{-2}} - \log t - 2\log\log t \leq \frac{1}{e+1} + \log(e+1) + x\right\} = 1 - R(-x) .$$

53.5. Denote by Δ_n the number of positive terms in the sequence $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n$. Then we have

$$\underset{\sim}{P}\{\tau_1 > n\} = \underset{\sim}{P}\{\zeta_r \leqq 0 \quad \text{for} \quad 0 \leqq r \leqq n\} = \underset{\sim}{P}\{\Delta_n = 0\}$$

for $n = 0,1,\ldots$. By Theorem 23.1 we have

$$\sum_{n=0}^{\infty} \underset{\sim}{P}\{\Delta_n = 0\}\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{P}\{\zeta_n \leqq 0\}\}$$

for $|\rho| < 1$. By (42.192) we have

$$\underset{\sim}{P}\{\zeta_n \leqq 0\} = 1 - q = \frac{1}{2} - \frac{\gamma}{2\alpha}$$

for $n = 1,2,\ldots$ where

$$\gamma = \frac{2}{\pi} \text{arc tan}(\beta \tan \frac{\alpha\pi}{2})$$

and $-1 < \gamma < 1$. Thus it follows that

$$\underset{\sim}{P}\{\tau_1 > n\} = \underset{\sim}{P}\{\Delta_n = 0\} = (-1)^n\binom{q-1}{n} = \binom{n-q}{n}$$

for $n = 0,1,2,\ldots$ and $0 < q < 1$.

By the solution of Problem 46.16 we have

$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\tau_n}{n^{1/q}} \leqq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(q,1,\cos \frac{q\pi}{2}, 0)$.

Hence by Theorem 49.2 we have

$$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\nu(t)}{t^q} \leqq x\} = 1 - R(x^{-1/q})$$

for $x > 0$ or

$$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\nu(t)}{t^q} \leq x\} = G_q(x)$$

where $G_q(x)$ is defined by (42.178) .

53.6. The random variables $\chi_1 = \zeta_{\tau_1}, \chi_2 = \zeta_{\tau_2} - \zeta_{\tau_1}, \ldots$ are mutually independent and identically distributed positive random variables and $\zeta_{\tau_n} = \chi_1 + \chi_2 + \ldots + \chi_n$ for $n = 1,2,\ldots$. By Theorem 19.4 we have

$$\phi(s) = \underset{\sim}{E}\{e^{-s\chi_1}\} = 1 - e^{-\sum_{n=1}^{\infty} \frac{1}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n > 0)\}}$$

for $Re(s) \geq 0$. The random variable ζ_n has a stable distribution of type $S(\alpha,\beta,nc,0)$ and thus by the solution of Problem 46.8 we have

$$\underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n > 0)\} = \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi} \int_0^{\infty} \frac{e^{-nc^* x^\alpha s^\alpha}}{1-2x\sin\frac{\gamma\pi}{2\alpha} + x^2} dx$$

for $Re(s) \geq 0$ where $c^* = c/\cos\frac{\gamma\pi}{2}$. Since $q = \frac{1}{2} + \frac{\gamma}{2\alpha}$ we have

$$\sin\frac{\gamma\pi}{2\alpha} = -\cos q\pi \quad \text{and} \quad \cos\frac{\gamma\pi}{2\alpha} = \sin q\pi ,$$

and

$$1-\phi(s) = \exp\{ \frac{\sin q\pi}{\pi} \int_0^{\infty} \frac{\log(1-e^{-c^* x^\alpha s^\alpha})}{1+2x\cos q\pi + x^2} dx \}$$

for $Re(s) \geq 0$. If we write

$$\log(1-e^{-c^* x^\alpha s^\alpha}) = \log(c^* x^\alpha s^\alpha) + \log\left(\frac{1-e^{-c^* x^\alpha s^\alpha}}{c^* x^\alpha s^\alpha}\right)$$

in the above integral and if we take into consideration that

$$\int_0^\infty \frac{dx}{1+2x\cos q\pi+x^2} = 2\int_0^1 \frac{dx}{1+2x\cos q\pi+x^2} = \frac{q\pi}{\sin q\pi}$$

and

$$\int_0^\infty \frac{\log x}{1+2x\cos q\pi+x^2} = \int_0^1 \frac{\log x}{1+2x\cos q\pi+x^2}\,dx + \int_1^\infty \frac{\log x}{1+2x\cos q\pi+x^2}\,dx = 0\ ,$$

then we can easily see that

$$\lim_{s\to 0} \frac{1-\phi(s)}{s^{\alpha q}} = (c^*)^q = \left[\frac{c}{\cos\frac{\gamma\pi}{2}}\right]^q = c^q(1+\beta^2\tan^2\frac{\alpha\pi}{2})^{\frac{q}{2}}\ .$$

If either $0 < \alpha < 1$ or $1 < \alpha < 2$ and $-1 < \beta \leq 1$, then $0 < \alpha q < 1$ and consequently

$$\lim_{x\to\infty} x^{\alpha q}\, P\{\chi_1 > x\} = \frac{(c^*)^q}{\Gamma(1-\alpha q)}\ .$$

Thus $P\{\chi_1 \leq x\}$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha q,1,\bar{c},0)$ where $\bar{c} > 0$. We have

$$\lim_{n\to\infty} P\left\{\frac{\chi_1+\chi_2+\ldots+\chi_n}{(bn)^{\alpha q}} \leq x\right\} = R(x)$$

if

$$b = \frac{(c^*)^q\,\pi}{\Gamma(1-\alpha q)2\bar{c}\Gamma(\alpha q)\sin\frac{\alpha q\pi}{2}} = \frac{(c^*)^q\cos\frac{\alpha q\pi}{2}}{\bar{c}}\ .$$

If $\bar{c} = (c^*)^q \cos \frac{\alpha q \pi}{2}$, then $b = 1$ and we have

$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{X_1 + X_2 + \ldots + X_n}{n^{\alpha q}} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type

$$S(\alpha q, 1, c^q (1+\beta^2 \tan^2 \frac{\alpha\pi}{2})^{\frac{q}{2}} \cos \frac{\alpha q \pi}{2} , 0) .$$

We note that if $1 < \alpha < 2$ and $\beta = -1$, then $\gamma = 2-\alpha$, and $\alpha q = 1$. In this case

$$\lim_{s \to 0} \frac{1-\phi(s)}{s} = \left(\frac{c}{|\cos \frac{\alpha\pi}{2}|}\right)^{1/\alpha}$$

that is

$$\underset{\sim}{E}\{X_1\} = \left(\frac{c}{|\cos \frac{\alpha\pi}{2}|}\right)^{1/\alpha} .$$

CHAPTER VIII

58.1. Let us suppose that for each $i = 1,2,\ldots, m$ independently of the others we perform the following random trial: We distribute a_i points on the interval $(0,1)$ in such a way that the points are distributed independently and uniformly on $(0,1)$. In the i-th trial denote by $\chi_i(u)$ the number of points in the interval $(0,u)$ for $0 \leqq u \leqq 1$. Then the processes $\{\chi_i(u), 0 \leqq u \leqq 1\}$ are independent for $i = 1,2,\ldots, m$ and we can easily see that the probability sought is

$$P = \underset{\sim}{P}\{\chi_1(u)+c_1 < \chi_2(u)+c_2 <\ldots< \chi_m(u)+c_m \text{ for } 0 \leqq u \leqq 1\} .$$

On the other hand if we suppose that $\{\nu_i(u), 0 \leqq u < \infty\}$ $(i = 1,2,\ldots, m)$ are independent Poisson processes of density λ, then obviously we can write that

$$P = \underset{\sim}{P}\{\nu_1(u)+c_1 < \nu_2(u)+c_2 <\ldots< \nu_m(u)+c_m \text{ for } 0 \leqq u \leqq 1 | \nu_1(1) =$$
$$= a_1, \nu_2(1) = a_2,\ldots, \nu_m(1) = a_m\} .$$

This latter probability is given by Theorem 56.9 .

58.2. Let us define

$$p_k(a) = \underset{\sim}{P}\{\nu(i) < i \quad \text{for} \quad 0 < i \leqq k | \nu(a+k) = k\}$$

for $k = 0,1,2,\ldots$ and $a \geqq 0$ where $p_0(a) = 1$, and

$$p_k^*(a) = \underset{\sim}{P}\{\nu(i) < i+1 \text{ for } 0 \leqq i < k | \nu(a+k-i) = k\}$$

for $k = 0,1,2,\ldots$ and $a \geqq 0$ where $p_0^*(a) = 1$ and $p_1^*(a) = 1$.

Then we have $W(t,0,k) = p_k(t-k)$ for $0 \leq k \leq t$ and $W(t,1,k) = p_k^*(t+1-k)$ for $0 \leq k \leq t+1$.

We can see immediately that

$$p_k(a) = \underset{\sim}{P}\{\nu(a+i) > i \quad \text{for} \quad 0 \leq i < k | \nu(a+k) = k\}$$

for $k = 0,1,2,\ldots$ and $a \geq 0$ where $p_0(a) = 1$.

If $k = 1,2,\ldots$ and $a \geq 0$, then we have

$$p_k(a) = 1 - \sum_{j=0}^{k-1} \underset{\sim}{P}\{\nu(a+j) = j | \nu(a+k) = k\} p_j(a)$$

where

$$\underset{\sim}{P}\{\nu(a+j) = j | \nu(a+k) = k\} = \binom{k}{j} \frac{(a+j)^j (k-j)^{k-j}}{(a+k)^k} .$$

Let

$$P_k(a) = \frac{p_k(a)(a+k)^k}{k!}$$

for $k = 0,1,2,\ldots$ and $a \geq 0$. Then we have $P_0(a) = 1$ and

$$P_k(a) = \frac{(a+k)^k}{k!} - \sum_{j=0}^{k-1} P_j(a) \frac{(k-j)^{k-j}}{(k-j)!}$$

for $k = 1,2,\ldots$. Hence

$$\sum_{k=0}^{\infty} P_k(a) z^k = \frac{\sum_{k=0}^{\infty} \frac{(a+k)^k}{k!} z^k}{\sum_{k=0}^{\infty} \frac{k^k}{k!} z^k} = \frac{\frac{e^{a\rho(z)}}{1-\rho(z)}}{\frac{1}{1-\rho(z)}} = e^{a\rho(z)} = \sum_{k=0}^{\infty} \frac{a(a+k)^{k-1}}{k!} z^k$$

for $|z| < 1/e$ where $w = \rho(z)$ is the only root of the equation $we^{-w} = z$ in the circle $|w| < 1$. (See (39.148) and (39.149).) Thus

$$p_k(a) = \frac{a}{a+k}$$

for $k = 0,1,\ldots$ and $a \geqq 0$ where $p_0(0) = 1$, and

$$W(t,0,k) = 1 - \frac{k}{t}$$

for $0 \leqq k \leqq t$ and $t > 0$. This is in agreement with (56.83).

Second, we can write equivalently that

$$p_k^*(a) = \underset{\sim}{P}\{\nu(a+i) \geqq i+1 \text{ for } 0 \leqq i < k \mid \nu(a+k-1) = k\}$$

for $k = 0,1,2,\ldots$ and $a \geqq 0$ where $p_0^*(a) = 1$ and $p_1^*(a) = 1$.

If $k = 1,2,\ldots$ and $a \geqq 0$, then we have

$$p_k^*(a) = 1 - \sum_{j=0}^{k-1} \underset{\sim}{P}\{\nu(a+j-1) = j, \nu(a+j) = j \mid \nu(a+k-1) = k\} p_j^*(a)$$

where

$$\underset{\sim}{P}\{\nu(a+j-1) = j,\ \nu(a+j) = j \mid \nu(a+k-1) = k\} = \frac{k!\,(a+j-1)^j (k-j-1)^{k-j}}{j!\,(k-j)!\,(a+k-1)^k} .$$

Let

$$P_k^*(a) = \frac{p_k^*(a)(a+k-1)^k}{k!}$$

for $k = 0,1,2,\ldots$ and $a \geqq 0$. Then we have $P_0^*(a) = 1$, $P_1^*(a) = 1$ and

$$P_k^*(a) = \frac{(a+k-1)^k}{k!} - \sum_{j=0}^{k-1} P_j^*(a) \frac{(k-j-1)^{k-j}}{(k-j)!}$$

for $k = 1,2,\ldots$. Hence

$$\sum_{k=0}^{\infty} P_k^*(a)z^k = \frac{\sum_{k=0}^{\infty} \frac{(a+k-1)^k}{k!} z^k}{\sum_{k=0}^{\infty} \frac{(k-1)^k}{k!} z^k} = \frac{\frac{e^{(a-1)\rho(z)}}{1-\rho(z)}}{\frac{e^{-\rho(z)}}{1-\rho(z)}} = e^{a\rho(z)} = \sum_{k=0}^{\infty} \frac{a(a+k)^{k-1}}{k!} z^k$$

for $|z| < 1/e$. Thus

$$p_k^*(a) = \frac{a(a+k)^{k-1}}{(a+k-1)^k}$$

for $k = 0,1,\ldots,$ and $a \geqq 0$ where $p_0^*(a) = 1$ and $p_1^*(a) = 1$, and

$$W(t,1,k) = \frac{(t+1-k)(t+1)^{k-1}}{t^k}$$

for $0 \leqq k \leqq t+1$ and $t > 0$. This is in agreement with (56.88).

58.3. If we take into consideration that in the underlying Poisson process in the interval $(0,\Delta t)$ one event occurs with probability $\lambda\Delta t + o(\Delta t)$ and more than one event occurs with probability $o(\Delta t)$, then we can write that

$$W(t+\Delta t,x) = (1-\lambda\Delta t)W(t,x+\Delta t) + \lambda\Delta t \int_{-\infty}^{x} W(t,x-y)dH(y) + o(\Delta t) .$$

Hence by the limiting procedure $\Delta t \to 0$ we obtain that $W(t,x)$ satisfies the integro-differential equation

$$\frac{\partial W(t,x)}{\partial t} = \frac{\partial W(t,x)}{\partial x} - \lambda W(t,x) + \lambda \int_{-\infty}^{x} W(t,x-y)dH(y)$$

for almost all (t,x) . The probaiblity $W(t,x)$ can be determined by solving this equation.

58.4. In this case

$$\mathbf{P}\{\chi(u) \leq x\} = e^{-\lambda u} + \sum_{n=1}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} \int_0^x e^{-\mu y} \frac{(\mu y)^{n-1}}{(n-1)!} \mu dy =$$

$$= e^{-\lambda u} [1 + \lambda\mu u \int_0^x e^{-\mu y} J'(\lambda\mu u y)dy]$$

for $x \geq 0$ and $\mathbf{P}\{\chi(u) \leq x\} = 0$ for $x < 0$ where

$$J(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

is a Bessel function. Hence $\mathbf{P}\{\chi(u) = 0\} = e^{-\lambda u}$ and

$$\frac{\partial \mathbf{P}\{\chi(u) \leq x\}}{\partial x} = \lambda\mu u\, e^{-\lambda u - \mu x} J'(\lambda\mu u x)$$

for $x > 0$.

By Theorem 55.6 we have

$$W(t,0) = e^{-\lambda t} [1 + \lambda\mu \int_0^t (t-y)e^{-\mu y} J'(\lambda\mu t y)dy]$$

for $t \geq 0$ and

$$W(t,x) = \mathbf{P}\{\chi(t) \leq t+x\} - \lambda\mu e^{-\mu x} \int_0^t u\, e^{-(\lambda+\mu)u} J'(\lambda\mu u(u+x))W(t-u,0)du$$

for $t \geq 0$ and $x \geq 0$. In another form we can write that

$$W(t,x) = 1 - \lambda e^{-\mu x} \int_0^t \frac{e^{-(\lambda+\mu)y}}{x+y} [xJ(\lambda\mu y(x+y)) + yJ'(\lambda\mu y(x+y))]dy$$

for $t \geq 0$ and $x \geq 0$.

58.5. By using the same notation as in Problem 58.4 we have

$$V(t,x) = 1 - e^{-\lambda x} - \lambda\mu x\, e^{\mu x} \int_x^t e^{-(\lambda+\mu)y}\, J'(\lambda\mu y(y-x))dy$$

for $0 < x \leq t$. This follows immediately from Theorem 55.9 .

58.6. If we suppose that $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ and $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ are numerical (non-random) quantities for which $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ and $\tau_n \to \infty$ as $n \to \infty$, then we have $\eta(t) = \eta_n$ for $\tau_n < t < \tau_{n+1}$ $(n = 0,1,\ldots)$ where $\eta_0 = 0$ and

$$\eta_n = \max(0, \chi_1 - \tau_1, \chi_1 + \chi_2 - \tau_2, \ldots, \chi_1 + \ldots + \chi_n - \tau_n) .$$

Thus

$$q \int_0^\infty e^{-qt-s\eta(t)}\, dt = \sum_{n=0}^\infty q e^{-s\eta_n} \int_{\tau_n}^{\tau_{n+1}} e^{-qt} dt = \sum_{n=0}^\infty e^{-s\eta_n}(e^{-q\tau_n} - e^{-q\tau_{n+1}})$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$. If $\{\chi_n\}$ and $\{\tau_n\}$ are random variables, then the above identity holds for almost all realizations of the process $\{\chi(u) , 0 \leq u < \infty\}$. If we form the expectation of the above expression, then we get

$$q \int_0^\infty e^{-qt}\, E\{e^{-s\eta(t)}\}dt = [1-\phi(q)]U(q,s)$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$ where by Theorem 4.1

$$U(q,s) = \sum_{n=0}^\infty E\{e^{-s\eta_n - q\tau_n}\} = e^{-T\{\log[1-\psi(s)\phi(q-s)]\}} .$$

The same result can also be obtained by Theorem 54.1 . The distribution function $P\{\eta(t) \leq x\}$ can be obtained by inversion from the above transform.

58.7. First, let us suppose that $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ and $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ are numerical (non-random) quantities for which $\chi_n \geq 0$ $(n=1,2,\ldots)$, $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ and $\tau_n \to \infty$ as $n \to \infty$. Let us write $\gamma_n = \chi_1 + \chi_2 + \ldots + \chi_n$ for $n = 1,2,\ldots$ and

$$\eta_n^* = \max(0,\ \tau_2 - \tau_1 - \gamma_1, \ldots, \tau_{n+1} - \tau_1 - \gamma_n)$$

for $n = 1,2,\ldots$ and $\eta_0^* = 0$. Then $\eta^*(\tau_n) = \tau_1 + \eta_{n-1}^*$ for $n = 1,2,\ldots$, and

$$\eta^*(t) = \max(\eta^*(\tau_n),\ t-\gamma_n) = t-\gamma_n + [\eta^*(\tau_n) + \gamma_n - t]^+$$

for $\tau_n \leqq t \leqq \tau_{n+1}$. If we calculate

$$q \int_{\tau_n}^{\tau_{n+1}} e^{-qt-s\eta^*(t)}dt$$

by using (54.17) and if we add these integrals for $n = 0,1,2,\ldots$, then we obtain that

$$q \int_0^\infty e^{-qt-s\eta^*(t)}dt = \frac{q}{q+s} + \frac{s\, e^{-(q+s)\tau_1}}{q+s} \sum_{n=0}^{\infty} e^{-q\gamma_n-(q+s)\eta_n^*}(1-e^{-q\chi_{n+1}})$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geqq 0$. If $\{\chi_n\}$ and $\{\tau_n\}$ are random variables, then the above identity holds for almost all realizations of the process $\{\chi(u),\ 0 \leqq u < \infty\}$. If we form the expectation of the above expression, then we get

$$q \int_0^\infty e^{-qt}\, \mathbf{E}\{e^{-s\eta^*(t)}\}dt = \frac{q}{q+s} + \frac{s}{q+s}\, \phi(q+s)[1-\psi(q)]U^*(q,q+s)$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geqq 0$ where by Theorem 4.1

$$U^*(q,s) = \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-q\gamma_n - s\eta_n^*}\} = e^{-\underset{\sim}{T}\{\log[1-\phi(s)\psi(q-s)]\}} .$$

The distribution function $\underset{\sim}{P}\{\eta^*(t) \leqq x\}$ can be obtained by inversion from the above transform.

58.8. If $\gamma \leqq \alpha$, then

$$(s-q) \int_\alpha^\beta e^{-qt} dt = \frac{(q-s)}{q} (e^{-q\beta} - e^{-q\alpha}) ,$$

if $\gamma \geqq \beta$, then

$$(s-q) \int_\alpha^\beta e^{-qt-s(\gamma-t)} dt = e^{-q\beta-s(\gamma-\beta)} - e^{-q\alpha-s(\gamma-\alpha)} ,$$

and if $\alpha \leqq \gamma \leqq \beta$, then

$$(s-q) \int_\alpha^\gamma e^{-qt-s(\gamma-t)} dt + (s-q) \int_\gamma^\beta e^{-qt} dt =$$

$$= [e^{-q\gamma} - e^{-q\alpha-s(\gamma-\alpha)}] - (1 - \frac{s}{q})(e^{-q\beta} - e^{-q\gamma}) .$$

These formulas prove the identity in question in each case.

CHAPTER IX.

61.1. In this case

$$\psi(s) = \int_0^\infty e^{-sx} dH(x) = \frac{\mu}{\mu+s}$$

for $\mathrm{Re}(s) > -\mu$ and by (59.12) we have

$$\int_0^\infty e^{-sx}\, d_x P\{\beta(a+x) \leq x\} = e^{-\frac{\lambda a s}{\mu+s}}$$

for $\mathrm{Re}(s) \geq 0$ and $a > 0$. Hence by inversion we get

$$P\{\beta(a+x) \leq x\} = e^{-\lambda a}[1 + \sqrt{\lambda\mu a} \int_0^x e^{-\mu u} u^{-1/2} I_1(2\sqrt{\lambda\mu a u})du]$$

for $a > 0$ and $x \geq 0$ where

$$I_1(x) = \sum_{n=0}^\infty \frac{(x/2)^{2n+1}}{n!(n+1)!}$$

is a Bessel function. Thus we have

$$P\{\beta(t) \leq x\} = e^{-\lambda(t-x)}[1 + \sqrt{\lambda\mu(t-x)} \int_0^x e^{-\mu u} u^{-1/2} I_1(2\sqrt{\lambda\mu(t-x)u})\, du]$$

for $0 \leq x < t$.

61.2. If we use the notation of Example 1 in Section 59, then $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are mutually independent random variables for which

$$P\{\alpha_n = 2j-1\} = P\{\beta_n = 2j-1\} = \frac{1}{2j-1}\binom{2j}{j}\frac{1}{2^{2j}} \sim \frac{1}{\sqrt{4\pi j^3}}$$

as $j \to \infty$ $(j = 1,2,\ldots)$. Hence

$$\lim_{x\to\infty} [1-G(x)]x^{1/2} = \lim_{x\to\infty} [1-H(x)]x^{1/2} = \sqrt{\frac{2}{\pi}} ,$$

and by (59.62) we obtain that

$$\lim_{t\to\infty} \underset{\sim}{P}\{\beta(t) \leq tx\} = \frac{2}{\pi} \text{ arc sin } \sqrt{x}$$

for $0 \leq x \leq 1$. For a direct proof see (37.166).

61.3. Denote by $\Delta_n(t)$ the number positive elements in the sequence $\xi(\frac{rt}{n})$ $(r = 1,2,\ldots, n)$. Since $\xi(\frac{rt}{n}) - \xi(\frac{(r-1)t}{n})$ $(r = 1,2,\ldots, m)$ are mutually independent, identically distributed, symmetric random variables for which $\underset{\sim}{P}\{\xi(\frac{rt}{n}) = 0\} = 0$, by the solution of Problem 27.1 we have

$$\underset{\sim}{P}\{\Delta_n(t) = j\} = \binom{2j}{j}\binom{2n-2j}{n-j} \frac{1}{2^{2n}}$$

for $j = 0,1,\ldots, n$. Thus by (37.166) we have

$$\lim_{n\to\infty} \underset{\sim}{P}\{ \frac{\Delta_n(t)}{n} \leq x\} = \frac{2}{\pi} \text{ arc sin } \sqrt{x}$$

for $0 \leq x \leq 1$. Now by Theorem 52.3 we can conclude **that**

$$\underset{\sim}{P}\{ \frac{\beta(t)}{t} \leq x\} = \lim_{n\to\infty} \underset{\sim}{P}\{ \frac{\Delta_n(t)}{n} \leq x\}$$

for $0 \leq x \leq 1$ and therefore

$$\underset{\sim}{P}\{\beta(t) \leq tx\} = \frac{2}{\pi} \text{ arc sin} \sqrt{x}$$

for $0 \leq x \leq 1$.

61.4. Let us use the notation of Example 1 in Section 59. In this case $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed random variables. We have

$$\mathbf{P}\{\alpha_n = 2j-1\} = \frac{1}{2j-1}\binom{2j}{j}\frac{1}{2^{2j}}$$

for $j = 1,2,\ldots$ and $\mathbf{E}\{\beta_n\} = m$. Hence

$$\lim_{x\to\infty} [1-G(x)]x^{1/2} = \sqrt{\frac{2}{\pi}}$$

and by (59.52) we can conclude that

$$\lim_{t\to\infty} \mathbf{P}\{\sqrt{\frac{2}{\pi}}\ \frac{\beta(t)}{mt^{1/2}} \leq x\} = \mathbf{P}\{\gamma^{-1/2} \leq x\}$$

where γ has a stable distribution of type $S(\frac{1}{2}, 1, \sqrt{\frac{\pi}{2}}, 0)$. Thus we can write that $\gamma = \pi/2\gamma^{*2}$ where $\mathbf{P}\{\gamma^{*} \leq x\} = \Phi(x)$, the normal distribution function. Thus

$$\lim_{t\to\infty} \mathbf{P}\{\beta(t) \leq x\sqrt{t}\} = \mathbf{P}\{|\gamma^{*}| \leq \frac{x}{m}\} = 2\Phi(\frac{x}{m}) - 1$$

for $x \geq 0$.

61.5. We shall use the same notation as in the proof of Theorem 59.2. In this case

$$\lim_{n\to\infty} \mathbf{P}\{\frac{\delta_n}{n^{1/\alpha}\ r(n)} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha, 1, \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}, 0)$ and $r(t)$ satisfies the relation

$$\lim_{t \to \infty} \frac{h(t^{1/\alpha} r(t))}{(r(t))^{\alpha}} = 1 .$$

(See Problem 46.13.) We note that $\lim_{t \to \infty} r(\omega t)/r(t) = 1$ for any $\omega > 0$. If we define $\rho(t)$ by (59.5), then we have

$$\frac{\rho(t)}{t} \Rightarrow \frac{1}{A}$$

as $t \to \infty$. Thus by Theorem 45.4 we have

$$\lim_{t \to \infty} \mathbf{P}\{\frac{A^{1/\alpha}}{t^{1/\alpha}} \frac{\delta_{\rho}(t)}{r(t)} \leq x\} = R(x)$$

regardless of whether $\{\alpha_n\}$ depends on $\{\beta_n\}$ or not. If we define

$$u = t + xr(t)(t/A)^{1/\alpha} ,$$

then

$$\lim_{u \to \infty} \frac{t[r(u^{\alpha})]^{\alpha}}{A\, u^{\alpha}} = \frac{1}{x^{\alpha}}$$

for $x > 0$ and thus by (59.6) we have

$$\lim_{u \to \infty} P\{\beta(u) \leq u - t\} = R(x)$$

for $x > 0$. Accordingly,

$$\lim_{u \to \infty} P\{ \frac{[u - \beta(u)][r(u^{\alpha})]^{\alpha}}{A\, u^{\alpha}} \leq \frac{1}{x^{\alpha}}\} = 1-R(x)$$

for $x > 0$, or

$$\lim_{t \to \infty} P\{ \frac{[t-\beta(t)][r(t^{\alpha})]^{\alpha}}{A\, t^{\alpha}} \leq x\} = 1-R(\frac{1}{x^{1/\alpha}}) = G_{\alpha}(\Gamma(1-\alpha)x)$$

for $x > 0$ where $G_\alpha(x)$ is defined by (42.178) .

61.6. For each $t \geqq 0$ let us define $\omega(t)$ as a discrete random variable which takes on positive integers only and which satisfies

$$\{\omega(t) \leqq n\} \equiv \{\delta_n > t\}$$

for $t \geqq 0$ and $n = 0,1,2,\ldots$. Then by (59.1) we can write that

$$\underset{\sim}{P}\{\beta(t) \leqq x\} = 1-\underset{\sim}{P}\{\gamma_{\omega(x)} < t-x\}$$

for $0 \leqq x \leqq t$.

In our case

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\gamma_n}{n^{1/\alpha} r(n)} \leqq x\right\} = R(x)$$

where $R(x)$ and $r(x)$ have the same meaning as in the solution of Problem 61.4 . Furthermore, we have

$$\frac{\omega(t)}{t} \Rightarrow \frac{1}{B}$$

as $t \to \infty$. Thus by Theorem 45.4 we have

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{B^{1/\alpha}\, \gamma_{\omega(t)}}{t^{1/\alpha} r(t)} \leqq x\right\} = R(x)$$

regardless of whether $\{\alpha_n\}$ depends on $\{\beta_n\}$ or not.

If we define

$$u = t + xr(t)(t/B)^{1/\alpha} ,$$

then

$$\lim_{u \to \infty} \frac{t[r(u^{\alpha})]^{\alpha}}{A u^{\alpha}} = \frac{1}{x^{\alpha}}$$

for $x > 0$, and consequently

$$\lim_{u \to \infty} \underset{\sim}{P}\{\beta(u) \leq t\} = 1-R(x)$$

for $x > 0$. Hence we get

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\beta(t)[r(t^{\alpha})]^{\alpha}}{B t^{\alpha}} \leq x \right\} = 1 - R\left(\frac{1}{x^{1/\alpha}}\right) = G_{\alpha}(\Gamma(1-\alpha)x)$$

for $x > 0$ where $G_{\alpha}(x)$ is defined by (42.178).

61.7. By Theorem 59.6 we obtain that

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\beta(t) - \frac{B_1 t}{A_1+B_1}}{\left(\frac{A_1}{A_1+B_1}\right)^{3/2} t^{1/2}} \leqq x \right\} = \underset{\sim}{P}\left\{ \frac{A_1 B_2 \delta - B_1 A_2 \gamma}{A_1^{3/2}} \leqq x \right\}$$

where $\underset{\sim}{P}\{\delta \leqq x, \gamma \leqq y\} = F(x,y)$. Hence it follows that

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\beta(t) - M_1 t}{M_2 t^{1/2}} \leqq x \right\} = \Phi(x)$$

where $M_1 = B_1/(A_1 + B_1)$,

$$M_2 = \frac{(A_1^2 B_2^2 + B_1^2 A_2^2 - 2r A_1 B_1 A_2 B_2)^{1/2}}{(A_1 + B_1)^{3/2}},$$

and $\Phi(x)$ is the normal distribution function of type $N(0,1)$.

61.8. (i) If $\Phi(s,q) = e^{-s^{\alpha} - q^{\alpha}}$ for $Re(s) \geqq 0$ and $Re(q) \geqq 0$, and $0 < \alpha < 1$, then by (59.131) we have

$$V(x) = \frac{x^{\alpha}}{1 + x^{\alpha}}$$

for $x \geqq 0$ and therefore

$$\frac{d\,Q(x)}{dx} = \frac{x^{\alpha-1} \sin \alpha\pi}{\pi(1+2x^{\alpha}\cos\alpha\pi + x^{2\alpha})}$$

for $x > 0$. Thus by (59.109)

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\beta(t)}{t} \leqq x \right\} = 1 - Q\left(\frac{B_2(1-x)}{A_2 x}\right)$$

for $0 < x \leqq 1$.

(ii) If $\Phi(s,q) = e^{-(s+q)^{\alpha}}$ for $Re(s+q) \geqq 0$, and $0 < \alpha < 1$, then by (59.131) we have

$$V(x) = 1 - \frac{1}{(1+x)^{\alpha}}$$

for $x \geqq 0$ and therefore

$$\frac{dQ(x)}{dx} = \begin{cases} \dfrac{\sin\alpha\pi}{\pi x(x-1)^{\alpha}} & \text{for } x > 1, \\ 0 & \text{for } x \leqq 1. \end{cases}$$

Thus by (59.109)

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\beta(t)}{t} \leqq x \right\} = 1 - Q\left(\frac{B_2(1-x)}{A_2 x}\right)$$

for $0 < x \leqq 1$.

CHAPTER X .

65.1. First, let us consider a general single-server queue in which customers arrive at a counter at times $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ where $\tau_0 = 0$. Denote by χ_n the service time of the customer arriving at time τ_n $(n = 0,1,2,\ldots)$. Let η_0 be the initial occupation time of the server immediately before $t = 0$. Let

$$\chi(u) = \sum_{0<\tau_n \leq u} \chi_n$$

for $u \geq 0$.

Now we shall prove that

$$\theta(t) = \sup\{0 \text{ and } u - \chi(u) - \eta_0 - \chi_0 \text{ for } 0 \leq u \leq t\}$$

for $t \geq 0$.

Define $\gamma_n = \eta_0 + \chi_0 + \ldots + \chi_{n-1}$ for $n = 1,2,\ldots$ and $\gamma_0 = 0$.

Let $\tau_n \leq t \leq \tau_{n+1}$. If at time t the server is busy, then $\theta(t) = \theta(\tau_n)$ and $\theta(\tau_n) \geq t - \gamma_{n+1}$. If at time t the server is idle, then $\theta(t) = t - \gamma_{n+1}$ and $t - \gamma_{n+1} \geq \theta(\tau_n)$. Thus we have

$$\theta(t) = \max(\theta(\tau_n) , t - \gamma_{n+1})$$

for $\tau_n \leq t \leq \tau_{n+1}$ and $n = 0,1,2,\ldots$. In particular, $\theta(\tau_{n+1}) = \max(\theta(\tau_n) , \tau_{n+1} - \gamma_{n+1})$ for $n = 0,1,\ldots$ and $\theta(\tau_0) = 0$, and consequently

$$\theta(\tau_n) = \max(0, \tau_1 - \gamma_1, \ldots, \tau_n - \gamma_n)$$

for $n = 1,2,\ldots$.

These relations are valid for any single-server queue.

Now let us suppose that $\tau_0, \tau_1, \ldots, \tau_n, \ldots, \chi_0, \chi_1, \ldots, \chi_n, \ldots$ and η_0 are numerical (non-random) quantities for which $\tau_0 = 0 < \tau_1 < \ldots < \tau_n < \ldots$ and $\tau_n \to \infty$ as $n \to \infty$. If we write $\theta(t) = t - \gamma_{n+1} + [\theta(\tau_n) + \gamma_{n+1} - t]^+$ for $\tau_n \leqq t \leqq \tau_{n+1}$ $(n = 0,1,\ldots)$ and if we use (54.17) in calculating the integral

$$q \int_{\tau_n}^{\tau_{n+1}} e^{-qt-s\theta(t)}\, dt$$

for $n = 0,1,2,\ldots$, then we obtain that

$$q \int_0^\infty e^{-qt-s\theta(t)} dt = 1 - \frac{s}{q+s} e^{-q(\eta_0 + \chi_0)} +$$

$$+ \frac{s}{q+s} \left\{ \sum_{n=0}^{\infty} e^{-q\gamma_{n+1} - (q+s)\theta(\tau_{n+1})} (1 - e^{-q\chi_n}) \right\}$$

for $Re(q) > 0$ and $Re(s) \geqq 0$.

Now let us suppose that $\underset{\sim}{P}\{\eta_0 = 0\} = 1$ and χ_n $(n = 0,1,\ldots)$ and $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots)$ are independent / and identically distributed positive random variables. Let $\underset{\sim}{E}\{e^{-s\chi_n}\} = \psi(s)$ and $\underset{\sim}{E}\{e^{-s(\tau_n - \tau_{n-1})}\} = \phi(s)$ for $Re(s) \geqq 0$. Then the above identity holds for almost all realizations of the queuing process. If we form its expectation, then we obtain that

$$q \int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\theta(t)}\}\, dt = \frac{q}{q+s} + \frac{s}{q+s} [1 - \psi(q)] V(q, q+s)$$

for $Re(q) > 0$ and $Re(s) \geqq 0$ where

/ sequences of mutually independent

$$V(q,s) = 1 + \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-q\gamma_{n+1} - s\theta(\tau_{n+1})}\} =$$

$$= e^{-\underset{\sim}{T}\{\log[1-\phi(s)\psi(q-s)]\}} .$$

This last equation can be proved by using Theorem 4.1 . The distribution function $\underset{\sim}{P}\{\theta(t) \leq x\}$ can be obtained by inversion from the above transform.

We note that if $\underset{\sim}{P}\{\eta_0 = 0\} = 1$ and $\underset{\sim}{P}\{\chi_0 = 0\} = 1$, then

$$\theta(t) = \sup_{0 \leq u \leq t} [u - \chi(u)]$$

for $t \geq 0$, and $\underset{\sim}{P}\{\theta(t) \leq x\}$ can be obtained by the solution of Problem 58.7 .

65.2. Since $\theta(t)$ is a nondecreasing function of t , the limit $\lim_{t \to \infty} \underset{\sim}{P}\{\theta(t) \leq x\} = V(x)$ exists for every x and by the solution of Problem 65.1 we have

$$\Omega^*(s) = \int_0^{\infty} e^{-sx}\, dV(x) = \lim_{q \to +0} q \int_0^{\infty} e^{-qt}\, \underset{\sim}{E}\{e^{-s\theta(t)}\} dt$$

for $Re(s) > 0$. Thus

$$\Omega^*(s) = \lim_{q \to +0} [1 - \psi(q)]V(q, q+s) .$$

Since

$$[1-\psi(q)]V(q,q+s) = \exp\{-\sum_{n=1}^{\infty} \frac{1}{n} e^{-q\gamma_n} (1-\underset{\sim}{E}\{e^{-(q+s)[\tau_n-\gamma_n]^+}\})\}$$

for $Re(q) > 0$ and $Re(q+s) \geq 0$, it follows that

$$\Omega^*(s) = \exp\{-\sum_{n=1}^{\infty} \frac{1}{n}(1 - \underset{\sim}{E}\{e^{-s[\tau_n - \gamma_n]^+}\})\}$$

for $Re(s) > 0$. If $a < b$, then by Theorem 43.13 we have $\lim_{s\to+0} \Omega^*(s) = 1$, that is, $V(x)$ is a proper distribution function and its Laplace-Stieltjes transform is given by $\Omega^*(s)$ for $Re(s) \geq 0$. It is interesting to point out that by Theorem 62.2 we can conclude that $V(x)$ is the limiting distribution of the actual waiting time of the n-th arriving customer in the inverse queuing process, that is, in the queuing process in which the inter-arrival times and service times are interchanged.

65.3. By Theorem 62.2 we have

$$\lim_{n\to\infty} \underset{\sim}{P}\{\eta_n \leq x\} = \underset{\sim}{P}\{\sup\ (0, \chi_0 - \tau_1, \chi_0 + \chi_1 - \tau_2, \ldots) \leq x\}$$

and obviously,

$$\sup_{0\leq u<\infty} [\chi(u) - u] = \sup(0, \chi_1 - \tau_1, \chi_1 + \chi_2 - \tau_2, \ldots) .$$

Since $\{\tau_n - \tau_{n-1}\ , n = 1,2,\ldots\}$ and $\{\chi_n\ , n = 0,1,2,\ldots\}$ are independent sequences of mutually independent and identically distributed random variables the assertion follows.

65.4. Since

$$\underset{\sim}{P}\{\eta_n \leq x|\eta_0 = 0\} = \underset{\sim}{P}\{\chi_0 - \tau_1 \leq x,\ \chi_0 + \chi_1 - \tau_2 \leq x, \ldots,\ \chi_0 + \ldots + \chi_{n-1} - \tau_n \leq x\}$$

and

$$\underset{\sim}{P}\{\rho_0^* > n|\eta_0^* = x\} = \underset{\sim}{P}\{\chi_0 \leq \tau_1 + x,\ \chi_0 + \chi_1 \leq \tau_2 + x, \ldots,\ \chi_0 + \ldots + \chi_{n-1} \leq \tau_n + x\}$$

for $n = 1,2,\ldots$, the assertion follows immediately. We note that

$$\lim_{n \to \infty} \mathbf{P}\{\eta_n \leq x \mid \eta_0 = 0\} = 1 - \mathbf{P}\{\rho_0^* < \infty \mid \eta_0^* = x\}$$

for $x > 0$.

65.5. We can interpret $G^{(r)}(x)$ as the probability that the length of the initial busy period is $\leq x$ provided that the initial queue size is r. Denote by $\chi_1, \chi_2, \ldots, \chi_n$ the lengths of the first n service times and by $\nu_1, \nu_2, \ldots, \nu_n$ the number of customers arriving during the 1-st, 2-nd, ..., n-th service time respectively. If we use Lemma 20.2, then we obtain that the probability that the initial busy period has length $\leq x$ and consists of n services is given by

$$G_n^{(r)}(x) = \mathbf{P}\{\chi_1 + \ldots + \chi_n \leq x,\ \nu_1 + \ldots + \nu_i > i - r \text{ for } i = r, \ldots, n-1$$

$$\text{and } \nu_1 + \ldots + \nu_n = n - r\} = \mathbf{P}\{\chi_1 + \ldots + \chi_n \leq x,\ \nu_1 + \ldots + \nu_i < i \text{ for } i = 1, \ldots, n-r$$

$$\text{and } \nu_1 + \ldots + \nu_n = n - r\} = \frac{r}{n} \mathbf{P}\{\nu_1 + \ldots + \nu_n = n - r \text{ and } \chi_1 + \ldots + \chi_n \leq x\} =$$

$$= \frac{r}{n} \int_0^x e^{-\lambda u} \frac{(\lambda u)^{n-r}}{(n-r)!} \, dH_n(u)$$

for $x \geq 0$. Finally,

$$G^{(r)}(x) = \sum_{n=r}^{\infty} G_n^{(r)}(x).$$

65.6. Let us define ξ_n $(n = 1, 2, \ldots)$ by (62.9) and let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots,$ and $\zeta_0 = 0$. Then $\mathbf{E}\{\xi_n\} = 0$ and $\mathbf{Var}\{\xi_n\} = \sigma_a^2 + \sigma_b^2$. By (62.12) we can conclude that η_n has the same asymptotic distribution as $\max_{0 \leq k \leq n} \zeta_k$ regardless of the distribution of η_0. Thus by the Theorem 45.6 we have

$$\lim_{n \to \infty} P\{ \frac{\eta_n}{\sqrt{(\sigma_a^2 + \sigma_b^2)n}} \leq x\} = 2\Phi(x) - 1$$

for $x \geq 0$ where $\Phi(x)$ is the normal distribution function.

Denote by $\nu(t)$ the number of arrivals in the time interval $(0, t)$. Then $\nu(t)/t \Rightarrow 1/a$ as $t \to \infty$. We can easily see that $\eta(t)$ has the same asymptotic distribution as $\eta_{\nu(t)}$. Thus by Theorem 45.5 we obtain that

$$\lim_{t \to \infty} P\{ \frac{\eta(t)}{\sqrt{(\sigma_a^2 + \sigma_b^2)t/a}} \leq x\} = 2\Phi(x) - 1$$

for $x \geq 0$.

65.7. Let us define ξ_n $(n = 1,2,\ldots)$ by (62.9) and let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. By (62.12) we can conclude that η_n has the same asymptotic distribution as $\max_{0 \leq k \leq n} \zeta_k$ regardless of the distribution of η_0. In our case

$$\lim_{n \to \infty} P\{ \frac{\chi_1 + \ldots + \chi_n - na}{n^{1/\alpha}\rho(n)} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha, 1, \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}, 0)$ and

$$\lim_{t \to \infty} t[1 - H(t^{1/\alpha}\rho(t))] = 1 .$$

Furthermore, we have

$$\frac{\tau_n - na}{n^{1/\alpha}} \Longrightarrow 0$$

as $n \to \infty$. Thus it follows that

$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\zeta_n}{n^{1/\alpha} \rho(n)} \leq x\} = R(x) .$$

Now by Theorem 45.10 it follows that

$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\eta_n}{n^{1/\alpha} \rho(n)} \leq x\} = Q(x)$$

where

$$Q(x) = \underset{\sim}{P}\{ \sup_{0 \leq u \leq 1} \xi(u) \leq x\}$$

and $\{\xi(u) , 0 \leq u \leq 1\}$ is a separable stable process of type $S(\alpha,1,\Gamma(1-\alpha)\cos \frac{\alpha\pi}{2},0$ The distribution function $Q(x)$ can be determined by (45.232) .

If $\nu(t)$ denotes the number of arrivals in the time interval $(0,t)$, then $\nu(t)/t \Longrightarrow 1/a$ as $t \to \infty$. Since $\eta(t)$ has the same asymptotic distribution as $\eta_{\nu(t)}$, by Theorem 45.5 it follows that

$$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\eta(t)\, a^{1/\alpha}}{t^{1/\alpha} \rho(t)} \leq x\} = Q(x)$$

also holds.

65.8. By (62.167) we have

$$\eta(t) = \eta_0 + \chi_0 + \chi(t) - \sigma(t)$$

where $\chi(t)$ is defined by (62.166) and $0 \leq \sigma(t) \leq t$. If

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\chi(t) - D_1(t)}{D_2(t)} \leq x \right\} = Q(x)$$

exists and $\lim_{t \to \infty} D_2(t)/t = \infty$, then obviously

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\eta(t) - D_1(t)}{D_2(t)} \leq x \right\} = Q(x)$$

also holds. In our case

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\tau_n}{(na_1)^{1/\alpha_1}} \leq x \right\} = R_1(x)$$

where $R_1(x)$ is a stable distribution function of type $S(\alpha_1, 1, \Gamma(1-\alpha_1)\cos\frac{\alpha_1\pi}{2}, 0)$ and

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\chi_1 + \ldots + \chi_n}{(na_2)^{1/\alpha_2}} \leq x \right\} = R_2(x)$$

where $R_2(x)$ is a stable distribution function of type $S(\alpha_2, 1, \Gamma(1-\alpha_2)\cos\frac{\alpha_2\pi}{2}, 0)$.
Thus by (49.205) we obtain that

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\chi(t)}{(a_2 t^{\alpha_1}/a_1)^{1/\alpha_2}} \leq x \right\} = Q(x)$$

where $Q(x) = \underset{\sim}{P}\{\chi\theta^{-\alpha_1/\alpha_2} \leq x\}$ and θ and χ are independent random variables for which $\underset{\sim}{P}\{\theta \leq x\} = R_1(x)$ and $\underset{\sim}{P}\{\chi \leq x\} = R_2(x)$. Since $\alpha_1/\alpha_2 > 1$, it follows that $\eta(t)$ has the same asymptotic distribution as $\chi(t)$ as $t \to \infty$.

65.9. Since $b < a$ and $0 < \sigma_a^2 + \sigma_b^2 < \infty$, it follows that $E\{\theta_n\}$, $Var\{\theta_n\}$ and $E\{\sigma_n\}$, $Var\{\sigma_n\}$ exist. Thus by Theorem 59.6 and by (59.107) we have

$$\lim_{t \to \infty} P\left\{ \frac{\sigma(t) - \frac{B_1 t}{A_1 + B_1}}{\left(\frac{A_1}{A_1 + B_1}\right)^{3/2} t^{1/2}} \leq x \right\} = P\left\{ \frac{A_1 B_2 \delta - B_1 A_2 \gamma}{A_1^{3/2}} \leq x \right\}$$

where $A_1 = E\{\theta_n\}$, $A_2 = \sqrt{Var\{\theta_n\}}$, $B_1 = E\{\sigma_n\}$, $B_2 = \sqrt{Var\{\sigma_n\}}$ and (δ, γ) has a normal distribution of type

$$N\left(\begin{Vmatrix} 0 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 1 & r \\ r & 1 \end{Vmatrix} \right)$$

where $r = Cov\{\theta_n, \sigma_n\}/A_2 B_2$. Accordingly (62.175) holds with

$$M_1 = \frac{B_1}{A_1 + B_1}$$

and

$$M_2^2 = \frac{E\{(A_1 \sigma_n - B_1 \theta_n)^2\}}{(A_1 + B_1)^3} .$$

Denote by ν_n the number of customers served in the n-th busy period. If $b < a$, then $E\{\nu_n\}$ is finite and by Theorem 62.2 we have

$$E\{\nu_n\} = 1/W(0) = \exp\left\{ \sum_{n=1}^{\infty} \frac{P\{\chi_1 + \ldots + \chi_n > \tau_n\}}{n} \right\}.$$

Thus by Theorem 6.1 in the Appendix we have

$$E\{\sigma_n + \theta_n\} = A/W(0)$$

and

$$\mathbf{E}\{\sigma_n\} = b/W(0) ,$$

and by Theorem 6.2 and Theorem 6.3 in the Appendix we have

$$\mathbf{E}\{(\sigma_n + \theta_n - \nu_n a)^2\} = \sigma_a^2/W(0) ,$$

$$\mathbf{E}\{(\sigma_n - \nu_n b)^2\} = \sigma_b^2/W(0)$$

and

$$\mathbf{E}\{(\sigma_n + \theta_n - \nu_n a)(\sigma_n - \nu_n b)\} = \mathbf{Cov}\{\tau_n - \tau_{n-1}, \chi_n\}/W(0) = 0 .$$

Thus $A_1 + B_1 = a/W(0)$, $B_1 = b/W(0)$ and

$$\mathbf{E}\{[a\sigma_n - b(\sigma_n + \theta_n)]^2\} = (a^2\sigma_b^2 + b^2\sigma_a^2)/W(0) .$$

In the last equation we used that

$$a\sigma_n - b(\sigma_n + \theta_n) = a(\sigma_n - \nu_n b) - b(\sigma_n + \theta_n - \nu_n a) .$$

The above formulas prove that (62.175) holds if M_1 is given by (62.176) and M_2 by (62.177) .

65.10. Let us use the same notation as in Theorem 62.9 and denote by ν_n the number of customers served in the n-th busy period. Then by (62.106) we have

$$1-\mathbf{E}\{e^{-w\sigma_n - s\theta_n} \rho^{\nu_n}\} = \exp\{-\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{E}\{e^{-w\gamma_n - s(\tau_n - \gamma_n)} \delta(\tau_n \geq \gamma_n)\}\}$$

for $\text{Re}(s) \geq 0$, $\text{Re}(w) \geq 0$ and $|\rho| \leq 1$. Hence it follows that

$$E\{\theta_n \rho^{\nu_n}\} = \frac{1}{2}[1-E\{\rho^{\nu_n}\}] \sum_{n=1}^{\infty} \frac{\rho^n}{n} E\{|\tau_n - \gamma_n|\}$$

for $|\rho| < 1$. Here we used that $E\{\tau_n - \gamma_n\} = 0$.

Since

$$\frac{E\{(\tau_n - \gamma_n)^2\}}{n(\sigma_a^2 + \sigma_b^2)} = 1 ,$$

it follows that

$$\lim_{n \to \infty} \frac{E\{|\tau_n - \gamma_n|\}}{\sqrt{n(\sigma_a^2 + \sigma_b^2)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-x^2/2}\, dx = \sqrt{\frac{2}{\pi}} .$$

Thus by Theorem 9.3 in the Appendix we can conclude that

$$\lim_{\rho \to 1-0} (1-\rho)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\rho^n}{n} E\{|\tau_n - \gamma_n|\} = \lim_{\rho \to 1-0} (1-\rho)^{\frac{1}{2}} \frac{2E\{\theta_n \rho^{\nu_n}\}}{1-E\{\rho^{\nu_n}\}} =$$

$$= [2(\sigma_a^2 + \sigma_b^2)]^{1/2} .$$

Since

$$\frac{(1-\rho)^{\frac{1}{2}}}{1-E\{\rho^{\nu_n}\}} = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} [P\{\tau_n \geqq \gamma_n\} - \frac{1}{2}]\}$$

for $|\rho| < 1$, it follows that

$$E\{\theta_n\} = A = \left(\frac{\sigma_a^2 + \sigma_b^2}{2}\right)^{\frac{1}{2}} \exp\{-\sum_{n=1}^{\infty} \frac{1}{n} [P\{\tau_n \geqq \gamma_n\} - \frac{1}{2}]\} .$$

If we use the notation $\psi(s) = E\{e^{-s(\tau_n - \tau_{n-1})}\}$ for $Re(s) \geqq 0$, then $\psi(s) = 1-as+o(s)$ as $s \to +0$.

Since

$$\frac{1-E\{e^{-s\sigma_n}\}}{s^{1/2}} = \left(\frac{1-\psi(s)}{s}\right)^{\frac{1}{2}} \exp\{-\sum_{n=1}^{\infty} \frac{1}{n} [E\{e^{-s\gamma_n}\delta(\tau_n \geqq \gamma_n)\} - \frac{1}{2} E\{e^{-s\gamma_n}\}]\}$$

for $\mathrm{Re}(s) > 0$, we obtain that

$$\lim_{s\to+0} \frac{1-E\{e^{-s\sigma_n}\}}{s^{1/2}} = A\left(\frac{2a}{\sigma_a^2+\sigma_b^2}\right)^{1/2} .$$

Hence

$$\lim_{x\to\infty} P\{\sigma_n > x\}\, x^{1/2} = \frac{A}{\pi^{1/2}} \left(\frac{2a}{\sigma_a^2+\sigma_b^2}\right)^{\frac{1}{2}}$$

and

$$\lim_{n\to\infty} P\left\{\frac{\sigma_1+\sigma_2+\dots+\sigma_n}{n^2A^2a/(\sigma_a^2+\sigma_b^2)} \leqq x\right\} = 2\left[1-\Phi\left(\frac{1}{\sqrt{x}}\right)\right]$$

for $x > 0$. This limit theorem and the relation

$$\frac{\theta_1+\theta_2+\dots+\theta_n}{n} \Longrightarrow A$$

as $n \to \infty$, by the solution of Problem 61.5 or by the 7-th statement of Theorem 59.2, imply that

$$\lim_{t\to\infty} P\left\{\frac{a^{1/2}\,\theta(t)}{[(\sigma_a^2+\sigma_b^2)t]^{1/2}} \leqq x\right\} = 2\Phi(x) - 1$$

for $x \geqq 0$ where $\Phi(x)$ is the normal distribution function.

65.11. Let us use the same notation as in the solution of Problem 65.10 . In this case by (61.191) and (61.192) we have

$$\lim_{n \to \infty} \mathbf{P}\{ \frac{\tau_n - \gamma_n}{(nh)^{1/\alpha}} \leqq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha,-1,\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2},0)$. Hence it follows that

$$\lim_{n \to \infty} \frac{\mathbf{E}\{|\tau_n - \gamma_n|\}}{(nh)^{1/\alpha}} = \int_{-\infty}^{\infty} |x| dR(x) = \frac{2[-\Gamma(1-\alpha)]^{1/\alpha}}{\Gamma(\frac{1}{\alpha})}$$

(See (42.198).) Thus by Theorem 9.3 in the Appendix it follows that

$$\lim_{\rho \to 1-0} (1-\rho)^{\frac{1}{\alpha}} \sum_{n=0}^{\infty} \frac{\rho^n}{n} \mathbf{E}\{|\tau_n - \gamma_n|\} =$$

$$= \lim_{\rho \to 1-0} (1-\rho)^{\frac{1}{\alpha}} \frac{2\mathbf{E}\{\theta_n \rho^{\nu_n}\}}{1-\mathbf{E}\{\rho^{\nu_n}\}} = 2h^{1/\alpha} [-\Gamma(1-\alpha)]^{1/\alpha} .$$

Since

$$\frac{(1-\rho)^{\frac{1}{\alpha}}}{1-\mathbf{E}\{\rho^{\nu_n}\}} = \exp\{ \sum_{n=1}^{\infty} \frac{\rho^n}{n} [\mathbf{P}\{\tau_n \geqq \gamma_n\} - \frac{1}{\alpha}]\}$$

for $|\rho| < 1$, it follows that

$$\mathbf{E}\{\theta_n\} = A = h^{1/\alpha}[-\Gamma(1-\alpha)]^{\frac{1}{\alpha}} \exp\{- \sum_{n=1}^{\infty} \frac{1}{n} [\mathbf{P}\{\tau_n \geqq \gamma_n\} - \frac{1}{\alpha}]\} .$$

If we use the notation $\psi(s) = \underset{\sim}{E}\{e^{-s(\tau_n - \tau_{n-1})}\}$ for $\mathrm{Re}(s) \geq 0$, then we have

$$1 - \psi(s) = as + \Gamma(1-\alpha)hs^\alpha + o(s^\alpha)$$

as $s \to +0$. Since

$$\frac{1-\underset{\sim}{E}\{e^{-s\sigma_n}\}}{s^{1/\alpha}} = [\frac{1-\psi(s)}{s}]^{\frac{1}{\alpha}} \exp\{-\sum_{n=1}^{\infty} \frac{1}{n} [\underset{\sim}{E}\{e^{-s\gamma_n} \delta(\tau_n \geq \gamma_n)\} - \frac{1}{\alpha} \underset{\sim}{E}\{e^{-s\gamma_n}\}]\}$$

for $\mathrm{Re}(s) > 0$, we obtain that

$$\lim_{s \to +0} \frac{1-\underset{\sim}{E}\{e^{-s\sigma_n}\}}{s^{1/\alpha}} = \frac{A\, a^{1/\alpha}}{h^{1/\alpha}[-\Gamma(1-\alpha)]^{1/\alpha}} .$$

Accordingly, we have

$$\lim_{x \to \infty} \underset{\sim}{P}\{\sigma_n > x\}x^{\frac{1}{\alpha}} = \frac{A\, a^{1/\alpha}}{\Gamma(1-\frac{1}{\alpha})h^{1/\alpha}[-\Gamma(1-\alpha)]^{1/\alpha}}$$

and thus

$$\lim_{n \to \infty} \underset{\sim}{P}\{\frac{(\sigma_1+\sigma_2+\ldots+\sigma_n)h[-\Gamma(1-\alpha)]}{a\, A^\alpha n^\alpha} \leq x\} = R^*(x)$$

where $R^*(x)$ is a stable distribution function of type $S(\frac{1}{\alpha}, 1, \cos\frac{\pi}{2\alpha}, 0)$. Furthermore, we have

$$\frac{\theta_1 + \theta_2 + \ldots + \theta_n}{n} \Longrightarrow A$$

as $n \to \infty$. Thus by the solution of Problem 61.5 or by the 7-th statement of Theorem 59.2 we obtain that

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\theta(t)a^{1/\alpha}}{h^{1/\alpha}[-\Gamma(1-\alpha)]^{1/\alpha}t^{1/\alpha}} \leq x \right\} = 1 - R^{*}\left(\frac{1}{x^{\alpha}}\right) = G_{1/\alpha}(x)$$

for $x > 0$ where $G_{1/\alpha}(x)$ is defined by (42.178). This result is in agreement with (62.194).