Option contracts for power system balancing Part 2: Geometric solution of optimal stopping problems

John Moriarty (Queen Mary University of London)

YEQT XI: "Winterschool on Energy Systems", Eindhoven

14th December 2017

Recall that the hiring problem was purely combinatorial. Suppose instead that we wish to stop a **Brownian motion** optimally.

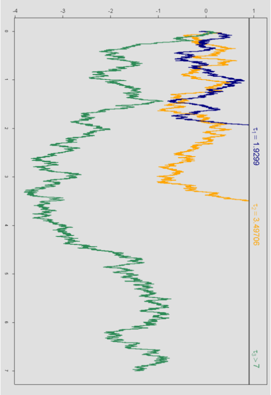


Figure: Some simulated hitting times for Brownian motion

(source: Thomas Steiner)

[The following formulation is modified from Pedersen (2005) and Dayanik and Karatzas (2003).]

Let $X = (X_t)_{t \geq 0}$ be

- a standard Brownian motion (ie $dX_t = dW_t$),
- taking values in an interval $\mathscr{I}$ with endpoints a and b,
- with initial value x,

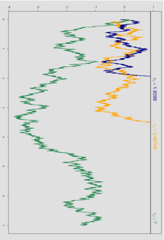
defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$.

Let $\mathscr{S}$ be the set of all *stopping times*. That is, each τ is

- a nonnegative random variable,
- non-ancitipative: that is, for each $t \geq 0$ we have

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathscr{F}_t.$$

A stopping time can be interpreted as *the time at which X exhibits a given behaviour of interest.*



A basic optimal stopping problem

Find

$$v(x) := \sup_{\tau \in \mathscr{S}} \mathbb{E}^x[h(X_\tau)], \qquad x \in \mathscr{I} \tag{1}$$

and, if it exists, an optimal stopping time τ_* satisfying

$$v(x) = \mathbb{E}^x[h(X_{\tau_*})].$$

Here

- v is called the **value function**
- h is the real-valued **gain function**

and for simplicity, we will take

- h continuous on $\mathbb{R}$, and
- define $h(X_\tau) = 0$ on $\{\tau = +\infty\}$: *never stopping* $\Rightarrow$ *zero gain.*

The function $f : \mathbb{R} \to \mathbb{R}$ is said to be **excessive for** X if

$$f(x) \geq \mathbb{E}^x[f(x_t)], \qquad \forall\, t \geq 0, \forall\, x \in \mathscr{I}, \tag{2}$$

and **superharmonic for** X if

$$f(x) \geq \mathbb{E}^x[f(x_\tau)], \qquad \forall\, \tau \in \mathscr{S}, \forall\, x \in \mathscr{I}. \tag{3}$$

- Clearly, if f is superharmonic for X then it is also excessive for X (take $\tau = t$ a.s.)
- Let $\mathscr{L}(X)$ be the class of all lower semicontinuous real functions f such that either $\mathbb{E}^x[\sup_{t\geq 0} f(X_t)] < \infty$ or $\mathbb{E}^x[\inf_{t\geq 0} f(X_t)] > -\infty$. Then excessivity and superharmonicity for X are equivalent on $\mathscr{L}(X)$.

Recall the optimal stopping problem:

$$v(x) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^x[h(X_\tau)], \qquad x \in \mathscr{I}. \tag{4}$$

- By the strong Markov property, v is superharmonic
- Trivially: v *majorises* h (that is, $v \geq h$; just take $\tau = 0$)
- If a superharmonic function f majorises h then it majorises v

This actually characterises v...

First define the **continuation region**

$$C = \{x \in \mathscr{I} : h(x) < v(x)\}$$

and let τ_* be the first exit time of X from C:

$$\tau_* = \inf\{t > 0 : X_t \notin C\}.$$

Theorem (Dynkin 1963)

Suppose that $h \in \mathscr{L}(Z)$. Then:

1. *The value function v is the smallest nonnegative superharmonic majorant of the gain function h with respect to the process X.*
2. *τ_* is an optimal stopping time*
3. *If an optimal stopping time σ exists then $\tau_* \leq \sigma$ $\mathbb{P}_x$–a.s. for all x and τ_* is also an optimal stopping time.*

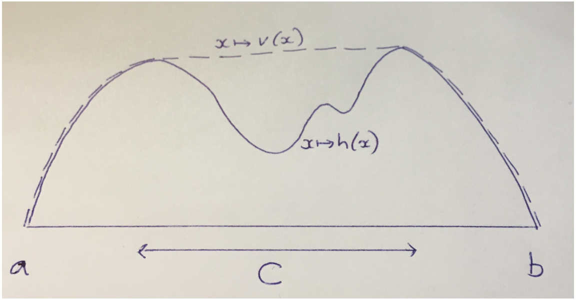


Figure: An example to fix ideas. The continuation region $C = \{x \in [a, b] : h(x) < v(x)\}$.

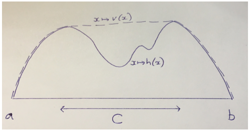
Solutions can be obtained by a geometric method:

Theorem (Dynkin and Yushkevich, 1969)

Every excessive function for one-dimensional Brownian motion X is concave, and vice-versa.

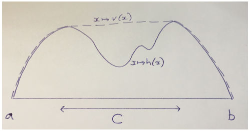
Corollary

Let X be a standard Brownian motion in a closed bounded interval $\mathscr{I} = [a, b]$ and absorbed at its boundaries. Then the value function v is the **smallest nonnegative concave majorant** of h.



Remarks:

- The value function v resembles a rope stretched over the gain function h
- The continuation region C has two boundary points in this example, but there can be many
- These are referred to as **free boundaries** since their position is not specified a priori
- The value function v is linear (that is, **harmonic** for the Brownian motion X) on the open set C
- v is concave (that is, superharmonic for X) on its complement $\overline{C}$, which is the closed **stopping set**



The principle of smooth fit

This famous principle (also called 'smooth pasting' or the 'high contact principle') was first applied in Mikhalevich (1958) and later independently in Chernoff (1961) and Lindley (1961). It asserts that the value function v should be **continuously differentiable across the free boundaries**.

This principle is:

- often used in analytic solution methods:
  - a candidate solution is constructed
  - this candidate is verified analytically
- not necessary, but typically holds in 'nice' problems...

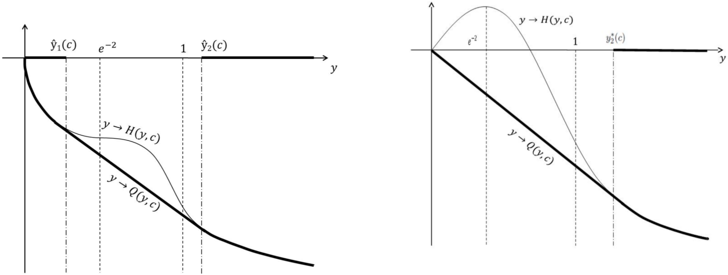


Figure: Example optimal stopping problems (NB of minimisation, not of maximisation). Left: Smooth fit holds at both boundaries. Right: Smooth fit holds only at the right boundary.

This method can be extended to more general optimal stopping problems, with

- time discounting of the gain function:

$$V(x) = \sup_{\tau \in \mathscr{S}} \mathbb{E}^x[e^{-r\tau} h(X_\tau)], \qquad x \in \mathscr{I}, \tag{5}$$

  where $r \geq 0$ is a discount rate (which may be state-dependent $x \mapsto r(x)$)
- taking X as any time-homogeneous regular diffusion: that is,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

This is achieved by:

- Applying a nonlinear scaling to the previous picture
- Equivalently, using a generalised concept of concavity.

Let

- the infinitesimal generator of X be $\mathscr{A}u = \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2} + \mu(x)\frac{\partial u}{\partial x}$
- the equation $\mathscr{A}u = ru$ have fundamental solutions ψ and ϕ (linearly independent, positive, ϕ decreasing, ψ increasing; eg. for Brownian motion and $r = 0$ we have $\phi(x) = 1$, $\psi(x) = x$).

The generalised method is:

Proposition

Let $F = \frac{\psi}{\phi}$ and let W be the smallest nonnegative concave majorant of $H := \frac{h}{\phi} \circ F^{-1}$ on $[F(a), F(b)]$. Then $V(x) = \phi(x)W(F(x))$, for every $x \in [a, b]$.

Too good to be true?

We need to perform the previous procedure of finding the smallest nonnegative concave majorant, taking the gain function $H := \frac{h}{\phi} \circ F^{-1}$ (where $F = \frac{\psi}{\phi}$).

- For Brownian motion (BM) and $r = 0$ we have $\phi(x) = 1$, $\psi(x) = x$
- For geometric Brownian motion (GBM) we have $\phi(x) = e^{-\sqrt{2r}x}$, $\psi(x) = e^{\sqrt{2r}x}$
- However in general, and eg. for the Ornstein-Uhlenbeck process, no explicit forms for $\phi(x)$ or $\psi(x)$ - so don't know the geometry of H precisely

In M. & Palczewski (*EJOR* 2016) we solve an optimal stopping problem for a battery operator providing grid support services under option-type contracts. There, X is Brownian motion with constant discounting and the gain function is non-smooth:

$$-f(x) + p_c + h_c(x),$$

where

$$h_c(x) = \begin{cases} K, & x < x^*, \\ Ke^{-a(x-x^*)}, & x \geq x^*. \end{cases} \tag{6}$$

This produces a surprising variety of solutions!

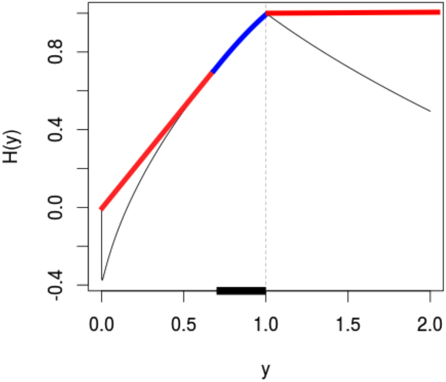


Figure: An example with a bounded interval stopping region (thick black line) with one smooth fit point

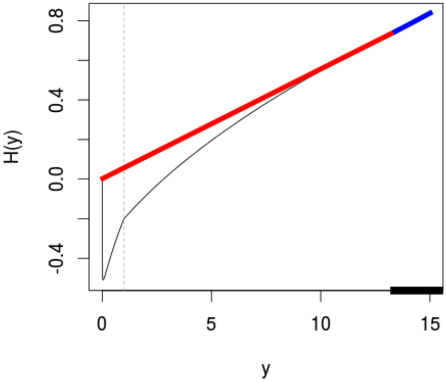


Figure: An unbounded interval stopping region and smooth fit

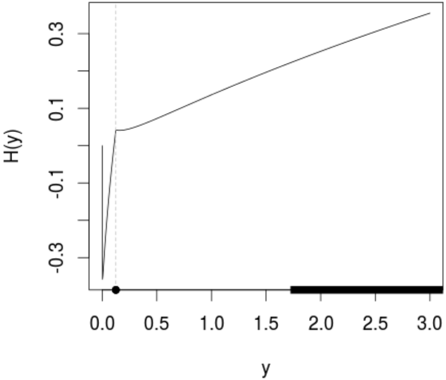


Figure: Stopping region given by union of isolated point and unbounded interval

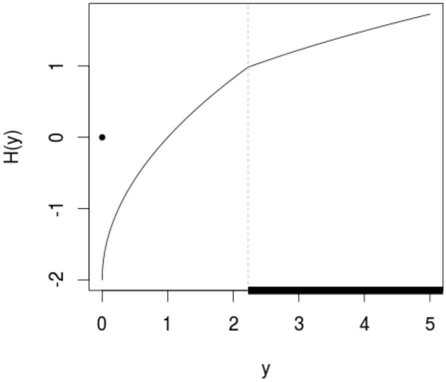


Figure: Stopping region given by unbounded interval, no smooth fit

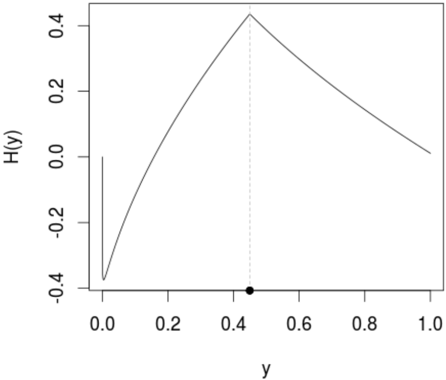


Figure: Stopping region given by an isolated point

Figure: Now to the spaceship

Too simple to be useful?

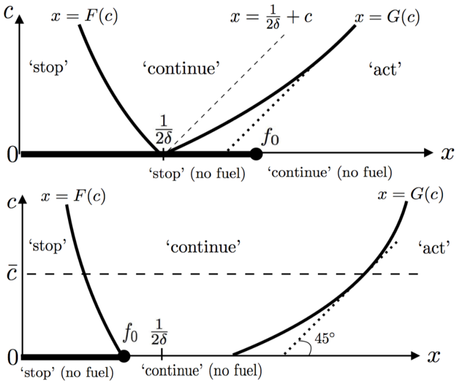


Figure: A new control solution found in M. (2015) to a problem of singular stochastic control with stopping (Karatzas, Ocone, Wang and Zervos, 2000)

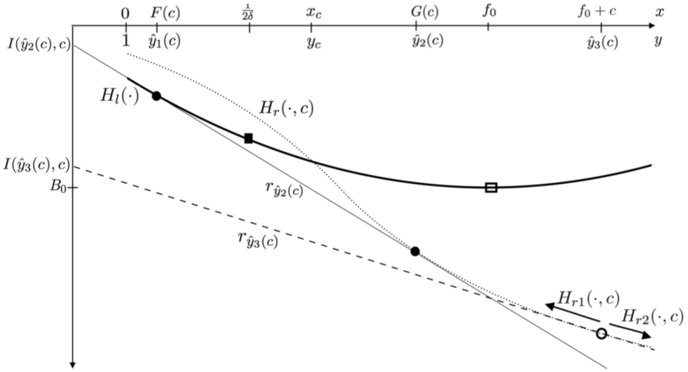


Figure: A related optimal stopping problem from M. (2015)

- When X takes values in R^d **for d $\geq$ 1** (and h is still assumed real-valued and continuous) there are iterative methods:

Iterative solution method 1

Let h satisfy $\mathbb{E}^x[\sup_{t\geq 0} h(X_s)] < \infty$. Define the operator

$$Q_j(h)(x) = h(x) \vee \mathbb{E}^x[h(X_{2^{-j}})]$$

and, writing Q_j^n for its nth power, set

$$h_{j,n}(x) = Q_j^n[h](x).$$

Then the least superharmonic majorant of h is

$$\hat{h}(x) := \lim_{j\to\infty} \lim_{n\to\infty} h_{j,n}(x)$$

Iterative solution method 2

Let X be a Feller process and let h be bounded from below. Set

$$h_j(x) = \sup_{t \geq 0} \mathbb{E}^x(h_{j-1}(X_t))$$

for $j \geq 1$ and $h_0 = h$. Then

$$\hat{h}(x) = \lim_{j \to \infty} h_j(x)$$

is the least superharmonic majorant of h.