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Abstract

In this paper, we study a single-server polling model with two queues. Customers arrive at the queues according to two independent Poisson processes. The server spends random amounts of time in each queue, regardless of the amounts of work present at the queues. The service speed is not constant; it is assumed that the server works at speed $r_i x_i$ at queue i when its current workload equals x_i, $i = 1, 2$. We first focus on the case that all visit times are constant. In the two-queue case we then compute the LST of the steady-state joint workload distribution. Using a different method in which we exploit the independence of the workloads at visit endings, we compute the joint LST of workloads in the case of an arbitrary number of queues with constant visit times.

Next, we consider a two-queue polling model with constant visit times at the first queue and general visit times at the second, and we derive the marginal workload distribution at the first queue. We also investigate the case of a two-queue polling model with exponentially distributed visit times. We determine the steady-state marginal workload distributions, and we formulate a two-dimensional Volterra integral equation for the LST of the steady-state joint workload distribution. We finally show that this equation can be solved by a fixed-point iteration.
Keywords: polling model; time-limited; workload; workload-dependent service speed.

1 Introduction

In this paper, we study a single-server polling model with two queues, in which customers arrive at the queues according to two independent Poisson processes and bring along certain service requirements. Our model has two special features. The first one concerns the visit times of the server to the queues. It is usually assumed in polling models (see [7] for a survey) that the server follows one of the key service disciplines (namely exhaustive, gated or k-limited); and if, for the latter disciplines, the visited queue becomes idle, the server immediately moves to another queue. However, in this paper, we assume that the server spends a certain amount of time (either deterministic or random) at a queue, and stays there until that time is up, even if the queue becomes empty beforehand. When the server moves to the next queue, that switch is assumed to be instantaneous.

The second special feature of the model is, that the service speed at each queue is not constant but workload dependent; it is assumed that the server works at speed $r_i x_i$ at queue i when its current workload equals x_i, $i = 1, 2$. Such a workload process with exponential decay in between upward jumps is called a *shot-noise* process.

Our motivation for studying this polling model is twofold. Our first motivation is an intrinsic interest in the mathematical complexity of polling systems. A service speed which is proportional to the workload has mathematically pleasing properties. That raises the question whether polling models with such a workload-dependent (shot-noise) service speed might be more tractable than classic polling models. For classic polling models, a sharp division is known to exist between "easy" and "hard" models. Service disciplines of so-called branching type give rise to "easy" models [16], for which explicit expressions for (the transform of) the joint queue length distribution can be obtained. However, if the service discipline in at least one queue is not of branching type, then the joint queue length distribution is almost never known; see Section 5 of [7] for a discussion of some exceptions to this rule. When it comes to joint workloads, a very similar sharp division holds; now multi-type Jirina processes (continuous-state discrete-time branching processes) give rise to explicit expressions for (the transform of) the steady-state joint workload distribution [9]. We shall show that polling systems with workload-proportional service speed are quite tractable, even if the service discipline at the queues is not of branching type.

Our second motivation is provided by the increasingly important topic of balancing energy consumption of processors and performance for users. Dynamic scaling techniques like frequency scaling or voltage scaling enable individual computers to adjust their processing speed in accordance with their workload [17]. Since energy consumption is an increasing function of the processor speed, less energy is consumed when the processor has a smaller workload and, accordingly, a lower speed. Another societal argument for studying workload-dependent server speeds is that the service speed of human servers typically is influenced by their workload.

Related literature. In the recent literature, shot-noise processes have been widely studied in the context of queueing, mathematical finance, and insurance, see for example [1,3,4,6,11,13]. However, as far as we know, in polling systems, we are applying the shot-noise process for the first time. In [18] a single server polling model with two queues and random (exponentially distributed) visit times was analyzed, with constant arrival and service rates. In the present paper, we analyze the same model where the service rates are not constant but workload-dependent. Other research that is relevant to this work is [17], where the authors study a queueing system in which the service speed is a function of the workload, and in which the server switches off when the system becomes empty, only to be activated again when the workload reaches a certain threshold. For this system, the authors obtained the steady-state workload distribution.

Main contributions. In Theorem 4.1, we obtain an explicit expression for the joint workload LST, for the two-queue polling model in the case of constant visit times. Using a different method, we generalize this result in Theorem 5.1 to the case of an arbitrary number of queues with constant visit times and Lévy subordinator inputs. This provides a rare example of a polling system in which an exact joint workload distribution can be obtained even though the so-called branching property is violated. When one of the visit times is not constant, we are no longer able to obtain the joint workload LST. However, for the case of one constant and one general visit time, we derive detailed exact and asymptotic results for the marginal workload LST at the queue with constant visit times. Moreover, for the case of two queues with exponentially distributed visit times, we are able to determine the marginal workload LST and to obtain a two-dimensional Volterra integral equation for the joint workload LST. The associated Volterra operator has an unbounded kernel, and is therefore of non-standard type. It has the joint workload LST as an eigenfunction with eigenvalue 1. While we have not been able to find the joint workload LST analytically, we show that it can be found numerically by a fixed-point iteration, the Volterra operator being a weak contraction.

Structure of the paper. In Section 2, we present some results for a single server queue with workload-proportional server speed – a shot-noise queue. The two-queue polling model under consideration in this paper is described in Section 3. Section 4 contains the derivation of the joint workload LST in the two-queue case with constant visit times. This result is extended to the case of an arbitrary number of queues in Section 5. Sections 6 and 7 are successively devoted to the case of constant visit times at queue 1 and general visit times at queue 2, and to the case of exponential visit times at both queues.

2 Preliminaries and notation

In this section we briefly review the case of an $M/G/1$ queue with the special feature that the server works at speed rx when the workload is x, with $r > 0$. Suppose that the Poisson arrival process $\{A(t),\ t \geq 0\}$ has rate λ and that the service requirements $(B_i)_{i\in\mathbb{N}}$ are i.i.d. with distribution $B(\cdot)$ and LST $\beta(\cdot)$. Denote the workload of the system at time t by $\{X(t),\ t \geq 0\}$. Due to the linear service speed assumption, it holds that (see [15, Eqn. (1)])

$$X(t) = X(0)e^{-rt} + \sum_{i=1}^{A(t)} B_i e^{-r(t-t_i)}, \quad t \geq 0, \tag{1}$$

with t_i the arrival epoch of customer i, $i = 1, \ldots, A(t)$. Using the well-known property that, if n events occur in $[0, t]$ according to a Poisson process, the event times are independent and uniformly distributed on $[0, t]$, it easily follows (see, e.g., Chapter 2 of [14]) that

$$\mathbb{E}(e^{-sX(t)}) = \exp\left(-sX(0)e^{-rt} - \frac{\lambda}{r}\int_{se^{-rt}}^{s} \frac{1-\beta(u)}{u}\,\mathrm{d}u\right), \quad \mathrm{Re}\, s \geq 0. \tag{2}$$

We shall strongly rely on this expression in the remainder of this paper.

For any $r > 0$, the process $\{X(t), t \geq 0\}$ has a steady-state distribution. Denoting by X a random variable with that distribution, with LST $\phi(s)$, it follows from (2) that (see already [12])

$$\phi(s) = \exp\left(-\frac{\lambda}{r}\int_0^s \frac{1-\beta(u)}{u}\,\mathrm{d}u\right). \tag{3}$$

In the special case that B is exponentially distributed with mean $1/\mu$, we substitute $\beta(s) = \frac{\mu}{\mu+s}$ to obtain

$$\phi(s) = \left(\frac{\mu}{\mu+s}\right)^{\lambda/r},$$

which corresponds to a Gamma distribution with shape parameter λ/r and scale parameter $1/\mu$.

3 Model description and notation

In this section, we introduce the single-server two-queue polling model that will be studied in this paper. Customers arrive at the two queues, say Q_1 and Q_2, according to two independent Poisson processes at rates λ_1, λ_2. There is a single server, that alternately visits the two queues for random periods. The service times of customers in Q_1 (respectively in Q_2) are independent and identically, generally, distributed positive random variables; a generic service time at Q_i is denoted by B_i, $i = 1, 2$. The service times in Q_1 are also independent of those in Q_2, and the service times are also independent of the interarrival times. We denote the LST of B_i by $\beta_i(s) = \mathbb{E}(e^{-sB_i})$, with $\mathrm{Re}\, s \geq 0$, $i = 1, 2$.

Just like in the shot-noise queue described in Section 2 the service speed is a linear function of the workload of the queue that it visits, i.e., if the workload at Q_i is x_i then the service rate at that queue is $r_i x_i$, $i = 1, 2$. A special feature of this service speed and the ensuing exponential decrement of the workload is that neither queue ever becomes empty.

We make the following assumption about the visit times of the server at the two queues. The server alternately spends independent random times $T_{11}, T_{21}, T_{12}, T_{22}, \ldots$ at Q_1, Q_2, $Q_1, Q_2, \ldots$. Upon completion of a visit time at Q_i, the server instantaneously switches to the other queue, i.e., there is no switch-over time. Furthermore if, upon completion of the visit time, the server is providing service to a customer, this service is interrupted and resumed at the next visit of the server to the queue. More explicitly, we assume that if a server resumes the service after being interrupted, the server continues from where the service stopped instead of starting from the beginning, i.e., the service is *preemptive–resume*.

4 Model 1: Two queues with constant visit times

In this section we consider the single-server two-queue polling model as described in Section 3, with the following additional assumption: the visit periods to Q_1 and Q_2 are *both constant*, of size T_1 and T_2, respectively. We focus on the steady-state joint workload of Q_1 and Q_2.

We let $V_i(t)$ denote the workload at time $t \geq 0$ at Q_i, $i = 1, 2$, and V_i the steady-state workload at an arbitrary epoch.

Theorem 4.1. *The LST of the steady-state joint workload of Q_1 and Q_2 is*

$$\begin{aligned}
&\mathbb{E}(e^{-s_1V_1-s_2V_2})\\
=&\frac{\exp\left(-\frac{\lambda_1}{r_1}\int_0^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u\right)}{T_1+T_2}\times\\
&\left[\exp\left(-\lambda_2T_1\sum_{j=0}^{\infty}(1-\beta_2(s_2e^{-(j+1)r_2T_2}))\right)\right.\\
&\int_0^{T_1}\exp\left(-\lambda_2(1-\beta_2(s_2))x-\lambda_1T_2\sum_{j=0}^{\infty}(1-\beta_1(s_1e^{-jr_1T_1-r_1x}))\right)\mathrm{d}x\\
&+\exp\left(-\lambda_1T_2\sum_{j=0}^{\infty}(1-\beta_1(s_1e^{-(j+1)r_1T_1}))\right)\\
&\left.\int_0^{T_2}\exp\left(-\lambda_1(1-\beta_1(s_1))x-\lambda_2T_1\sum_{j=0}^{\infty}(1-\beta_2(s_2e^{-jr_2T_2-r_2x}))\right)\mathrm{d}x\right]. \qquad (4)
\end{aligned}$$

Proof. The proof of (4) consists of five steps. In Step 1, we express the joint workload LST at time T_1 in that at time 0. In Step 2, we express the joint workload LST at time $T_1 + T_2$ in that at time 0. In Step 3, we observe that, in steady-state, the latter two LST's should coincide. That leads to a recursive relation, which is solved. In Step 4 we determine the joint workload LST at time T_1. In Step 5, finally, we obtain the joint workload LST at an *arbitrary* epoch.
Step 1. Calculation of $\mathbb{E}(e^{-s_1V_1(T_1)-s_2V_2(T_1)})$.
Because of the constant visit times and the independent arrival processes, we have

$$\begin{aligned}
&\mathbb{E}(e^{-s_1V_1(T_1)-s_2V_2(T_1)})\\
&=\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\mathbb{E}(e^{-s_1V_1(T_1)-s_2V_2(T_1)}|V_1(0)=x_1,V_2(0)=x_2)\\
&\qquad \mathrm{d}\,\mathbb{P}(V_1(0)<x_1,V_2(0)<x_2)\\
&=\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\mathbb{E}(e^{-s_1V_1(T_1)}|V_1(0)=x_1)\mathbb{E}(e^{-s_2V_2(T_1)}|V_2(0)=x_2)\\
&\qquad \mathrm{d}\,\mathbb{P}(V_1(0)<x_1,V_2(0)<x_2). \qquad (5)
\end{aligned}$$

In the time interval $[0, T_1)$, the first queue behaves as a shot-noise queue, hence from (2) we know that

$$\mathbb{E}(e^{-s_1V_1(t)}|V_1(0)=x_1)=e^{-s_1x_1e^{-r_1t}-\frac{\lambda_1}{r_1}\int_{s_1e^{-r_1t}}^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u},\quad 0\leq t\leq T_1. \qquad (6)$$

In the time interval $[0, T_1)$, the second queue behaves as a vacation queue so the workload in that system only increases by the sum of the service times of all the customers that arrived in this interval. The increments occur according to a compound Poisson process, hence we have

$$\mathbb{E}(e^{-s_2 V_2(t)}|V_2(0) = x_2) = e^{-s_2 x_2 - \lambda_2(1-\beta_2(s_2))t}, \quad 0 \le t \le T_1. \tag{7}$$

So combining (5), (6) and (7), we get

$$\begin{aligned}
&\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)}) \\
=&e^{-\lambda_2(1-\beta_2(s_2))T_1 - \frac{\lambda_1}{r_1}\int_{s_1 e^{-r_1 T_1}}^{s_1} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \times \\
&\int_{x_1=0}^{\infty}\int_{x_1=0}^{\infty} e^{-s_1 x_1 e^{-r_1 T_1} - s_2 x_2}\, \mathrm{d}\,\mathbb{P}(V_1(0) < x_1, V_2(0) < x_2) \\
=&e^{-\lambda_2(1-\beta_2(s_2))T_1 - \frac{\lambda_1}{r_1}\int_{s_1 e^{-r_1 T_1}}^{s_1} \frac{1-\beta_1(u)}{u}\mathrm{d}u}\mathbb{E}(e^{-s_1 e^{-r_1 T_1} V_1(0) - s_2 V_2(0)}).
\end{aligned} \tag{8}$$

Step 2. Calculation of $\mathbb{E}(e^{-s_1 V_1(T_1+T_2) - s_2 V_2(T_1+T_2)})$.
Because of the symmetry of our polling model, performing a similar step as in Step 1, it is obvious that

$$\begin{aligned}
&\mathbb{E}(e^{-s_1 V_1(T_1+T_2) - s_2 V_2(T_1+T_2)}) \\
&= e^{-\lambda_1(1-\beta_1(s_1))T_2 - \frac{\lambda_2}{r_2}\int_{s_2 e^{-r_2 T_2}}^{s_2} \frac{1-\beta_2(u)}{u}\mathrm{d}u}\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 e^{-r_2 T_2} V_2(T_1)}).
\end{aligned} \tag{9}$$

Substituting $\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 e^{-r_2 T_2} V_2(T_1)})$ from (8) in the above equation yields

$$\begin{aligned}
&\mathbb{E}(e^{-s_1 V_1(T_1+T_2) - s_2 V_2(T_1+T_2)}) \\
=&e^{-\lambda_1(1-\beta_1(s_1))T_2 - \lambda_2(1-\beta_2(s_2 e^{-r_2 T_2}))T_1 - \frac{\lambda_1}{r_1}\int_{s_1 e^{-r_1 T_1}}^{s_1} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \\
&e^{-\frac{\lambda_2}{r_2}\int_{s_2 e^{-r_2 T_2}}^{s_2} \frac{1-\beta_2(u)}{u}\mathrm{d}u}\mathbb{E}(e^{-s_1 e^{-r_1 T_1} V_1(0) - s_2 e^{-r_2 T_2} V_2(0)}).
\end{aligned} \tag{10}$$

Step 3. Calculation of $\mathbb{E}(e^{-s_1 V_1(T_1+T_2) - s_2 V_2(T_1+T_2)})$ in steady-state.
In steady-state, $(V_1(0), V_2(0))$ has the same distribution as $(V_1(T_1+T_2), V_2(T_1+T_2))$. Introducing $G_i(s_1, s_2)$ as the joint workload LST at the end of a visit to Q_i, $i = 1, 2$, we have

$$G_2(s_1, s_2) = H(s_1, s_2)\; G_2(s_1 e^{-r_1 T_1}, s_2 e^{-r_2 T_2}), \tag{11}$$

with

$$\begin{aligned}
H(s_1, s_2) =&e^{-\lambda_1(1-\beta_1(s_1))T_2 - \lambda_2(1-\beta_2(s_2 e^{-r_2 T_2}))T_1 - \frac{\lambda_1}{r_1}\int_{s_1 e^{-r_1 T_1}}^{s_1} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \\
&e^{-\frac{\lambda_2}{r_2}\int_{s_2 e^{-r_2 T_2}}^{s_2} \frac{1-\beta_2(u)}{u}\mathrm{d}u}.
\end{aligned} \tag{12}$$

Iterating (11) yields after one step

$$G_2(s_1, s_2) = H(s_1, s_2) H(s_1 e^{-r_1 T_1}, s_2 e^{-r_2 T_2}) G_2(s_1 e^{-2r_1 T_1}, s_2 e^{-2r_2 T_2}).$$

Continuing in this way and observing that $G_2(s_1 e^{-jr_1 T_1}, s_2 e^{-jr_2 T_2}) \to G_2(0,0) = 1$ as $j \to \infty$, we obtain

$$G_2(s_1, s_2) = \prod_{j=0}^{\infty} H(s_1 e^{-jr_1 T_1}, s_2 e^{-jr_2 T_2}).$$

Here we observe that, since $1-e^{-x}<x$ for $x>0$, one has convergence of the infinite sums: $\sum_{j=0}^{\infty}(1-\beta_i(s_i e^{-jr_iT_i}))\leq s_i\,\mathbb{E}(B_i)\sum_{j=0}^{\infty}e^{-jr_iT_i}<\infty$ for $i=1,2$. By substituting $H(s_1,s_2)$ from (12) in the above equation, we get

$$\begin{aligned}&G_2(s_1,s_2)\\ =&\exp\left(-\frac{\lambda_1}{r_1}\int_0^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u\right)\\ &\prod_{j=0}^{\infty}\exp\left(-\lambda_1(1-\beta_1(s_1e^{-jr_1T_1}))T_2-\lambda_2(1-\beta_2(s_2e^{-(j+1)r_2T_2}))T_1\right)\\ =&\exp\left(-\frac{\lambda_1}{r_1}\int_0^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u\right)\\ &\exp\left(-\lambda_1T_2\sum_{j=0}^{\infty}(1-\beta_1(s_1e^{-jr_1T_1}))-\lambda_2T_1\sum_{j=0}^{\infty}(1-\beta_2(s_2e^{-(j+1)r_2T_2}))\right).\end{aligned}\tag{13}$$

Step 4. Calculation of $\mathbb{E}(e^{-s_1V_1(T_1)-s_2V_2(T_1)})$ in steady-state.
It follows from (8) that

$$G_1(s_1,s_2)=e^{-\lambda_2(1-\beta_2(s_2))T_1-\frac{\lambda_1}{r_1}\int_{s_1e^{-r_1T_1}}^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u}G_2(s_1e^{-r_1T_1},s_2).\tag{14}$$

Step 5. Calculation of $\mathbb{E}(e^{-s_1V_1-s_2V_2})$ in steady-state.
Firstly, let us denote by $\{S=1\}$ (respectively by $\{S=2\}$) the event that the server resides in the first (respectively second) queue. Then,

$$\begin{aligned}&\mathbb{E}(e^{-s_1V_1-s_2V_2})\\ &=\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=1)\mathbb{P}(S=1)+\mathbb{E}(e^{-s_1V_1-s_2V_1}|S=2)\mathbb{P}(S=2)\\ &=\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=1)\frac{T_1}{T_1+T_2}+\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=2)\frac{T_2}{T_1+T_2}.\end{aligned}\tag{15}$$

Using the stochastic mean value theorem on $[0,T_1]$ (cf. Chapter 1 of [10]), we determine the steady-state joint workload LST when the server is serving at Q_1 as

$$\begin{aligned}&\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=1)\\ &=\frac{1}{T_1}\,\mathbb{E}\left(\int_0^{T_1}e^{-s_1V_1(x)-s_2V_2(x)}\mathrm{d}x\right)\\ &=\frac{1}{T_1}\,\mathbb{E}\left(\int_0^{T_1}e^{-s_1V_1(0)e^{-r_1x}-\frac{\lambda_1}{r_1}\int_{s_1e^{-r_1x}}^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-s_2V_2(0)-\lambda_2(1-\beta_2(s_2))x}\mathrm{d}x\right)\\ &=\frac{1}{T_1}\int_0^{T_1}e^{-\frac{\lambda_1}{r_1}\int_{s_1e^{-r_1x}}^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\lambda_2(1-\beta_2(s_2))x}\,G_2(s_1e^{-r_1x},s_2)\mathrm{d}x.\end{aligned}\tag{16}$$

Substituting $G_2(s_1, s_2)$ from (13) in the above equation, we obtain

$$\begin{aligned}
&\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=1)\\
=&\frac{1}{T_1}\int_0^{T_1}\Bigg[e^{-\frac{\lambda_1}{r_1}\int_{s_1e^{-r_1x}}^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\lambda_2(1-\beta_2(s_2))x-\frac{\lambda_1}{r_1}\int_0^{s_1e^{-r_1x}}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u}\\
&\exp\left(-\lambda_1T_2\sum_{j=0}^{\infty}(1-\beta_1(s_1e^{-jr_1T_1-r_1x}))-\lambda_2T_1\sum_{j=0}^{\infty}(1-\beta_2(s_2e^{-(j+1)r_2T_2}))\right)\Bigg]\mathrm{d}x\\
=&\frac{1}{T_1}\exp\Bigg(-\frac{\lambda_1}{r_1}\int_0^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u\\
&-\lambda_2T_1\sum_{j=0}^{\infty}\Big(1-\beta_2\big(s_2e^{-(j+1)r_2T_2}\big)\Big)\Bigg)\\
&\int_0^{T_1}\exp\Big(-\lambda_2(1-\beta_2(s_2))x-\lambda_1T_2\sum_{j=0}^{\infty}(1-\beta_1(s_1e^{-jr_1T_1-r_1x}))\Big)\mathrm{d}x,
\end{aligned}\tag{17}$$

where in the last step we took all the terms not involving x outside the integral. In a completely similar way, or just using symmetry, we obtain

$$\begin{aligned}
&\mathbb{E}(e^{-s_1V_1-s_2V_2}|S=2)\\
=&\frac{1}{T_2}\exp\Bigg(-\frac{\lambda_1}{r_1}\int_0^{s_1}\frac{1-\beta_1(u)}{u}\mathrm{d}u-\frac{\lambda_2}{r_2}\int_0^{s_2}\frac{1-\beta_2(u)}{u}\mathrm{d}u\\
&-\lambda_1T_2\sum_{j=0}^{\infty}\Big(1-\beta_1\big(s_1e^{-(j+1)r_1T_1}\big)\Big)\Bigg)\\
&\int_0^{T_2}\exp\Big(-\lambda_1(1-\beta_1(s_1))x-\lambda_2T_1\sum_{j=0}^{\infty}(1-\beta_2(s_2e^{-jr_2T_2-r_2x}))\Big)\mathrm{d}x.
\end{aligned}\tag{18}$$

Combining (15), (17) and (18) concludes the proof of (4). □

Corollary 4.2. *It holds that*

$$\mathbb{E}(V_1)=\frac{\lambda_1\,\mathbb{E}(B_1)}{r_1}\left[1+\frac{T_2}{T_1+T_2}\left(1+\frac{r_1T_2}{2}\frac{1+e^{-r_1T_1}}{1-e^{-r_1T_1}}\right)\right],\tag{19}$$

$$\begin{aligned}
\mathbb{E}(V_1^2)=&\frac{\lambda_1\,\mathbb{E}(B_1^2)}{2r_1}+\left(\frac{\lambda_1\,\mathbb{E}(B_1)}{r_1}\right)^2\left[1+\frac{2T_2}{T_1+T_2}\left(1+\frac{r_1T_2}{2}\frac{1+e^{-r_1T_1}}{1-e^{-r_1T_1}}\right)\right]\\
&+\frac{\lambda_1\,\mathbb{E}(B_1)T_2}{2r_1(T_1+T_2)}\Bigg[\frac{\lambda_1\,\mathbb{E}(B_1)T_2(1+e^{-r_1T_1})}{1-e^{-r_1T_1}}+\frac{2r_1\lambda_1\,\mathbb{E}(B_1)T_2^2e^{-2r_1T_1}}{(1-e^{-r_1T_1})^2}\\
&+\frac{2\lambda_1\,\mathbb{E}(B_1)r_1T_2^2}{3}\frac{1+2e^{-r_1T_2}}{1-e^{-r_1T_1}}+\frac{\mathbb{E}(B_1^2)}{\mathbb{E}(B_1)}\left(1+r_1T_2\frac{1+e^{-2r_1T_1}}{1-e^{-2r_1T_1}}\right)\Bigg].
\end{aligned}\tag{20}$$

Symmetric formulas hold for $\mathbb{E}(V_2)$ and $\mathbb{E}(V_2^2)$. It also holds that

$$
\begin{aligned}
\mathbb{E}(V_1V_2) =& \frac{\lambda_1\lambda_2\,\mathbb{E}(B_1)\,\mathbb{E}(B_2)}{r_1r_2}\left[1+\frac{T_2}{T_1+T_2}\left(1+\frac{r_1T_2}{2}\frac{1+e^{-r_1T_1}}{1-e^{-r_1T_1}}\right)\right. \\
&\left.+\frac{T_1}{T_1+T_2}\left(1+\frac{r_2T_1}{2}\frac{1+e^{-r_2T_2}}{1-e^{-r_2T_2}}\right)\right] \\
&+\frac{\lambda_1\lambda_2\,\mathbb{E}(B_1)\,\mathbb{E}(B_2)}{T_1+T_2}\left[\frac{T_1}{r_2^2}+\frac{T_2}{r_1^2}+\frac{T_2T_1e^{-r_1T_1}}{1-e^{-r_1T_1}}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\right. \\
&\left.+\frac{T_1T_2e^{-r_2T_2}}{1-e^{-r_2T_2}}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right], \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{C}ov(V_1,V_2) =& \frac{\lambda_1\lambda_2\,\mathbb{E}(B_1)\,\mathbb{E}(B_2)}{r_1r_2}\frac{T_1T_2}{T_1+T_2}\left[\frac{r_1}{T_2r_2}+\frac{r_2}{T_1r_1}\right. \\
&+(r_1-r_2)\left(\frac{e^{-r_1T_1}}{1-e^{-r_1T_1}}-\frac{e^{-r_2T_2}}{1-e^{-r_2T_2}}\right) \\
&\left.-\frac{1}{T_1+T_2}\left(1+\frac{r_1T_2}{2}\frac{1+e^{-r_1T_1}}{1-e^{-r_1T_1}}\right)\left(1+\frac{r_2T_1}{2}\frac{1+e^{-r_2T_2}}{1-e^{-r_2T_2}}\right)\right]. \qquad (22)
\end{aligned}
$$

Proof. The moment expressions follow from the LST expression of Equation (4) after tedious but straightforward differentiations. □

Remark 4.3. It seems natural to expect that V_1 and V_2 are negatively correlated. In the special case $r_1 = r_2$, $T_1 = T_2$, we have been able to verify this. The term between square brackets in (22) in this case becomes, with $x := r_1T_1$:

$$\frac{2}{T}-\frac{1}{2T}\left(1+\frac{x}{2}\frac{1+\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}}\right).$$

This expression is non-positive if $x\frac{1+\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} \geq 2$, which is easily shown to hold (with equality for $x = 0$).

Remark 4.4. It is straightforward to extend Theorem 4.1 from compound Poisson inputs at the queues to Lévy subordinator inputs (i.e., nondecreasing Lévy processes). Indeed, consider a shot-noise process as in Section 2, but with input a Lévy subordinator process $\{L(t), t \geq 0\}$, with Laplace exponent $\eta(\cdot)$; i.e., $\mathbb{E}(\mathrm{e}^{-sL(t)}) = \mathrm{e}^{-\eta(s)t}$, $t \geq 0$. In the compound Poisson case, one has $\eta(s) = \lambda(1-\beta(s))$. Formula (2) now generalizes to (cf. Formula (12) of [19]):

$$\mathbb{E}(e^{-sX(t)}) = \exp\left(-sX(0)e^{-rt}-\frac{1}{r}\int_{s\mathrm{e}^{-rt}}^{s}\frac{\eta(u)}{u}\,\mathrm{d}u\right), \quad \mathrm{Re}[s] \geq 0, \qquad (23)$$

the only difference being that $\lambda(1-\beta(u))$ has been replaced by $\eta(u)$. Theorem 4.1 remains valid in the case of Lévy subordinators with Laplace exponents $\eta_i(\cdot)$ at Q_i, $i = 1,2$, if one also simply replaces $\lambda_i(1-\beta_i(\cdot))$ by $\eta_i(\cdot)$, $i = 1,2$; this is easily seen by carefully checking the five steps of the proof of Theorem 4.1.

5 N queues with constant visit times

The techniques employed in Section 4 could also be used to analyze the steady-state joint workload LST in the case of $N > 2$ queues with constant visit times, independent Lévy subordinator input processes (cf. Remark 4.4) and a service speed at each queue which is proportional to its workload. However, the analysis, and the bookkeeping of the various workload contributions, become quite involved. For this reason, we implement a slightly more straightforward approach. Inspection of Expression (13) for $G_2(s_1, s_2)$, which has a product form, reveals that the workloads in Q_1 and Q_2 at visit completion epochs are independent. Of course that is not surprising, because we are viewing the queues after fixed visit times. This independence at visit completion epochs also holds in the case of $N > 2$ queues with constant visit times. To obtain the steady-state joint workload LST at arbitrary epochs, we can now use the following procedure:

Step 1. Calculate the marginal workload LST of some queue Q_m at the end of its visit period.
Step 2. Calculate the marginal workload LST of Q_m at the end of a visit to Q_{i-1}.
Step 3. Multiply all those LST's of the independent marginal workloads, thus obtaining the joint workload LST at the end of a visit to Q_{i-1}.
Step 4. Use the latter result to obtain the joint workload LST at an arbitrary epoch during a visit to Q_i.
Step 5. Take a weighted average of all these LST's, for $i = 1, 2, \ldots, N$, over the visit intervals of lengths $T_1, T_2, \ldots, T_N$.

It should be noted that the N workloads at an arbitrary epoch are *not* independent. They are correlated because, when considering them at an arbitrary epoch in a visit period to some queue, say, Q_i, the length of the past part of T_i influences how the workload at Q_i has developed, and how the other workloads have grown.

Below we first formally describe the N-queue model; subsequently we follow the five outlined steps to arrive at Theorem 5.1 for the steady-state joint workload LST.

Model description. A single server cyclically visits N queues, having constant visit times $T_1, T_2, \ldots, T_N$ at these queues. Work arrives at the queues according to N independent Lévy subordinators, with Laplace exponents $\eta_1(\cdot), \eta_2(\cdot), \ldots, \eta_N(\cdot)$. When the server visits Q_i, it serves that queue at speed $r_i y_i$ when the workload is y_i, $i = 1, 2, \ldots, N$.

Step 1: The LST of the marginal workload $\hat{V}_m$ of Q_m at the end of its visit.
Take the visit periods at $Q_{m+1}, \ldots, Q_N, Q_1, \ldots, Q_{m-1}$ together as one new visit period in a two-queue model consisting of Q_m and the aggregated other queues. It follows from (13) for $G_2(s_1, s_2)$ with $s_2 = 0$ – while using the fact that, cf. Remark 4.4, we can replace $\lambda_i(1 - \beta_i(\cdot))$ by the Laplace exponent $\eta_i(\cdot)$ – that, for $m = 1, 2, \ldots, N$,

$$\mathbb{E}(\mathrm{e}^{-s_m \hat{V}_m}) = \mathrm{e}^{-\frac{1}{r_m} \int_0^{s_m} \frac{\eta_m(u)}{u} \mathrm{d}u} \mathrm{e}^{-\sum_{k \neq m} T_k \sum_{j=0}^{\infty} \eta_m(s_m \mathrm{e}^{-(j+1) r_m T_m})}. \tag{24}$$

Step 2: The marginal workload LST of Q_m at the end of a visit to Q_{i-1}.

The workload of Q_m does not decrease after the end of the last visit to Q_m until the end of the next visit to Q_{i-1}; it equals the workload present at the end of the visit to Q_m plus all the work that arrives during the visit periods of $Q_{m+1}, \dots, Q_{i-1}$. Hence, we let i runs from 1 to N, in view of Step 4 below, but $i-1=0$ should be replaced by $i-1=N$. Hence, denoting the workload of Q_m at the end of a visit to Q_{i-1} by $V_{m,i-1}$, we have $V_{m,m} = \hat{V}_m$ and

$$\mathbb{E}(\mathrm{e}^{-s_m V_{m,i-1}}) = \mathbb{E}(\mathrm{e}^{-s_m \hat{V}_m})\mathrm{e}^{-\sum_{k=m+1}^{i-1} T_k \eta_m(s_m)}, \quad m \neq i-1. \tag{25}$$

Here $\sum_{k=m+1}^{i-1} T_k \eta_m(s_m) = \sum_{k=m+1}^{N} T_k \eta_m(s_m) + \sum_{k=1}^{i-1} T_k \eta_m(s_m)$ if $i-1 < m$.

Step 3. Using the independence of the workloads of the N queues at the end of a visit to Q_{i-1} we can write:

$$\begin{aligned} & f_{i-1}(s_1, s_2, \dots, s_N) := \mathbb{E}(\mathrm{e}^{-s_1 V_{1,i-1} - s_2 V_{2,i-1} - \dots - s_N V_{N,i-1}}) \qquad (26) \\ = \quad & \exp\Big(- \sum_{m=1}^{N} \frac{1}{r_m} \int_0^{s_m} \frac{\eta_m(u)}{u} \mathrm{d}u - \sum_{m=1}^{N} \sum_{k \neq m} T_k \sum_{j=0}^{\infty} \eta_m(s_m \mathrm{e}^{-(j+1) r_m T_m}) \Big) \\ & \exp\Big(- \sum_{m \neq i-1} \sum_{k=m+1}^{i-1} T_k \eta_m(s_m) \Big). \end{aligned}$$

Step 4: The joint workload LST at an arbitrary epoch during a visit to Q_i.
Denoting the joint workload LST at some time $x \in (0, T_i)$ by $g_{i,x}(s_1, s_2, \dots, s_N)$, we can write (cf. (2)):

$$\begin{aligned} & g_{i,x}(s_1, s_2, \dots, s_N) \\ & = f_{i-1}(s_1, \dots, s_{i-1}, s_i \mathrm{e}^{-r_i x}, s_{i+1}, \dots, s_N) \mathrm{e}^{-\frac{1}{r_i} \int_{s_i \mathrm{e}^{-r_i x}}^{s_i} \frac{\eta_i(u)}{u} \mathrm{d}u} \mathrm{e}^{-\sum_{j \neq i} \eta_j(s_j) x}. \end{aligned} \tag{27}$$

Here $i = 1, 2, \dots, N$ but $f_0 = f_N$.

Step 5. To obtain the LST of the steady-state joint workload distribution, take a weighted average of all these LST's, for $i = 1, 2, \dots, N$, over the visit intervals of lengths $T_1, T_2, \dots, T_N$, using a stochastic mean value theorem to average over a T_i interval:

$$\mathbb{E}(\mathrm{e}^{-s_1 V_1 - s_2 V_2 - \dots - s_N V_N}) = \sum_{i=1}^{N} \frac{T_i}{T_1 + T_2 + \dots + T_N} \int_{x=0}^{T_i} g_{i,x}(s_1, s_2, \dots, s_N) \frac{\mathrm{d}x}{T_i}. \tag{28}$$

We thus arrive at the following theorem.

Theorem 5.1. *The LST of the steady-state joint workload distribution in the $N-$queue polling model with constant visit times, independent Lévy subordinator inputs and workload-proportional* *service speeds is given by*

$$\begin{aligned}
&\mathbb{E}(\mathrm{e}^{-s_1V_1-s_2V_2-\cdots-s_NV_N}) \\
&= \frac{1}{\sum_{i=1}^N T_i} \exp\Big(-\sum_{i=1}^N \frac{1}{r_i}\int_0^{s_i} \frac{\eta_i(u)}{u}\mathrm{d}u\Big)\times \\
&\sum_{i=1}^N \int_{x=0}^{T_i} \exp\Big(-\sum_{m\neq i}\sum_{k\neq m} T_k \sum_{j=0}^{\infty} \eta_m(s_m\mathrm{e}^{-(j+1)r_mT_m})\Big) \\
&\exp\Big(-\sum_{k\neq i} T_k \sum_{j=0}^{\infty} \eta_i(s_i\mathrm{e}^{-r_ix-(j+1)r_iT_i})\Big)\ \exp\Big(-\sum_{m\neq i}\sum_{k=m+1}^{i-1} T_k\eta_m(s_m)\Big) \\
&\exp\Big(-\sum_{k=i+1}^{i-1} T_k\eta_i(s_i\mathrm{e}^{-r_ix})\Big)\ \exp\Big(-\sum_{j\neq i}\eta_j(s_j)x\Big)\,\mathrm{d}x. \qquad (29)
\end{aligned}$$

Remark 5.2. Theorem 5.1 is readily seen to reduce to Theorem 4.1 when $N = 2$ and the arrival processes are compound Poisson processes.

6 Model 2: Constant visit times for Q_1, general visit times for Q_2

In this section, we consider the same single-server two-queue polling model as in Section 4, with one difference: the visit periods of Q_2 now have a *general* distribution with LST $\gamma_2(\cdot)$ (whereas the visit periods of Q_1 are still constant). In this section, we focus on the steady-state workload of Q_1; for this model we have not been able to determine the *joint* workload LST.

Let $V_1(t)$ denote the workload at time t, $t \geq 0$, and V_1 the steady-state workload at an arbitrary epoch.

Theorem 6.1. *The LST of the steady-state workload of Q_1 is*

$$\begin{aligned}
\mathbb{E}(e^{-sV_1}) = \frac{\exp\Bigg(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u\Bigg)}{T_1+\mathbb{E}(T_2)} &\Bigg[\int_0^{T_1} \prod_{j=0}^{\infty} \gamma_2(\lambda_1(1-\beta_1(se^{-r_1(x+jT_1)})))\mathrm{d}x \\
&+ \frac{1-\gamma_2(\lambda_1(1-\beta_1(s)))}{\lambda_1(1-\beta_1(s))} \prod_{j=1}^{\infty} \gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1})))\Bigg]. \qquad (30)
\end{aligned}$$

Proof. The proof contains the following steps:
Step 1. Calculation of $\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1) = y)$.
During $(T_1, T_1 + T_2)$ the server is on vacation, so the workload in the system only increases by the sum of the service times of all the customers that arrived in this interval. The increments occur according to a compound Poisson process. So,

$$\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1) = y) = e^{-sy}\gamma_2(\lambda_1(1-\beta_1(s))). \qquad (31)$$

Step 2. Calculation of $\mathbb{E}[e^{-sV_1(T_1)}|V_1(0)=x]$.
From (2), we know that

$$\mathbb{E}(e^{-sV_1(T_1)}|V_1(0)=x) = e^{-sxe^{-r_1T_1}-\frac{\lambda_1}{r_1}\int_{se^{-r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u}. \tag{32}$$

Step 3. Calculation of $\mathbb{E}[e^{-sV_1(T_1+T_2)}|V_1(0)=x]$. Let $f_{V_1(T_1)}(\cdot|V_1(0)=x)$ denote the density of $V_1(T_1)$ conditional on $\{V_1(0)=x\}$, then

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(0)=x)\\
&=\int_{y=0}^{\infty}\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1)=y)f_{V_1(T_1)}(y|V_1(0)=x)\mathrm{d}y\\
&=\gamma_2(\lambda_1(1-\beta_1(s)))\int_{y=0}^{\infty}e^{-sy}f_{V_1(T_1)}(y|V_1(0)=x)\mathrm{d}y\\
&=\gamma_2(\lambda_1(1-\beta_1(s)))\mathbb{E}(e^{-sV_1(T_1)}|V_1(0)=x)\\
&=\gamma_2(\lambda_1(1-\beta_1(s)))e^{-sxe^{-r_1T_1}-\frac{\lambda_1}{r_1}\int_{se^{-r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u},
\end{aligned} \tag{33}$$

where the last equality comes from Equation (32).
Step 4. Calculation of $\mathbb{E}(e^{-sV_1(T_1+T_2)})$ in steady-state.
Observe that

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)})\\
&=\int_{x=0}^{\infty}\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(0)=x)f_{V_1(0)}(x)\mathrm{d}x\\
&=\gamma_2(\lambda_1(1-\beta_1(s)))\int_{x=0}^{\infty}e^{-sxe^{-r_1T_1}-\frac{\lambda_1}{r_1}\int_{se^{-r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u}f_{V_1(0)}(x)\mathrm{d}x,
\end{aligned} \tag{34}$$

with $f_{V_1(0)}(x)$ the probability density function of $V_1(0)$. Now observe that in steady-state $V_1(T_1+T_2)$ has the same distribution as $V_1(0)$. So we can rewrite (34) as follows:

$$\mathbb{E}(e^{-sV_1(T_1+T_2)}) = \gamma_2(\lambda_1(1-\beta_1(s)))e^{-\frac{\lambda_1}{r_1}\int_{se^{-r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u}\,\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-r_1T_1}}).$$

Using the above equation, we compute $\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-r_1T_1}})$. Substituting this in the r.h.s. of the above equation yields

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)})\\
=&\gamma_2(\lambda_1(1-\beta_1(s)))\gamma_2(\lambda_1(1-\beta_1(se^{-r_1T_1})))\times\\
&e^{-\frac{\lambda_1}{r_1}\left(\int_{se^{-2r_1T_1}}^{se^{-r_1T_1}}\frac{1-\beta_1(u)}{u}\mathrm{d}u+\int_{se^{-r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u\right)}\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-2r_1T_1}})\\
=&\gamma_2(\lambda_1(1-\beta_1(s)))\gamma_2(\lambda_1(1-\beta_1(se^{-r_1T_1})))e^{-\frac{\lambda_1}{r_1}\int_{se^{-2r_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u}\\
&\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-2r_1T_1}})\\
&\vdots\\
=&e^{-\frac{\lambda_1}{r_1}\int_{se^{-kr_1T_1}}^{s}\frac{1-\beta_1(u)}{u}\mathrm{d}u}\prod_{j=0}^{k-1}\gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1})))\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-kr_1T_1}}).
\end{aligned}$$

Observing that $\mathbb{E}(e^{-sV_1(T_1+T_2)e^{-jr_1T_1}}) \to 1$ as $j \to \infty$, the r.h.s. of the above equation becomes

$$\mathbb{E}(e^{-sV_1(T_1+T_2)}) = e^{-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u} \prod_{j=0}^{\infty} \gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1}))), \tag{35}$$

assuming

$$\prod_{j=0}^{\infty} \gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1}))) < \infty. \tag{36}$$

In Step 6 we shall prove that (36) indeed holds.

Step 5. Calculation of $\mathbb{E}(e^{-sV_1})$ in steady-state.

Firstly, let us again denote by $\{S=1\}$ (respectively by $\{S=2\}$) the event of the server residing in the first (respectively second) queue. Then,

$$\begin{aligned}\mathbb{E}(e^{-sV_1}) &= \mathbb{E}(e^{-sV_1}|S=1)\mathbb{P}(S=1) + \mathbb{E}(e^{-sV_1}|S=2)\mathbb{P}(S=2)\\ &= \mathbb{E}(e^{-sV_1}|S=1)\frac{T_1}{T_1+\mathbb{E}(T_2)} + \mathbb{E}(e^{-sV_1}|S=2)\frac{\mathbb{E}(T_2)}{T_1+\mathbb{E}(T_2)}.\end{aligned} \tag{37}$$

Using a stochastic mean value theorem (see [10, Theorem 4.1]) we determine the LST of the steady-state workload when the server is serving at Q_1 as

$$\mathbb{E}(e^{-sV_1}|S=1) = \frac{1}{T_1}\,\mathbb{E}\left(\int_0^{T_1} e^{-sV_1(x)}\mathrm{d}x\right) = \frac{1}{T_1}\int_0^{T_1}\mathbb{E}(e^{-sV_1(x)})\mathrm{d}x. \tag{38}$$

Using (2), we obtain

$$\mathbb{E}(e^{-sV_1}|S=1) = \frac{1}{T_1}\int_0^{T_1}\mathbb{E}\left(e^{-sV_1(0)e^{-r_1x}}\right)e^{-\frac{\lambda_1}{r_1}\int_{se^{-r_1x}}^s \frac{1-\beta_1(u)}{u}\mathrm{d}u}\mathrm{d}x. \tag{39}$$

Since $V_1(0)$ and $V_1(T_1+T_2)$ have the same distribution in steady-state, substituting $\mathbb{E}(e^{-sV_1(T_1+T_2)})$ as given in (35) in the above equation yields

$$\begin{aligned}&\mathbb{E}(e^{-sV_1}|S=1)\\ &= \frac{\exp\left(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u\right)}{T_1}\int_0^{T_1}\prod_{j=0}^{\infty}\gamma_2(\lambda_1(1-\beta_1(se^{-r_1(x+jT_1)})))\mathrm{d}x.\end{aligned} \tag{40}$$

Again, using the stochastic mean value theorem for the given $\mathbb{E}(e^{-sV_1(T_1)})$, we determine the LST of the steady-state workload when the server is serving at Q_2,

$$\mathbb{E}(e^{-sV_1}|S=2) = \frac{1}{\mathbb{E}(T_2)}\,\mathbb{E}\left(\int_{T_1}^{T_1+T_2} e^{-sV_1(x)}\mathrm{d}x\right). \tag{41}$$

The workload $V_1(x)$ is the sum of two independent workloads, i.e., $V_1(T_1)$ and the workload (say $\mathcal{A}(x)$) that has arrived during the time period $[T_1, T_1+x)$; this yields

$$\mathbb{E}(e^{-sV_1}|S=2) = \frac{1}{\mathbb{E}(T_2)}\,\mathbb{E}\left(\int_0^{T_2}\mathbb{E}(e^{-s(V_1(T_1)+\mathcal{A}(x))}\mathrm{d}x\right). \tag{42}$$

During $[T_1, T_1 + T_2)$ the server serves only customers in the second queue, so the workload in the first queue only increases by the sum of the service times of all the customers that arrived in this interval. The increments occur according to a compound Poisson process. So,

$$\begin{aligned}\mathbb{E}(e^{-sV_1}|S=2) &= \frac{\mathbb{E}(e^{-sV_1(T_1)})}{\mathbb{E}(T_2)}\,\mathbb{E}\left(\int_0^{T_2} e^{-\lambda_1(1-\beta_1(s))x}\mathrm{d}x\right)\\ &= \frac{\mathbb{E}(e^{-sV_1(T_1)})}{\mathbb{E}(T_2)}\,\mathbb{E}\left(\frac{1-e^{-\lambda_1(1-\beta_1(s))T_2}}{\lambda_1(1-\beta_1(s))}\right)\\ &= \frac{\mathbb{E}(e^{-sV_1(T_1)})}{\mathbb{E}(T_2)}\,\frac{1-\gamma_2(\lambda_1(1-\beta_1(s)))}{\lambda_1(1-\beta_1(s))}. \end{aligned} \tag{43}$$

From (31), we obtain

$$\mathbb{E}(e^{-sV_1(T_1)}) = \frac{\mathbb{E}(e^{-sV_1(T_1+T_2)})}{\gamma_2(\lambda_1(1-\beta_1(s)))}. \tag{44}$$

Combining (35) and (44), we get

$$\mathbb{E}(e^{-sV_1(T_1)}) = \exp\left(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u\right)\prod_{j=1}^{\infty}\gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1}))). \tag{45}$$

Substituting (45) in (43) yields

$$\begin{aligned}\mathbb{E}(e^{-sV_1}|S=2) =& \frac{1-\gamma_2(\lambda_1(1-\beta_1(s)))}{\lambda_1(1-\beta_1(s))\,\mathbb{E}(T_2)}\exp\left(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u\right)\times\\ &\prod_{j=1}^{\infty}\gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1}))). \end{aligned} \tag{46}$$

Combining (37), (40) and (46) proves (30).

Step 6. Proof that $\prod_{j=0}^{\infty}\gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1}))) < \infty$.

It is well-known that, for $0 < a_j < 1$, the infinite product $\prod_{j=0}^{\infty} a_j$ converges iff $\sum_{j=0}^{\infty}(1-a_j) < \infty$. Since $\gamma_2(s)$ is the LST corresponding to the random variable T_2, we get

$$\begin{aligned}&\sum_{j=0}^{\infty}\Big(1-\gamma_2(\lambda_1(1-\beta_1(se^{-jr_1T_1})))\Big)\\ &= \sum_{j=0}^{\infty}\left(1-\int_0^{\infty} e^{-x(\lambda_1(1-\beta_1(se^{-jr_1T_1})))}\mathrm{d}\mathbb{P}(T_2<x)\right)\\ &= \sum_{j=0}^{\infty}\left(\int_0^{\infty}\left(1-e^{-x(\lambda_1(1-\beta_1(se^{-jr_1T_1})))}\right)\mathrm{d}\mathbb{P}(T_2<x)\right). \end{aligned} \tag{47}$$

Since $1 - e^{-x} < x$, for $x > 0$, we get

$$\sum_{j=0}^{\infty}\left(\int_0^{\infty}\left(1 - e^{-x(\lambda_1(1-\beta_1(se^{-jr_1T_1})))}\right)\mathrm{d}\mathbb{P}(T_2 < x)\right)$$
$$< \sum_{j=0}^{\infty}\left(\int_0^{\infty} x(\lambda_1(1-\beta_1(se^{-jr_1T_1})))\mathrm{d}\mathbb{P}(T_2 < x)\right)$$
$$= \lambda_1\,\mathbb{E}(T_2)\sum_{j=0}^{\infty}(1-\beta_1(se^{-jr_1T_1})). \tag{48}$$

Now we need to show that $\sum_{j=0}^{\infty}(1-\beta_1(se^{-jr_1T_1})) < \infty$. Since $\beta_1(s)$ is the LST of the random variable B_1, applying a similar analysis as in (47) and (48), one easily gets

$$\sum_{j=0}^{\infty}(1-\beta_1(se^{-jr_1T_1})) = \sum_{j=0}^{\infty}\left(\int_0^{\infty}(1-e^{-sye^{-jr_1T_1}})\mathrm{d}\mathbb{P}(B_1 < y)\right)$$
$$< \sum_{j=0}^{\infty}\left(\int_0^{\infty} sye^{-jr_1T_1}\mathrm{d}\mathbb{P}(B_1 < y)\right)$$
$$= s\,\mathbb{E}(B_1)\sum_{j=0}^{\infty}e^{-jr_1T_1} = \frac{s\,\mathbb{E}(B_1)}{1-e^{-r_1T_1}} < \infty. \tag{49}$$

□

Remark 6.2. In the case of constant T_2, the marginal workload LST of Q_1 can also be obtained by substituting $s_2 = 0$ in the joint workload LST obtained in Theorem 4.1.

Remark 6.3. The expression in Theorem 6.1 for the marginal workload LST allows us to study the tail behavior of the workload in case the service times at Q_1 and/or the visit times at Q_2 are regularly varying at infinity. First recall the definition of a regularly varying random variable/distribution:

Definition 6.4. The distribution function of a random variable B_1 on $[0, \infty)$ is called regularly varying of index $-\nu$, with $\nu \in \mathbb{R}$, if

$$\mathbb{P}(B_1 > x) \sim L(x)x^{-\nu},\ x \uparrow \infty, \tag{50}$$

with $L(x)$ a slowly varying function at infinity, i.e., $\lim\limits_{x\to\infty}\frac{L(\alpha x)}{L(x)} = 1$, for all $\alpha > 0$.

Using the Tauberian Theorem 8.1.6 of [5], which relates the behavior of a regularly varying function at infinity and the behavior of its LST near 0, one can prove the following. If B_1 is regularly varying of index $-\nu$ and T_2 is regularly varying of index $-\tau$, then the workload V_1 is regularly varying of index $-\min(\nu, \tau-1)$. We refrain from giving the details because the approach is fairly straightforward; cf. the survey [8].

7 Model 3: Exponential visit times

In this section, we consider the two-queue polling model of Section 3, but we now assume that the visit periods to Q_i are i.i.d. $\exp(c_i)$ distributed, $i = 1, 2$. In Subsection 7.1, we obtain the marginal workload LST for one of the queues. In Subsection 7.2, we show that this LST can be decomposed in three terms which are LST's of independent, non-negative random variables. That decomposition is exploited in Subsection 7.3 to obtain the asymptotic behavior of the workload at a queue in case its service time distribution is regularly varying. We derive a two-dimensional Volterra integral equation for the steady-state joint workload LST in Subsection 7.4, and we show, in Subsection 7.5, that this equation can be solved by implementing the so-called fixed-point iteration.

7.1 Marginal workload analysis

In this subsection, we derive the LST of the marginal workload in steady-state at an arbitrary epoch. The individual queues behave as vacation systems: from the perspective of one queue, the server is on vacation when it resides at the other queue.
We let $V_i(t)$ denote the workload at time t, $t \geq 0$, of Q_i, $i = 1, 2$, and let V_i denote the steady-state workload of Q_i at an arbitrary epoch, $i = 1, 2$. In this subsection, we prove the following theorem.

Theorem 7.1.

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1}) \\
&= \left(\frac{c_2}{c_1+c_2} + \frac{c_1}{c_1+c_2}\frac{c_2}{c_2+\lambda_1(1-\beta_1(s))}\right)\exp\left(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\mathrm{d}u\right) \\
&\exp\left(-\frac{c_1}{r_1}\int_0^s \frac{1-\frac{c_2}{c_2+\lambda_1(1-\beta_1(u))}}{u}\mathrm{d}u\right). \qquad (51)
\end{aligned}$$

$\mathbb{E}(e^{-sV_2})$ is given by the symmetric expression, with all indices 1 and 2 swapped.

Proof. We determine the marginal workload LST in the following five steps.
Step 1. Calculation of $\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1) = y)$.
During $(T_1, T_1 + T_2)$ the server serves only customers in the second queue, so the workload in the first queue only increases by the sum of the service times of all the customers that arrived in this interval. The increments occur according to a compound Poisson process. So,

$$\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1) = y) = e^{-sy}\frac{c_2}{c_2+\lambda_1(1-\beta_1(s))}. \qquad (52)$$

Step 2. Calculation of $\mathbb{E}[e^{-sV_1(T_1)}|V_1(0) = x]$.

$$\mathbb{E}(e^{-sV_1(T_1)}|V_1(0) = x) = \int_{t=0}^{\infty} c_1 e^{-c_1 t}\, \mathbb{E}(e^{-sV_1(t)}|V_1(0) = x)\mathrm{d}t. \qquad (53)$$

From (2) we know that

$$\mathbb{E}(e^{-sV_1(t)}|V_1(0) = x) = e^{-sxe^{-r_1 t} - \frac{\lambda_1}{r_1}\int_{se^{-r_1 t}}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u}. \qquad (54)$$

Combining (53) and (54), we get

$$\mathbb{E}(e^{-sV_1(T_1)}|V_1(0) = x) = \int_{t=0}^{\infty} c_1 e^{-c_1 t}\; e^{-sxe^{-r_1 t} - \frac{\lambda_1}{r_1}\int_{se^{-r_1 t}}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u}\,\mathrm{d}t. \qquad (55)$$

Simplifying the above equation by substituting $e^{-r_1 t} = z$ yields

$$\mathbb{E}(e^{-sV_1(T_1)}|V_1(0) = x) = \frac{c_1}{r_1} \int_{z=0}^{1} z^{\frac{c_1}{r_1}-1} \; e^{-sxz - \frac{\lambda_1}{r_1}\int_{sz}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \, \mathrm{d}z. \tag{56}$$

Step 3. Calculation of $\mathbb{E}[e^{-sV_1(T_1+T_2)}|V_1(0) = x]$.

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(0) = x) \\
&= \int_{y=0}^{\infty} \mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(T_1) = y) f_{V_1(T_1)}(y|V_1(0) = x)\mathrm{d}y \\
&= \frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \int_{y=0}^{\infty} e^{-sy} f_{V_1(T_1)}(y|V_1(0) = x)\mathrm{d}y \\
&= \frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \mathbb{E}(e^{-sV_1(T_1)}|V_1(0) = x) \\
&= \frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \frac{c_1}{r_1} \int_{z=0}^{1} z^{\frac{c_1}{r_1}-1} \; e^{-sxz - \frac{\lambda_1}{r_1}\int_{sz}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \, \mathrm{d}z,
\end{aligned} \tag{57}$$

where the second equation comes from Equation (52) and the fourth from Equation (56).

Step 4. Calculation of $\mathbb{E}(e^{-sV_1(T_1+T_2)})$ in steady-state.

Observe that

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)}) \\
&= \int_{x=0}^{\infty} \mathbb{E}(e^{-sV_1(T_1+T_2)}|V_1(0) = x) f_{V_1(0)}(x)\mathrm{d}x \\
&= \int_{x=0}^{\infty} \left[\frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \frac{c_1}{r_1} \int_{z=0}^{1} z^{\frac{c_1}{r_1}-1} \; e^{-sxz - \frac{\lambda_1}{r_1}\int_{sz}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \, \mathrm{d}z \right] f_{V_1(0)}(x)\mathrm{d}x,
\end{aligned} \tag{58}$$

with $f_{V_1(0)}(x)$ the probability density function of $V_1(0)$. Now observe that in steady-state $V_1(T_1+T_2)$ has the same distribution as $V_1(0)$. So we can rewrite (58) as follows:

$$\begin{aligned}
&\mathbb{E}(e^{-sV_1(T_1+T_2)}) \\
&= \frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \frac{c_1}{r_1} \int_{z=0}^{1} z^{\frac{c_1}{r_1}-1} \; e^{- \frac{\lambda_1}{r_1}\int_{sz}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \, \mathbb{E}(e^{-sV_1(T_1+T_2)z}) \, \mathrm{d}z.
\end{aligned}$$

Defining $G(s) := \mathbb{E}(e^{-sV_1(T_1+T_2)})$ and then substituting $sz = v$ in the integrand of the r.h.s. yields

$$G(s) = \frac{1}{s} \frac{c_2}{c_2 + \lambda_1(1-\beta_1(s))} \frac{c_1}{r_1} \int_{v=0}^{s} \left(\frac{v}{s}\right)^{\frac{c_1}{r_1}-1} \; e^{- \frac{\lambda_1}{r_1}\int_{v}^{s} \frac{1-\beta_1(u)}{u}\mathrm{d}u} \, G(v) \, \mathrm{d}v \tag{59}$$

Multiplying by s in the above equation and subsequently differentiating w.r.t. s, by using the Leibniz integral rule, yields

$$\begin{aligned}
s\frac{\mathrm{d}}{\mathrm{d}s}G(s) + G(s) = &\frac{\lambda_1 \beta_1'(s)}{c_2 + \lambda(1-\beta_1(s))} sG(s) + \frac{c_2}{c_2 + \lambda(1-\beta_1(s))} \frac{c_1}{r_1} G(s) \\
&+ \left(1 - \frac{c_1}{r_1}\right) G(s) - \frac{\lambda_1}{r_1} \frac{1-\beta_1(s)}{s} sG(s).
\end{aligned}$$

Arranging the terms of the above equation we get

$$\frac{\mathrm{d}}{\mathrm{d}s}G(s) = \left(\frac{\lambda_1\beta_1'(s)}{c_2+\lambda_1(1-\beta_1(s))} - \frac{c_1}{r_1}\frac{\lambda_1(1-\beta_1(s))}{s(c_2+\lambda_1(1-\beta_1(s)))} - \frac{\lambda_1(1-\beta_1(s))}{sr_1}\right)G(s),$$

which implies

$$\begin{aligned} G(s) =& \exp\left(-\int_0^s\Big(-\frac{\lambda_1\beta_1'(u)}{c_2+\lambda_1(1-\beta_1(u))}+\frac{c_1}{r_1}\frac{\lambda_1(1-\beta_1(u))}{u(c_2+\lambda(1-\beta_1(u)))}\right. \\ &\left.+\frac{\lambda_1(1-\beta_1(u))}{ur_1}\Big)\mathrm{d}u\right) \\ =&\frac{c_2}{c_2+\lambda_1(1-\beta_1(s))}\exp\left(-\int_0^s\Big(\frac{c_1}{r_1}\frac{\lambda_1(1-\beta_1(u))}{u(c_2+\lambda_1(1-\beta_1(u)))}\right. \\ &\left.+\frac{\lambda_1(1-\beta_1(u))}{ur_1}\Big)\mathrm{d}u\right). \end{aligned} \tag{60}$$

Step 5. Calculation of $\mathbb{E}(e^{-sV_1})$ in steady-state.
Firstly, let us again denote by $\{S=1\}$ (respectively by $\{S=2\}$) the event of the server residing in the first (respectively second) queue. Then,

$$\begin{aligned} \mathbb{E}(e^{-sV_1}) &= \mathbb{E}(e^{-sV_1}|S=1)\mathbb{P}(S=1)+\mathbb{E}(e^{-sV_1}|S=2)\mathbb{P}(S=2) \\ &= \mathbb{E}(e^{-sV_1}|S=1)\frac{c_2}{c_1+c_2}+\mathbb{E}(e^{-sV_1}|S=2)\frac{c_1}{c_1+c_2}. \end{aligned} \tag{61}$$

Because of the memoryless property of the exponential distribution it is obvious that

$$\mathbb{E}(e^{-sV_1}|S=1)=\mathbb{E}(e^{-sV_1(T_1)}),\ \mathbb{E}(e^{-sV_1}|S=2)=\mathbb{E}(e^{-sV_1(T_1+T_2)}).$$

The latter term is given by (60), while the former term is calculated using the same argument as in the derivation of Equation (56):

$$\mathbb{E}(e^{-sV_1(T_1+T_2)})=\mathbb{E}(e^{-sV_1(T_1)})\frac{c_2}{c_2+\lambda_1(1-\beta_1(s))}. \tag{62}$$

Substituting (62), for $\mathbb{E}(e^{-sV_1(T_1)})$, and (60) in Equation (61) yields (56). □

Remark 7.2. Equation (51) for $c_2\to\infty$ (zero visit time at Q_2) reduces to Equation (3) which gives the LST of the steady-state amount of work in the shot-noise queue.

7.2 Workload decomposition

In this subsection, we show that the steady-state workload V_1 can be written as the sum of three independent terms, one corresponding to the steady-state workload when the server is only serving Q_1 all the time, and the second plus third corresponding to the amount of work when the server is on ”vacation”. We focus on Q_1, but a symmetric result holds for Q_2 when the indices 1 and 2 are swapped.

Corollary 7.3. *The steady-state amount of work of the first queue, V_1, is distributed as a sum of three independent random variables X_1, X_2 and X_3, i.e.,*

$$V_1 \stackrel{d}{=} X_1 + X_2 + X_3, \tag{63}$$

where X_1 is the steady-state amount of work in Q_1 considered in isolation (see Section 2), X_2 is the steady-state amount of work in a shot-noise queue with arrival rate c_1, service speed r_1 and upward jumps having LST $\frac{c_2}{c_2+\lambda_1(1-\beta_1(u))}$, and X_3 is a weighted sum of 0 and the amount of work increment in Q_1 during a visit period of Q_2.

Proof. The decomposition immediately follows from the product form of the LST of V_1 in the righthand side of (51). The LST of X_1 was already given in (3). The LST's of X_2 and X_3 are given as

$$\mathbb{E}(e^{-sX_2}) = \exp\left(-\frac{c_1}{r_1}\int_0^s \frac{1-\frac{c_2}{c_2+\lambda_1(1-\beta_1(u))}}{u}\mathrm{d}u\right), \tag{64}$$

$$\mathbb{E}(e^{-sX_3}) = \frac{c_2}{c_1+c_2} + \frac{c_1}{c_1+c_2}\frac{c_2}{c_2+\lambda_1(1-\beta_1(s))}. \tag{65}$$

The LST of X_2 has exactly the same shape as that of X_1, but the arrival rate is c_1 instead of λ_1 (it corresponds to occurrences of ends of visits to Q_1) and the service requirement is the total amount of service requirement arriving in a visit period of Q_2, instead of B_1. The LST of X_3 is a weighted sum of 1 (the LST of 0) and of the LST of the work increment in Q_1 during an exponential visit period of Q_2, with weights the fractions of time spent in Q_1 and on vacation (i.e., in Q_2). □

We now use the decomposition result (63) to determine the mean and the variance of V_1. A straightforward computation yields:

Corollary 7.4. *The expectation of the steady-state workload of the first queue, $\mathbb{E}(V_1)$, is*

$$\mathbb{E}(V_1) = \frac{\lambda_1\,\mathbb{E}(B_1)}{r_1}\left[1+\frac{c_1}{c_2}+\frac{c_1}{c_2}\frac{r_1}{c_1+c_2}\right], \tag{66}$$

and the corresponding variance, $\mathbb{V}\mathrm{ar}(V_1)$, is

$$\mathbb{V}\mathrm{ar}(V_1) = \frac{\lambda_1\,\mathbb{E}(B_1^2)}{2r_1}\left[1+\frac{c_1}{c_2}+\frac{c_1}{c_2}\frac{2r_1}{c_1+c_2}\right] + \frac{c_1}{r_1}\left(\frac{\lambda_1\,\mathbb{E}(B_1)}{c_2}\right)^2\left[1+\frac{r_1(c_1+2c_2)}{(c_1+c_2)^2}\right]. \tag{67}$$

7.3 Heavy-tail asymptotics

In this subsection, we discuss the tail behavior of the workload in the case of heavy-tailed service time distributions (cf. Definition 6.4).

Theorem 7.5. *If the service time distribution of the random variable B_1 is regularly varying of index $-\nu$, with $\nu \in (1,2)$, then the workload of the first queue is regularly varying at infinity of index $-\nu$. More precisely,*

$$\mathbb{P}(V_1 > x) \sim \frac{\lambda_1\Gamma(-\nu)}{r_1}\left[1+\frac{c_1}{c_2}+\frac{c_1}{c_2}\frac{\nu r_1}{c_1+c_2}\right]x^{-\nu}L(x), \qquad x\uparrow\infty. \tag{68}$$

Proof. To prove that V_1 is regularly varying at infinity, one can again use the decomposition property of the workload V_1. Corollary 7.3 implies that

$$\mathbb{P}(V_1 > x) = \mathbb{P}(X_1 + X_2 + X_3 > x). \tag{69}$$

Now we have to consider the behavior of $\mathbb{P}(X_1 + X_2 + X_3 > x)$ for $x \uparrow \infty$. Our main tool is the Tauberian Theorem 8.1.6 of [5], which relates the behavior of a regularly varying function at infinity and the behavior of its LST near 0. This theorem states that (50) holds iff

$$\beta_1(s) - 1 + s\mathbb{E}(B_1) \sim -\Gamma(1-\nu)\, s^\nu L\left(\frac{1}{s}\right), \qquad s \downarrow 0. \tag{70}$$

We successively consider the LST's of X_1, X_2 and X_3, each time using (70). One has

$$\begin{aligned}
\mathbb{E}(e^{-sX_1}) &= \exp\left(-\frac{\lambda_1}{r_1}\int_0^s \frac{1-\beta_1(u)}{u}\,\mathrm{d}u\right) \\
&\sim \exp\left(-\frac{\lambda_1}{r_1}\int_0^s \left(\mathbb{E}(B_1) + \Gamma(1-\nu)\, u^{\nu-1} L\left(\frac{1}{u}\right)\right)\mathrm{d}u\right) \\
&\sim 1 - \frac{\lambda_1\,\mathbb{E}(B_1)}{r_1}s - \frac{\lambda_1\Gamma(1-\nu)}{r_1\nu}s^\nu L\left(\frac{1}{s}\right) + O(s^2), \quad s \downarrow 0.
\end{aligned} \tag{71}$$

Hence

$$\mathbb{E}(e^{-sX_1}) - 1 + \mathbb{E}(X_1)s \sim -\frac{\lambda_1\Gamma(-\nu)}{r_1}s^\nu L\left(\frac{1}{s}\right), \text{ as } s \downarrow 0. \tag{72}$$

Similarly using (70) in (64) and (65) , we get

$$\mathbb{E}(e^{-sX_2}) - 1 + \mathbb{E}(X_2)s \sim -\frac{\lambda_1\Gamma(-\nu)}{r_1}\frac{c_1}{c_2}s^\nu L\left(\frac{1}{s}\right), \text{ as } s \downarrow 0, \tag{73}$$

$$\mathbb{E}(e^{-sX_3}) - 1 + \mathbb{E}(X_3)s \sim -\frac{\lambda_1\nu\Gamma(-\nu)}{c_1+c_2}\frac{c_1}{c_2}s^\nu L\left(\frac{1}{s}\right), \text{ as } s \downarrow 0. \tag{74}$$

From Equation (72), (73) and (74), we see that X_1, X_2 and X_3 are all regularly varying random variables of index $-\nu$. Using the workload decomposition property (63) and a well-known result regarding the tail behavior of the sum of independent regularly varying random variables of the same index, see [20], yields

$$\mathbb{P}(V_1 > x) \sim (C_1 + C_2 + C_3)x^{-\nu} L(x), \qquad x \uparrow \infty, \tag{75}$$

with C_1, C_2 and C_3 the coefficients of the tail $x^{-\nu}$ for X_1, X_2 and X_3 in (72), (73) and (74), respectively. Substituting the coefficients from (72), (73) and (74) concludes the proof of the theorem. □

7.4 Joint workload in the symmetric case

So far we have focused on the marginal workload distribution at the first queue. A much harder problem is to determine the steady-state joint workload distribution. In this subsection and the next one, we begin the exploration of this problem, outlining

a possible approach as well as the mathematical complications arising. Let us now restrict ourselves to the fully symmetric case $c_1 = c_2 = c$, $\lambda_1 = \lambda_2 = \lambda$, $\beta_1(s) = \beta_2(s) = \beta(s)$ and $r_1 = r_2 = r$.
Step 1: Calculation of $\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)} | V_1(0) = x_1, V_2(0) = x_2)$.
In the first step we calculate the LST of the two-dimensional workload for $t = T_1$. Using the same arguments as in Subsection 7.1 we can easily see that

$$\begin{aligned}&\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)} | V_1(0) = x_1, V_2(0) = x_2)\\&= e^{-s_2 x_2} \int_{t=0}^{\infty} e^{-\lambda(1-\beta(s_2))t} \mathbb{E}(e^{-s_1 V_1(t)} | V_1(0) = x_1) c e^{-ct} \mathrm{d}t.\end{aligned}$$

Using (2) we get

$$\begin{aligned}&\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)} | V_1(0) = x_1, V_2(0) = x_2)\\&= e^{-s_2 x_2} \int_{t=0}^{\infty} e^{-\lambda(1-\beta(s_2))t} e^{-s_1 x_1 e^{-rt} - \frac{\lambda}{r} \int_{s_1 e^{-rt}}^{s_1} \frac{1-\beta(u)}{u} \mathrm{d}u} c e^{-ct} \mathrm{d}t.\end{aligned}$$

Step 2: Calculation of $\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)})$ in steady-state. Using the fact that, in steady-state, $(V_1(T_1 + T_2), V_2(T_1 + T_2))$ has the same distribution as $(V_1(0), V_2(0))$, we have

$$\begin{aligned}&\mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)})\\&= \int_{t=0}^{\infty} e^{-\lambda(1-\beta(s_2))t} e^{-\frac{\lambda}{r} \int_{s_1 e^{-rt}}^{s_1} \frac{1-\beta(u)}{u} \mathrm{d}u} c e^{-ct}\, \mathbb{E}(e^{-s_1 e^{-rt} V_1(T_1+T_2) - s_2 V_2(T_1+T_2)})\, \mathrm{d}t.\end{aligned}$$

Defining $G(s_1, s_2) := \mathbb{E}(e^{-s_1 V_1(T_1) - s_2 V_2(T_1)})$ and simplifying the above equation by substituting $e^{-rt} = z$ yields

$$G(s_1, s_2) = \frac{c}{r} \int_{z=0}^{1} z^{\frac{c+\lambda(1-\beta(s_2))}{r} - 1}\, e^{-\frac{\lambda}{r} \int_{s_1 z}^{s_1} \frac{1-\beta(u)}{u} \mathrm{d}u}\, G(s_2, s_1 z)\, \mathrm{d}z.$$

Substituting $s_1 z = v$ in the integrand of the r.h.s. yields

$$G(s_1, s_2) = \frac{c}{s_1 r} \int_{v=0}^{s_1} \left(\frac{v}{s_1} \right)^{\frac{c+\lambda(1-\beta(s_2))}{r} - 1} e^{-\frac{\lambda}{r} \int_{v}^{s_1} \frac{1-\beta(u)}{u} \mathrm{d}u}\, G(s_2, v)\, \mathrm{d}v. \tag{76}$$

Replacing s_2 by s_1 and s_1 by s_2 in the above equation gives

$$G(s_2, s_1) = \frac{c}{s_2 r} \int_{v=0}^{s_2} \left(\frac{v}{s_2} \right)^{\frac{c+\lambda(1-\beta(s_1))}{r} - 1} e^{-\frac{\lambda}{r} \int_{v}^{s_2} \frac{1-\beta(u)}{u} \mathrm{d}u}\, G(s_1, v)\, \mathrm{d}v. \tag{77}$$

Combining the above two equations yields a two-dimensional Volterra Integral equation, as

$$\begin{aligned}G(s_1, s_2) = \frac{c^2}{s_1 s_2 r^2} \int_{v=0}^{s_1} \int_{w=0}^{s_2} &\left(\frac{v}{s_1} \right)^{\frac{c+\lambda(1-\beta(s_2))}{r} - 1} \left(\frac{w}{s_2} \right)^{\frac{c+\lambda(1-\beta(v))}{r} - 1}\\&e^{-\frac{\lambda}{r} \left(\int_{v}^{s_1} \frac{1-\beta(u)}{u} \mathrm{d}u + \int_{w}^{s_2} \frac{1-\beta(u)}{u} \mathrm{d}u \right)}\, G(v, w)\, \mathrm{d}w \mathrm{d}v. \end{aligned}\tag{78}$$

In the next subsection, we shall prove that (78) can be solved by a fixed-point iteration.

7.5 Solution of (78) by fixed-point iteration

Our goal in this section is to show that (78) can be solved by a fixed-point iteration. We do this in three steps, after introducing some definitions. Let $S > 0$ be fixed, and consider the set F_S of all continuous functions $H : [0, S]^2 \to [0, 1]$ with $H(0, 0) = 1$. For $H \in F_S$ and $(s_1, s_2) \in (0, S]^2$, we put

$$RH(s_1, s_2) = \frac{c^2}{s_1 s_2 r^2} \int_{v=0}^{s_1} \int_{w=0}^{s_2} \left(\frac{v}{s_1}\right)^{\frac{c+\lambda(1-\beta(s_2))}{r}-1} \left(\frac{w}{s_2}\right)^{\frac{c+\lambda(1-\beta(v))}{r}-1} Z(s_1, v; s_2, w)\, H(v, w)\, \mathrm{d}w\mathrm{d}v, \tag{79}$$

where for $0 \le v \le s_1, 0 \le w \le s_2$,

$$Z(s_1, v; s_2, w) = \exp\left(-\frac{\lambda}{r}\Big(\int_v^{s_1} \frac{1-\beta(u)}{u}\mathrm{d}u + \int_w^{s_2} \frac{1-\beta(u)}{u}\mathrm{d}u\Big)\right). \tag{80}$$

Step 1. Let $H \in F_S$. Our aim in this step is to show that we can define RH for all $(s_1, s_2) \in [0, S]^2$, and that this extended RH is a member of F_S.
To this end, we start by noting that for all $(s_1, s_2) \in [0, S]^2$ and $0 \le v \le s_1, 0 \le w \le s_2$,

$$0 < Z(s_1, v; s_2, w) \le 1 = Z(s_1, s_1; s_2, s_2), \tag{81}$$

with equality in the second inequality iff $v = s_1$ and $w = s_2$. This follows from $\beta(u) \le \beta(0) = 1, u \ge 0$. Furthermore, by the boundedness of $\frac{1-\beta(u)}{u} = \mathbb{E}(\mathrm{e}^{-uB^{res}})\,\mathbb{E}(B), u \ge 0$, with B^{res} denoting the residual of a service requirement B, we have that

$$Z(s_1, v; s_2, w) \uparrow 1, \quad (s_1, s_2) \to (0, 0), \tag{82}$$

uniformly in v, w with $0 \le v \le s_1, 0 \le w \le s_2$.
A basic calculation shows that for $(s_1, s_2) \in (0, S]^2$,

$$\frac{c^2}{s_1 s_2 r^2} \int_{v=0}^{s_1} \int_{w=0}^{s_2} \left(\frac{v}{s_1}\right)^{\frac{c+\lambda(1-\beta(s_2))}{r}-1} \left(\frac{w}{s_2}\right)^{\frac{c+\lambda(1-\beta(v))}{r}-1} \mathrm{d}w\mathrm{d}v$$
$$= \frac{c^2}{s_1 r} \int_{v=0}^{s_1} \left(\frac{v}{s_1}\right)^{\frac{c+\lambda(1-\beta(s_2))}{r}-1} \frac{\mathrm{d}v}{c+\lambda(1-\beta(v))}. \tag{83}$$

The quantity on the r.h.s. of (83) is positive and less than 1, and tends to 1 as $(s_1, s_2) \to (0, 0)$, since $\beta(u) < 1, u > 0$ and $\beta(u) \uparrow \beta(0) = 1, u \downarrow 0$. As a consequence of $0 \le H(v, w) \le 1 = H(0, 0)$ and the continuity of H, we have

$$0 < RH(s_1, s_2) < 1, \quad (s_1, s_2) \in (0, S]^2, \tag{84}$$

and also

$$\lim_{(s_1,s_2)\to(0,0)} RH(s_1, s_2) = 1. \tag{85}$$

Similarly, by continuity of H and Z, we have for $s_1 > 0$,

$$\begin{aligned}
\lim_{s_2 \downarrow 0} RH(s_1, s_2) &= \lim_{s_2 \downarrow 0} \frac{c^2}{s_1 s_2 r^2} \int_{v=0}^{s_1} \int_{w=0}^{s_2} \left(\frac{v}{s_1}\right)^{\frac{c+\lambda(1-\beta(s_2))}{r}-1} \left(\frac{w}{s_2}\right)^{\frac{c+\lambda(1-\beta(v))}{r}-1} \\
&\qquad Z(s_1, v; s_2, w) H(v, w)\, \mathrm{d}w \mathrm{d}v \\
&= \frac{c^2}{s_1 r^2} \int_{v=0}^{s_1} \left(\frac{v}{s_1}\right)^{\frac{c}{r}-1} Z(s_1, v; 0, 0) H(v, 0) \\
&\qquad \lim_{s_2 \downarrow 0} \left[\int_{w=0}^{s_2} \left(\frac{w}{s_2}\right)^{\frac{c+\lambda(1-\beta(v))}{r}-1} \mathrm{d}\left(\frac{w}{s_2}\right) \right] \mathrm{d}v \\
&= \frac{c^2}{s_1 r} \int_{v=0}^{s_1} \left(\frac{v}{s_1}\right)^{\frac{c}{r}-1} Z(s_1, v; 0, 0) H(v, 0) \frac{\mathrm{d}v}{c + \lambda(1-\beta(v))}, \qquad (86)
\end{aligned}$$

and this limit is attained uniformly in $s_1 \in [\epsilon, S]$ for any $\epsilon > 0$. In the same way, we have for $s_2 > 0$,

$$\lim_{s_1 \downarrow 0} RH(s_1, s_2) = \frac{c^2}{s_2 r} \int_{w=0}^{s_2} \left(\frac{w}{s_2}\right)^{\frac{c}{r}-1} Z(0, 0; s_2, w) H(0, w) \frac{\mathrm{d}w}{c + \lambda(1-\beta(s_2))}, \qquad (87)$$

and this limit is attained uniformly in $s_2 \in [\epsilon, S]$ for any $\epsilon > 0$. Observe that the r.h.s. of (86) and (87) depend continuously on $s_1 > 0$ and $s_2 > 0$ and that their limit as $s_1 \downarrow 0$ and $s_2 \downarrow 0$ equals 1. Thus, when we define $RH(0, 0) = 1$ and $RH(s_1, 0)$, $RH(0, s_2)$ for $s_1 > 0, s_2 > 0$ by the r.h.s. of (86), (87), we get that $RH(s_1, s_2)$ is defined everywhere on $[0, S]^2$ as a continuous function. From (84) and (85) we then see that $RH \in F_S$.

Remark 7.6. Notice that (87) is in agreement with the integral equation (59) for the one-dimensional $G(s)$; both expressions concern the workload LST in a queue just before its visit begins. That one-dimensional equation can be solved explicitly, cf. (60).

Step 2. In this step, we show that R is a weak contraction of F_S in the sense that for $H_1, H_2 \in F_S$,

$$d(RH_1, RH_2) \leq d(H_1, H_2) := \max_{(s_1, s_2) \in [0, S]^2} |H_1(s_1, s_2) - H_2(s_1, s_2)|, \qquad (88)$$

with equality iff $H_1 = H_2$.
Indeed, we have for $H_1, H_2 \in F_S$ and $(s_1, s_2) \in (0, S]^2$ by the fact that the quantity in (83) is positive and less than 1,

$$\begin{aligned}
|RH_1(s_1, s_2) - RH_2(s_1, s_2)| &\leq \frac{c^2}{s_1 s_2 r^2} \int_{v=0}^{s_1} \int_{w=0}^{s_2} \left(\frac{v}{s_1}\right)^{\frac{c+\lambda(1-\beta(s_2))}{r}-1} \left(\frac{w}{s_2}\right)^{\frac{c+\lambda(1-\beta(v))}{r}-1} \\
&\qquad Z(s_1, v; s_2, w)\, |H_1(v, w) - H_2(v, w)|\, \mathrm{d}w \mathrm{d}v \\
&\leq d(H_1, H_2), \qquad (89)
\end{aligned}$$

with strict inequality when $d(H_1, H_2) > 0$, since $|H_1(v, w) - H_2(v, w)|$ is continuous at

$(v,w)=(0,0)$ and vanishes there. Similarly, using (86), we have for $s_1>0$,

$$\begin{aligned}|RH_1(s_1,0)-RH_2(s_1,0)|\leq&\frac{c}{s_1 r}\int_{v=0}^{s_1}\left(\frac{v}{s_1}\right)^{\frac{c}{r}-1}Z(s_1,v;0,0)\\&\times|H_1(v,0)-H_2(v,0)|\,\frac{\mathrm{d}v}{c+\lambda(1-\beta(v))}\\\leq&d(H_1,H_2),\end{aligned}\tag{90}$$

with strict inequality when $d(H_1,H_2)>0$, since $|H_1(v,0)-H_2(v,0)|$ is continuous at $v=0$ and vanishes there. In the same way, we have from (87) for $s_2>0$,

$$|RH_1(0,s_2)-RH_2(0,s_2)|\leq d(H_1,H_2),\tag{91}$$

with strict inequality when $d(H_1,H_2)>0$. Hence, the continuous function $|RH_1-RH_2|$ is less than $d(H_1,H_2)$ everywhere on the compact set $[0,S]^2$ when $d(H_1,H_2)>0$, and so (88) holds with strict inequality when $d(H_1,H_2)>0$.

Step 3. In this step, we turn to the fixed-point iteration itself. It is easy to prove that there is at most one $H\in F_S$ such that $RH=H$; that such an H does exist in the present case is clear since the G that occurs in (78) satisfies $RG=G$ and is a member of F_S.
We intend to approximate G by iteration. Thus, we set

$$H_0(s_1,s_2)=1,\;\;(s_1,s_2)\in[0,S]^2;\;\;H_{k+1}=RH_k,\;\;k=0,1,\cdots.\tag{92}$$

The operator R has a positive kernel. Then from $RG=G$, (84) and the continuity of all H_k, we get by induction

$$1=H_0(s_1,s_2)\geq H_1(s_1,s_2)\geq\cdots\geq G(s_1,s_2)>0,\;\;(s_1,s_2)\in[0,S]^2.\tag{93}$$

Thus we have

$$H_\infty(s_1,s_2):=\lim_{k\to\infty}H_k(s_1,s_2)\in[G(s_1,s_2),1],\;\;(s_1,s_2)\in[0,S]^2.\tag{94}$$

Our goal in the remainder of Step 3 is to show that $H_\infty(s_1,s_2)=G(s_1,s_2)$, for all $(s_1,s_2)\in[0,S]^2$. We first do this for $(0,S]^2$. While it is conceivable that H_∞ is not continuous everywhere on $[0,S]^2$, so that definition of RH_∞ at the boundary of $[0,S]^2$ might be an issue, we do have that $RH_\infty(s_1,s_2)$ is well-defined per integral formula (79) for $(s_1,s_2)\in(0,S]^2$, and that, by dominated convergence,

$$RH_\infty(s_1,s_2)=H_\infty(s_1,s_2),\;\;(s_1,s_2)\in(0,S]^2.\tag{95}$$

Also, by (94) since $G\in F_S$,

$$\lim_{(s_1,s_2)\to(0,0)}H_\infty(s_1,s_2)=1.\tag{96}$$

We shall show now that (95) and (96) imply that $G(s_1,s_2)=H_\infty(s_1,s_2)$ for $(s_1,s_2)\in(0,S]^2$. To this end, let

$$\delta:=\sup_{(s_1,s_2)\in(0,S]^2}(H_\infty(s_1,s_2)-G(s_1,s_2)),\tag{97}$$

and suppose that $\delta > 0$. We can find $\epsilon > 0$ such that

$$0 \leq H_\infty(v, w) - G(v, w) \leq \frac{1}{2}\delta,\ 0 < v, w \leq \epsilon. \tag{98}$$

Let L be a continuous function on $[0, S]^2$ such that

$$\mathrm{L}(v, w) \begin{cases} = \frac{1}{2}\delta, & v^2 + w^2 \leq \epsilon^2, \\ \in [\frac{1}{2}\delta, \delta], & \epsilon^2 \leq v^2 + w^2 \leq 2\epsilon^2, \\ = \delta, & v^2 + w^2 \geq 2\epsilon^2, (v, w) \in [0, S]^2. \end{cases} \tag{99}$$

Then $\mathrm{L} \geq H_\infty - G$ everywhere on $(0, S]^2$, and $R\mathrm{L}$ is well-defined and continuous on $[0, S]^2$, using the same limits as in (85)-(87). Furthermore,

$$R\mathrm{L}(s_1, s_2) \leq \frac{1}{2}\delta, \quad s_1^2 + s_2^2 \leq \epsilon^2, \tag{100}$$

$$R\mathrm{L}(s_1, s_2) < \delta, \quad s_1^2 + s_2^2 \geq \epsilon^2, (s_1, s_2) \in [0, S]^2, \tag{101}$$

with the strict inequality in (101) following from the continuity of L and $\mathrm{L}(0, 0) = \frac{1}{2}\delta < \delta$, compare (84). Therefore,

$$C := \max_{(s_1, s_2) \in [0, S]^2} R\mathrm{L}(s_1, s_2) < \delta, \tag{102}$$

by the continuity of $R\mathrm{L}$ and compactness of $[0, S]^2$. Then, for $(s_1, s_2) \in (0, S]^2$,

$$0 \leq H_\infty(s_1, s_2) - G(s_1, s_2) = R(H_\infty - G)(s_1, s_2) \leq R\mathrm{L}(s_1, s_2) \leq C < \delta, \tag{103}$$

since $H_\infty - G \leq \mathrm{L}$ everywhere on $(0, S]^2$. This contradicts the definition of δ in (97), and so $\delta = 0$.

We next show that $H_\infty(s_1, 0) = G(s_1, 0)$, $s_1 \in (0, S]$. Let $F_{S,1}$ be the set of all continuous functions $I : [0, S] \to [0, 1]$ with $I(0) = 1$, and define

$$R_1 I(s_1) = \frac{c^2}{s_1 r} \int_0^{s_1} \left(\frac{v}{s_1}\right)^{\frac{c}{r} - 1} Z(s_1, v; 0, 0) I(v) \frac{\mathrm{d}v}{c + \lambda(1 - \beta(v))}, \quad s_1 \in (0, S], \tag{104}$$

$$R_1 I(0) = \lim_{s_1 \downarrow 0} R_1 I(s_1) = 1. \tag{105}$$

Then, repeating the earlier steps for the weak contraction R of F_S,

- R_1 maps $F_{S,1}$ into $F_{S,1}$ and is a weak contraction of $F_{S,1}$,
- $R_1 H_1(s_1) = (RH)_1(s_1)$ when $H \in F_S$ and $H_1 = H(\cdot, 0)$,
- $G_1 = G(\cdot, 0)$ is the unique fixed point of R_1 in $F_{S,1}$,
- the iterands $H_{k1} = H_k(\cdot, 0)$ decrease pointwise to $H_{\infty 1} = H_\infty(\cdot, 0)$,
- $H_{\infty 1}(s_1) = G_1(s_1)$, i.e., $H_\infty(s_1, 0) = G(s_1, 0)$ for $s_1 \in (0, S]$.

Similarly, we can show that $H_\infty(0, s_2) = G(0, s_2)$ for $s_2 \in (0, S]$, and since $H_\infty(0, 0) = 1 = G(0, 0)$, we have shown now that $H_k \downarrow G$ pointwise on all of $[0, S]^2$. Since $[0, S]^2$ is compact, and G and all H_k are continuous, it follows from Dini's theorem [2, Theorem 8.2.6] that $H_k \downarrow G$ uniformly on $[0, S]^2$.

8 Conclusions and possible extensions

In this paper, we studied a two-queue single-server polling model with workload-dependent service speed. For the case of constant visit times of the server, we derived the LST of the steady-state joint workload distribution. We have also extended the results to the case of an arbitrary number of queues with constant visit times. In the case of constant visit times at the first queue and general visit times at the second, we derived the marginal workload distribution at the first queue. For the two-queue case of exponentially distributed visit times, we determined the steady-state marginal workload distributions, and the LST of the steady-state joint workload distribution was analyzed solving a two-dimensional Volterra integral equation by fixed-point iteration. An interesting open problem is to provide an analytic solution to that two-dimensional Volterra integral equation.

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