Kinetically constrained models

Cristina Toninelli

Ceremade, Univ. Paris Dauphine

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Overview of Lecture 1

- Our first KCM: the FA-2f model
- More examples: the most popular KCM
- Motivations from physics: the liquids/glass transition
- A related deterministic dynamics: Bootstrap Percolation
- Ergodicity and mixing for KCM
- Spectral gap, persistence and mean infection time
- A first tool to upper bound time-scales: the BC technique

Overview of Lecture 2

- East model:scaling for $q \downarrow 0$
- FA-1f model: scaling for $q \downarrow 0$
- The general definition of KCM
- Universality results in $d = 2$:
  - universality classes and results for BP
  - universality classes and results for KCM
  - open issue: the case of sub-critical KCM and BP models

Overview of Lecture 3

- Sharp threshold for FA-2f
  - results
  - heuristics
  - sketch of the proof
- Out of equilibrium
  - key questions
  - results for East model
  - partial results for FA-1f
  - open issues
  - more on East model: aging

Fredrickson Andersen 2 spin facilitated model (FA-2f)

An interacting particle system on $\{0,1\}^{\mathbb{Z}^d}$, $d \geq 2$.

0=empty, 1=occupied.

Dynamics: birth and death of particles

- Fix a parameter $q \in [0,1]$
- at rate 1 each site gets a proposal to update its state to empty at rate q and to occupied at rate $1-q$.
- the proposal is accepted iff the site has at least 2 empty nearest neighbours = *iff the kinetic constraint is satisfied*

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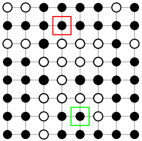

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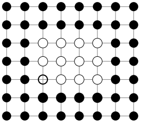

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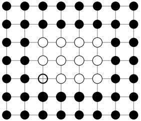

$\rightarrow$ subtle relaxation mechanism

$\rightarrow$ sharp slowdown for $q \downarrow 0$

Several IPS tools fail $\rightarrow$ new tools needed!

Let's change the constraint: other popular KCM

x can be updated iff ...

- FA-jf:
  ... there are at least j empty sites in $\{x \pm \vec{e}_1, \ldots, x \pm \vec{e}_d\}$
- East model:
  ... there is at least 1 empty site in $\{x + \vec{e}_1, \ldots, x + \vec{e}_d\}$
- North-East model ($d = 2$):
  ... both $x + \vec{e}_1$ and $x + \vec{e}_2$ are empty
- Duarte model ($d = 2$):
  ... there are at least 2 empty sites in $\{x + \vec{e}_2, x - \vec{e}_1, x - \vec{e}_2\}$

KCM: motivations from physics

Introduced in the '80's to model the liquid/glass transition

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- no significant structural changes.

$\Rightarrow$ kinetic constraints of KCM dynamics mimic *cage effect* :
if temperature is lowered free volume shrinks ($q \leftrightarrow e^{-1/T}$)

$\Rightarrow$ trivial equilibrium and yet sharp divergence of timescales when $q \downarrow 0$, aging, heterogeneities, ... $\rightarrow$ glassy dynamics

$\Rightarrow$ Key question: how do KCM time-scales diverge for $q \downarrow 0$?

$\Rightarrow$ Sharp divergence $\rightarrow$ numerical simulations do not give clear-cut answers, some of the conjectures were wrong!

Some notation

- $\Omega := \{0,1\}^{\mathbb{Z}^d}$ is the configuration space
- for $\sigma \in \Omega$ and $x \in \mathbb{Z}^d$, σ_x is the occupation variable at x
- for $\Lambda \subset \mathbb{Z}^d$, $\Omega_\Lambda := \{0,1\}^\Lambda$ and σ_Λ is the restriction of σ to Λ
- $\mu := \mu^{(q)} =$ Bernoulli $(1-q)$ product measure on $\mathbb{Z}^d$
- $\mu_\Lambda := \mu_\Lambda^{(q)} =$ Bernoulli $(1-q)$ product measure on Λ
- for $f : \Omega \to \mathbb{R}$, we let $\mu_\Lambda(f) : \Omega_{\mathbb{Z}^d \setminus \Lambda} \to \mathbb{R}$ be the mean of f w.r.t. μ_Λ with the other variable held fixed
- analogous definition for $\mathrm{Var}_\Lambda(f)$

Formal definition of the Markov process

The generator acts on local functions $f : \Omega \to \mathbb{R}$ as

$$\mathcal{L}f(\sigma) := \sum_{x \in \mathbb{Z}^d} c_x(\sigma)(\mu_x(f) - f(\sigma)) =$$

$$= \sum_{x \in \mathbb{Z}^d} c_x(\sigma)(q\sigma_x + (1-q)(1-\sigma_x))(f(\sigma^x) - f(\sigma))$$

with

$$\sigma^x(y) := \begin{cases} \sigma(y) & \text{if } y \neq x \\ 1 - \sigma(x) & \text{if } y = x \end{cases}$$

$$c_x(\sigma) := \begin{cases} 1 & \text{if there constraint is satisfied at x} \\ 0 & \text{otherwise} \end{cases}$$

The corresponding Dirichlet form is:

$$\mathcal{D}(f) := -\mu(f \cdot \mathcal{L}f) = \sum_{x \in \mathbb{Z}^d} \mu\left(c_x \mathrm{Var}_x(f)\right).$$

Focus questions

Q. *Is μ ergodic for the infinite volume process? Is it also mixing? And if so, how fast does converge to equilibrium in $L^2(\mu)$ occur?*

Recall that, if we denote by P_t is the Markov semigroup,

- μ *is ergodic* if for all $f \in L_2(\mu)$ the condition $P_t f = f \;\; \forall t \geq 0$ implies f constant a.s. in μ
- μ *is mixing* if $\forall f, g \in L^2(\mu)$ it holds $\lim_{t \to \infty} \mu(f P_t g) = \mu(f)\mu(g)$.

Thus mixing is stronger than ergodicity.

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Thus mixing is stronger than ergodicity.

$\Rightarrow$ To answer the above questions we should first introduce a related model: Bootstrap Percolation (BP)

2-neighbour Bootstrap Percolation

- At time $t = 0$ sites are i.i.d., empty with probability q, occupied with probability $1 - q$
- At time $t = 1$
  - each empty site remains empty
  - each occupied site is emptied iff it has at least 2 empty n.n.
- Iterate

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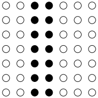
$\Rightarrow$ Final configuration: completely empty or $\exists$ clusters of mutually blocked particles



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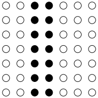

- BP blocked clusters $\leftrightarrow$ blocked particles under FA-2f
- BP is a discrete time deterministic *monotone* dynamics $\rightarrow$ easier to study

Critical density and Infection time

- *Will the whole lattice become empty?*

- $q_c := \inf\{q \in [0,1] : \mu_q(\text{origin is emptied eventually}) = 1\}$

- *How many steps are needed to empty the origin?*
- $\tau^{\text{BP}}(q) := \mu_q(\text{first time at which origin is empty})$

Critical density and Infection time

- *Will the whole lattice become empty?*

  $\rightarrow$ Yes $\forall q > 0$ (Van Enter '87)

- $q_c := \inf\{q \in [0,1] : \mu_q(\text{origin is emptied eventually}) = 1\}$

  $\rightarrow q_c = 0$

- *How many steps are needed to empty the origin?*

- $\tau^{\text{BP}}(q) := \mu_q(\text{first time at which origin is empty})$

$$\text{for } \; q \downarrow 0 \; \text{ w.h.p. } \; \tau_0^{\text{BP}} = \exp\left(\frac{\lambda(d)}{q^{1/(d-1)}}(1 - o(1))\right)$$

  - scaling (Aizenmann, Lebowitz '88)
  - $\lambda(2) = \pi^2/18$ (Holroyd '08)
  - $\lambda(d) = \ldots \; d > 2$ (Balogh, Bollobas, Duminil-Copin, Morris '12)

Bootstrap percolation

- Define analogously the BP processes corresponding to the constraints of FA-jf, Duarte, North-East and East
- $q_c = 0$ for East, Duarte and FA-jf $\forall j \in [1, d]$
- $q_c = 1 - p_c^{OP}$ for North-East
- for $q \downarrow 0$, w.h.p. $\tau_0^{\text{BP}} \sim q^{-1/d}$ for FA-1f and East
- the scalings for FA-jf with $j > 1$ and for Duarte model are more complicate and diverge more rapidly as $q \downarrow 0$ (see Lecture 2)

KCM: ergodicity and mixing

Theorem (Cancrini, Martinelli, Roberto, C.T. '08)

(i) $\forall q > q_c$, μ is mixing (and therefore ergodic);

(ii) $\forall q < q_c$, μ is not ergodic (and therefore not mixing).

Sketch of the proof

- for $q > q_c$ we prove that 0 is a simple eigenvalue of $\mathcal{L}$. Key ingredient: fix $x \in \mathbb{Z}^d$ and $\sigma \sim \mu$, then μ-a.s. there exists a *legal path* from σ to σ^x;
- for $q < q_c$ blocked structures percolate $\rightarrow f := 1_{\mathcal{E}}$ is left invariant by the dynamics and it is not constant a.s. w.r.t. μ where

$$\mathcal{E} := \{\eta : \text{the origin cannot be emptied by BP}\}.$$

Spectral gap and relaxation time

$$\text{gap} := \inf_{\substack{f \in \text{Dom}(\mathcal{L}) \\ \text{Var}(f) \neq 0}} \frac{\mathcal{D}(f)}{\text{Var}(f)}$$

i.e. $\text{gap} = T_{\text{rel}}^{-1}$, where T_{rel} is the smallest constant such that

$$\text{Var}(f) \leq T_{\text{rel}} \, \mathcal{D}(f) \quad \forall f$$

Thus, if gap > 0, it holds

$$\text{Var}\,(P_t f) = \mu(f P_t f) - \mu(f)^2 \leq \exp(-2\,t\,\text{gap})\text{Var}(f) \qquad \forall f \in L^2(\mu)$$

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Theorem (Cancrini, Martinelli, Roberto, C.T. '08)

For FA-jf, East, North-East and Duarte models it holds

$$\text{gap} > 0 \quad \forall q > q_c$$

Persistence function

$$F(t) := \int d\mu(\sigma)\, \mathbb{P}_\sigma(\sigma_s(0) = \sigma(0)\, \forall s \le t)$$

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Recall that $\forall \mathcal{A} \subset \Omega$ it holds

$$\mathbb{P}_\mu(\tau_{\mathcal{A}} > t) \le \exp(-t\lambda_{\mathcal{A}})$$

with $\tau_{\mathcal{A}}$ the hitting time of $\mathcal{A}$, and

$$\lambda_{\mathcal{A}} := \inf\left\{\mathcal{D}(f) :\ \mu(f^2) = 1,\ f \equiv 0 \text{ on } \mathcal{A}\right\}.$$

Thus

$$F(t) = \mathbb{P}(\tau_{\{\sigma(0)=1\}} > t) + \mathbb{P}(\tau_{\{\sigma(0)=0\}} > t) \le e^{(-(1-q)\, t\, \text{gap})} + e^{(-q\, t\, \text{gap})}$$

Mean infection time

$\mathbb{E}_\mu(\tau_0)$ with $\tau_0 =$ hitting time of $\{\sigma(0) = 0\}$.
The upper bound on $F(t)$ implies

$$\mathbb{E}_\mu(\tau_0) \leq (1 + o(q))\frac{T_{\rm rel}}{q}$$

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An easy lower bound

For any KCM there exists $\delta > 0$ s.t. for q small enough it holds

$$\mathbb{E}_\mu(\tau_0) \geq \delta \tau^{\mathrm{BP}}(q)$$

General but usually very far from the correct scaling.

Key idea: BP features only infecting moves while KCM has both infecting and healing moves $\rightarrow$ BP infects the origin at least as fast as the corresponding KCM.

East model in $d = 1$: gap > 0 via the BC technique

- gap:=spectral gap for East on $\mathbb{Z}$
- for $\Lambda \subset \mathbb{Z}$, $\text{gap}_\Lambda :=$ spectral gap for East on Λ with 0 b.c.
- $\Lambda_k := [0, 2^k]$ and $\gamma_k := 1/\text{gap}_{\Lambda_k}$
- $\text{gap} \geq \inf_k \text{gap}_{\Lambda_k}$

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$\rightarrow$ if we prove

$$\gamma_\mathbf{k} \leq \mathbf{a_k} \gamma_\mathbf{k-1} \quad \textbf{with} \prod_{\mathbf{k_0}}^{\infty} \mathbf{a_k} < \infty \quad \textbf{for a finite } \mathbf{k_0}$$

we have proven gap > 0

First idea

- Devide Λ_k into two B_1, B_2 each of the form Λ_{k-1}
- define an auxiliary block dynamics: ...
- $\rightarrow \quad T_{\mathrm{rel},\Lambda_k} \leq T_{\mathrm{rel}}^{\mathrm{block}} \max T_{\mathrm{rel},B_1} T_{\mathrm{rel},B_2}$
- $T_{\mathrm{rel}}^{\mathrm{block}} = 1$ (product measure) $\rightarrow \gamma_k \leq \gamma_{k-1}$
- ...so easy?!
- $\max T_{\mathrm{rel},B_1} T_{\mathrm{rel},B_2} = \infty$!

A general two-site constrained Poincaré inequality

Lemma

- $(\mathbb{X}_1, \nu_1)$ and $(\mathbb{X}_2, \nu_2)$ = finite probability spaces
- $(\mathbb{X}, \nu)$ = the associated product space
- $\mathcal{H} \subset \mathbb{X}_2$ with $\nu_2(\mathcal{H}) > 0$.

Then, for any $f : \mathbb{X} \to \mathbb{R}$, it holds

$$\mathrm{Var}_\nu(f) \leq \left(1 - \sqrt{1 - \nu_2(\mathcal{H})}\right)^{-1} \nu\left(1_{\{X_2 \in \mathcal{H}\}} \mathrm{Var}_{\nu_1}(f) + \mathrm{Var}_{\nu_2}(f)\right).$$

- Let $\vec{X} := (X_1, X_2)$. The inequality is an upper bound on T_{rel} for the Markov process reversible w.r.t. ν with generator:

$$\mathcal{L}f(\vec{X}) = 1_{\{X_2 \in \mathcal{H}\}}\left[\nu_1(f) - f(\vec{X})\right] + \left[\nu_2(f) - f(\vec{X})\right]$$

- easy proof via direct calculation on eigenvectors

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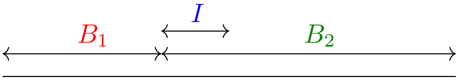

$\Lambda_k = [0, 2^k] = B_1 \sqcup B_2, \quad B_1 := [0, 2^{k-1} - 1], \quad B_2 := [2^{k-1}, 2^k],$

$I := [2^{k-1}, 2^{k-1} + 2^{k/3}], \quad \mathcal{H} \subset \Omega_{B_2} := \{\eta : \exists \text{ at least one zero in } I\}$

Use the two-site constrained Poincaré inequality to get

$$\mathbf{Var}_{\Lambda_k}(\mathbf{f}) \leq \epsilon_{\mathbf{k}} \mu_{\Lambda_k} \left(\mathbf{1}_{\mathcal{H}} \mathbf{Var}_{\mathbf{B_1}}(\mathbf{f}) + \mathbf{Var}_{\mathbf{B_2}}(\mathbf{f}) \right)$$

with $\epsilon_k = \left(1 - \sqrt{(1-q)^{2^{k/3}}} \right)^{-1}$

East model in $d = 1$: gap > 0 via the BC technique

Our goal: upper bound the r.h.s with $a_k(\text{gap}_{[0,2^{k-1}]})^{-1}\,\mathcal{D}_{\Lambda_k}(f)$

$$r.h.s. := \epsilon_k\,\mu_{\Lambda_k}\left(1_{\mathcal{H}}\,\mathrm{Var}_{B_1}(f) + \mathrm{Var}_{B_2}(f)\right)$$

Via the Poincaré inequality (i.e. the definition of gap) for East:

$$\mu_{\Lambda_k}(\mathrm{Var}_{B_i}(f)) \le (\text{gap}(\mathcal{L}_{B_i}))^{-1}\sum_{x\in B_i}\mu_{\Lambda_k}\left(c_{x,B_i}\mathrm{Var}_x(f)\right)$$

$$c_{x,B_i}(\sigma) := \begin{cases} 1-\sigma_{x+1} & \text{if } x \neq \text{ rightmost site of } B_i \\ 1 & \text{otherwise}\end{cases}$$

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$$c_{x,B_{\mathbf{2}}}(\sigma) = x,\Lambda_k(\sigma), \text{ but } c_{x,B_{\mathbf{1}}}(\sigma) \ge c_{x,\Lambda_k}(\sigma)\ !$$

$\rightarrow$ r.h.s. $\neq \epsilon_k\,\gamma_{k-1}\,\mu(\mathcal{D}_{B_1}(f) + \mathcal{D}_{B_2}(f)) = \epsilon_k\,\gamma_{k-1}\mathcal{D}_{\Lambda_k}(f)$

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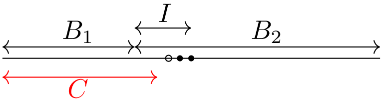

"Enlarge" Var_{B_1} by convexity to the random interval C:=...

$$c_{x,C}(\sigma)1_{\mathcal{H}} = c_{x,\Lambda_k}(\sigma)1_{\mathcal{H}} \quad \forall x \in C, \sigma \in \Omega_{\Lambda_k}$$

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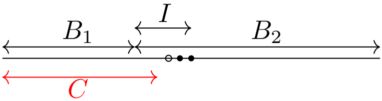

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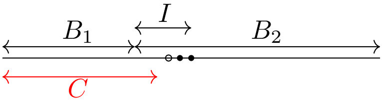

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$$\Rightarrow \; \text{Var}_{\Lambda_k} \leq \epsilon_k \gamma_{k-1} \mathcal{D}_{\Lambda_k} + \sum_{x \in I} \mu_{\Lambda_k}\left(c_{x,\Lambda_k}\text{Var}_x(f)\right)$$

- Technical point: "move" I and average over its positions
- use the variational definition of the spectral gap

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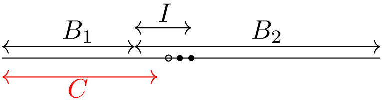

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$$c_{x,C}(\sigma)1_{\mathcal{H}} = c_{x,\Lambda_k}(\sigma)1_{\mathcal{H}} \quad \forall x \in C, \sigma \in \Omega_{\Lambda_k}$$

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$$\Rightarrow \ \mathrm{Var}_{\Lambda_k} \leq \epsilon_k \gamma_{k-1} \mathcal{D}_{\Lambda_k} + \sum_{x \in I} \mu_{\Lambda_k}\left(c_{x,\Lambda_k}\mathrm{Var}_x(f)\right)$$

- Technical point: "move" I and average over its positions
- use the variational definition of the spectral gap

$$\Rightarrow \ \gamma_{\mathbf{k}} \leq \mathbf{a_k}\gamma_{\mathbf{k-1}} \qquad \text{with} \quad \prod_{\mathbf{k_0}}^{\infty} \mathbf{a_k} < \infty \quad \text{for a finite } \mathbf{k_0}$$

East model in $d = 1$*: gap* > 0 *via the BC technique*

Theorem (Cancrini, Martinelli, Roberto, C.T '08)

For all $\delta > 0$ there exists C_δ s.t.

$$T_{\text{rel}} = \frac{1}{\text{gap}} \leq C_\delta \exp\left(\frac{|\log q|^2}{(2-\delta)\log 2}\right)$$

East model in $d = 1$: gap > 0 via the BC technique

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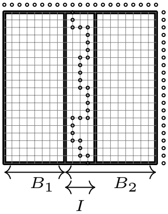
Remark

- The conjectures in physics were incorrect, claiming $T_{rel} \sim \exp^{\left(\frac{|\log q|^2}{\log 2}\right)}$
- The additional factor $1/2$ is due to the fact that (unexpectedly!) energy and entropy contributions are of the same order (more on Lecture 2)

Can BC be used to upper bound T_{rel} for all KCM?

Consider FA-2f on $\mathbb{Z}^2$



$$\Lambda_k := [1, 2^k] \times [1, 2^k]$$

$$I := [2^{k-1}, 2^{k-1} + 2^{k/3}] \times [1, 2^k]$$

$$\mathcal{H} := \{\eta : \exists \text{ top-bottom empty crossing in } I\}$$

Following the lines of the proof for East on $\mathbb{Z}$ we get

$$\gamma_k := \text{ gap }_{\Lambda_k}^{-1} \leq a_k \, \gamma_{k-1}$$

with $a_k := \left(1 - \sqrt{1 - \mu(\mathcal{H})}\right)^{-1} \to \prod_{k_0}^{\infty} a_k < 1$ iff $q > q_c^{\text{site perc.}}$

Can BC be used to upper bound $T_{\rm rel}$ for all KCM?

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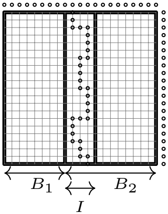

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$\to$ BC cannot be used (alone) to upper bound $T_{\rm rel}$ for all $q > 0$!

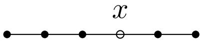
Renormalisation + other tools...

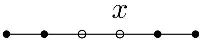
East model $d = 1$: combinatorics

Constraint = to update a site we need its right neighbour empty

East model $d = 1$: combinatorics

Constraint = to update a site we need its right neighbour empty

- If we start from a single vacancy 

  and we can create 1 zero we reach only 

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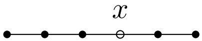

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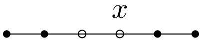
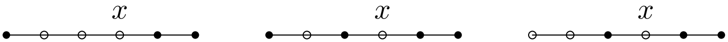


- if we can create up to 2 simultaneous additional zeros we reach also:



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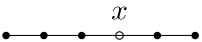

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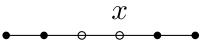

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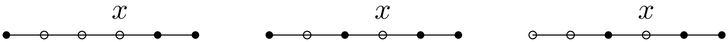

- if we can create up to n simultaneous additional zeros
  - one of the configurations that we can reach has its leftmost vacancy at $x - (2^n - 1)$;
  - all the others have leftmost vacancy in $[x, x - (2^n - 1)]$

East model $d = 1$: combinatorics

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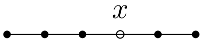

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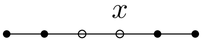

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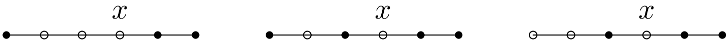

- if we can create up to n simultaneous additional zeros
  - one of the configurations that we can reach has its leftmost vacancy at $x-(2^n-1)$;
  - all the others have leftmost vacancy in $[x, x-(2^n-1)]$

$\Rightarrow$ the East model has logarithmic energy barriers [Evans Sollich '99, see also Chung Diaconis Graham '01]

East model $d = 1$: scaling for $q \downarrow 0$

- The first vacancy at the left of origin is at $\ell \sim 1/q$
- Trivially, $\tau_0^{\mathrm{BP}}(q) \sim 1/q$
- $\mathbb{E}_{\mu_q}(\tau_0) \sim$ time to create $\log_2(\ell)$ empty sites
- $\rightarrow \mathbb{E}_{\mu_q}(\tau_0) = 1/q^{\Theta(1)|\log q|}$ [Aldous, Diaconis JSP '02]

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- $\to \mathbb{E}_{\mu_q}(\tau_0) = 1/q^{\Theta(1)|\log q|}$ [Aldous, Diaconis JSP '02]
- Sharp result (taking entropy into account) in $d \geq 1$

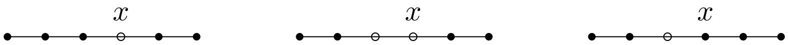
$$\lim_{q \to 0} \frac{\log \mathbb{E}_{\mu_q}(\tau_0)}{|\log q|^2} = (2d \log 2)^{-1}$$

[Cancrini, Martinelli, Roberto, C.T. PTRF '08] for $d = 1$

[Chleboun, Faggionato, Martinelli AoP '16] for $d \geq 2$

FA-1f: scaling for $q \downarrow 0$

Constraint = to be update we need an empty nearest neighbour



- a vacancy can move of one step by creating one additional vacancy $\rightarrow$ $\sim$ r.w. of rate q^{-1} $\rightarrow$ non-cooperative!

$\rightarrow$ $d = 1$ $\quad \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-1}(1/q)^2 = q^{-3};$

$\rightarrow$ $d = 2$ $\quad q^{-2} \leq \mathbb{E}_{\mu_q}(\tau_0) \leq q^{-2}|\log q|;$

$\rightarrow$ $d \geq 3$ $\quad \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-1}(1/q^{1/d})^d = q^{-2}$

[Cancrini, Martinelli, Roberto, C.T. PTRF ’08 + Shapira JSP ’20]

KCM: the general definition

Configurations : $\eta \in \Omega := \{0,1\}^{\mathbb{Z}^d}$, $0 =$ empty, $1 =$ occupied

Fix a density parameter $q \in [0,1]$ and an update family $\mathcal{U}$ with

$$\mathcal{U} = \{U_1, \ldots, U_m\}, \quad U_i \subset \mathbb{Z}^d \setminus 0, \quad |U_i| < \infty, \quad m < \infty$$

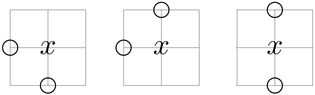
i.e. $\mathcal{U}$ is a finite collection of local neighbourhoods of the origin

Fix $\eta \in \Omega$ and $x \in \mathbb{Z}^d$: "the constraint is satisfied at x" iff at least one of the translated sets $U_i + x$ is completely empty

Dynamics: each site with the constraint satisfied is updated to empty at rate q and to occupied at rate $1 - q$

Our examples

- FA-jf model:
  $\mathcal{U}$ = all sets containing j nearest neighbours of the origin
- East model: $\mathcal{U} = \{\vec{e}_1, \ldots, \vec{e}_d\}$
- North-East model: $\mathcal{U} = \{U_1\}$ with $U_1 = \{(0,1),(1,0)\}$
- Duarte model: $\mathcal{U} = \{U_1, U_2, U_3\}$ with

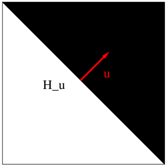


U_1 U_2 U_3

Universality class of $\mathcal{U}$ in $d = 2$

We need the notion of stable and unstable directions

- Fix a direction $\vec{u}$
- Start from a configuration which is
  - completely empty on the half plane perpendicular to $\vec{u}$ in the negative direction (H_u)
  - filled otherwise
- Run the bootstrap dynamics



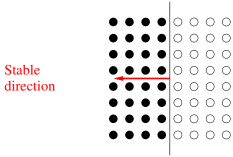
$$\vec{u} \text{ is } \begin{cases} \text{stable} & \text{if no other site can be emptied} \\ \text{unstable} & \text{otherwise} \end{cases}$$

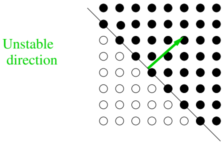
Stable and unstable directions: examples

Of course, the stability of a direction depends on $\mathcal{U}$

Ex. East model:

$\vec{u} = -\vec{e}_1$ is stable; $\vec{u} = \vec{e}_1 + \vec{e}_2$ is unstable



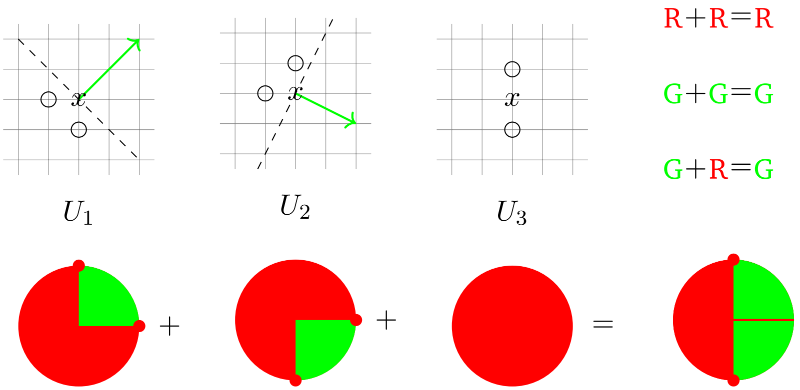


Instead :

- both directions are unstable for 1-neighbour bootstrap
- both directions are stable for North East

How to easily identify all stable and unstable directions

Draw the half planes H_u and $\mathbb{Z}^2 \setminus H_u$ so that the separation line contains the origin. $\vec{u}$ is unstable iff $U_i \subset H_u$ for at least one i



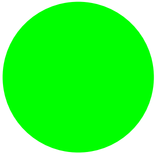
Easy-to-use criterion to determine the class of any $\mathcal{U}$
(m simple geometric checks, m= # of rules)

Supercritical universality class

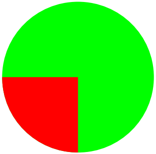
$\mathcal{U}$ is supercritical iff there exists an open semicircle $\mathcal{C}$ which does not contain stable directions.

A supercritical model is

- rooted if it has at least 2 non opposite stable directions
- unrooted otherwise



FA-1f
Unrooted

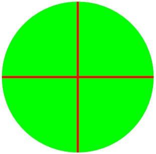


East
Rooted

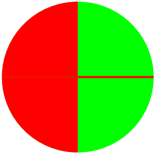
$\mathcal{U}$ is critical if it is not supercritical and there exists an open semicircle $\mathcal{C}$ with only a finite number of stable directions

A critical model is

$\left\{ \right.$
- finitely critical if it has a finite number of stable directions
- infinitely critical otherwise



FA-2f
Finitely critical



Duarte
Infinitely critical

Subcritical universality class

Two equivalent definitions

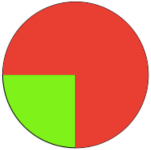
$\mathcal{U}$ is subcritical iff it is neather supercritical nor critical

or

$\mathcal{U}$ is subcritical iff each open semicircle has infinite stable directions

$\Rightarrow$ $q_c > 0$: blocked clusters percolate at $q < q_c$

Example: North East model



BP universality results in $d = 2$

Theorem [Bollobás, Smith, Uzzell '15 + Balister, Bollobás, Przykucki, Smith '16 + Bollobás, Duminil-Copin, Morris, Smith '16]

- Supercritical: $q_c = 0, \quad \tau_0^{\text{BP}}(q) = 1/q^{\Theta(1)}$ w.h.p. as $q \downarrow 0$
- Critical: $q_c = 0, \quad \tau_0^{\text{BP}}(q) = \exp(|\log q|^{O(1)}/q^{\alpha})$ w.h.p. as $q \downarrow 0$
- Subcritical: $q_c > 0$

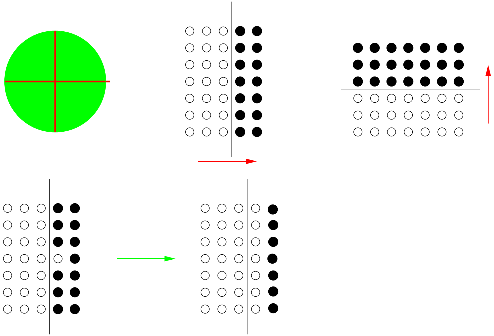
Definition of the "difficulty", α

$\alpha = \min_{\mathcal{C}} \max_{\vec{u} \in \mathcal{C}} d(\vec{u})$ with

$d(\vec{u})$ = minimal number of empty sites to unstabilize $\vec{u}$

Difficulty: examples

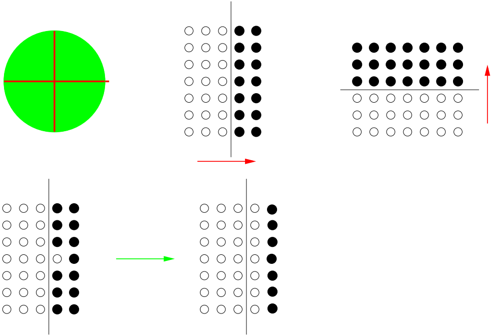
FA-2f :



- $\vec{e}_1$ is stable and $d(\vec{e}_1) = 1$
- same for $-\vec{e}_1$ and $\pm\vec{e}_2$
- all other directions are unstable

Difficulty: examples

FA-2f :



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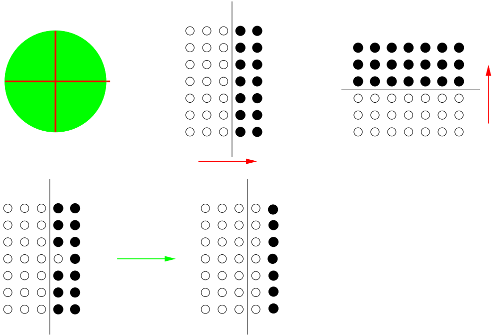
$\rightarrow$ $\alpha = 1$ for FA-$2f$

Difficulty: examples

FA-2f :



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- all other directions are unstable

$\rightarrow$ $\alpha = 1$ for FA-$2f$

Exercice: check that Duarte model also has difficulty $\alpha = 1$

KCM: universality results in $d = 2$

Theorem [Martinelli,Morris,C.T. '19; Marêché,Martinelli,C.T. '20, Marêché, Hartarsky, Toninelli '20, Hartarsky, Martinelli, C.T.'21]

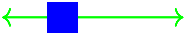
1. Supercritical unrooted: $\tau(q) = \frac{1}{q^{\Theta(1)}}$ and $\tau_0^{\mathrm{BP}} \sim \frac{1}{q^{\Theta(1)}}$ (FA-1f)
2. Supercritical rooted: $\tau_0(q) = \frac{1}{q^{\Theta(1)|\log q|}} \gg \tau_0^{\mathrm{BP}} = \frac{1}{q^{\Theta(1)}}$ (East)
3. Finitely critical: $\tau_0(q)$ and $\tau_0^{\mathrm{BP}} \sim \exp\left(\frac{\Theta(1)(\log q)^{\Theta(1)}}{q^{\nu}}\right)$ (FA-2f)
4. Infinitely critical:
   $\tau(q) = \exp\left(\frac{(\log q)^c}{q^{2\nu}}\right) \gg \tau_0^{\mathrm{BP}} = \exp\left(\frac{(\log q)^c}{q^{\nu}}\right)$ (Duarte)

$\longrightarrow$ For supercritical rooted and infinitely critical models
$\mathbb{E}_{\mu_q}(\tau_0) \gg \tau_0^{\mathrm{BP}}(q)^{\nu}$ for all ν.

Hartarsky Marêché $'21^{+}$: log corrections !

Supercritical models

- Unrooted: large empty droplet can move back and forth



→ renormalise to an FA-1f with effective density $q_{\text{eff}} = q^{\Theta(1)}$

→ $\mathbb{E}_{\mu_q}(\tau_0) \sim q^{-\Theta(1)}$

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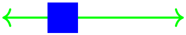
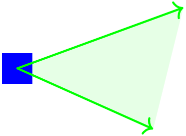


$\rightarrow$ renormalise to an FA-1f with effective density $q_{\text{eff}} = q^{\Theta(1)}$
$\rightarrow$ $\mathbb{E}_{\mu_q}(\tau_0) \sim q^{-\Theta(1)}$

- Rooted: any empty droplet can move only inside a cone

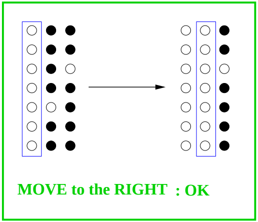


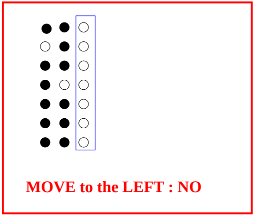
$\Longrightarrow$ logarithmic energy barriers as for East [Marêché '20]

$\rightarrow$ renormalise to an East with effective density $q_{\text{eff}} = q^{\Theta(1)}$
$\rightarrow$ $\mathbb{E}_{\mu_q}(\tau_0) \sim q_{\text{eff}}^{\Theta(|\log q_{\text{eff}}|)} = e^{\Theta(\log q)^2}$

Duarte model: heuristics

Constraint at x: at least 2 vacancies in $\{x - \vec{e}_1, x + \vec{e}_2, x - \vec{e}_2\}$





An empty segment of length $\ell = 1/q\,|\log q|$ can (typically) create an empty segment to its right, but never to its left!

$\rightarrow$ it is a mobile droplet with East-like dynamics and

density $q_{\text{eff}} = q^{\ell} = e^{-\Theta(\log q)^2/q}$

Duarte model: heuristics

- nearest empty droplet to the origin is at distance $L \sim q_{\text{eff}}^{-1}$

$\rightarrow$ $T^{\text{BP}} \sim L = \exp\left(\frac{\Theta(1)|\log q|^2}{q}\right)$
[Mountford '95]

Duarte model: heuristics

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$\rightarrow$ $T^{\text{BP}} \sim L = \exp\left(\frac{\Theta(1)|\log q|^2}{q}\right)$
[Mountford '95]

- Duarte droplets move East like $\rightarrow$ to empty the origin we have to create $\log(L)$ simultaneous droplets

$\rightarrow$ $\mathbb{E}_{\mu_q}(\tau_0) \sim q_{\text{eff}}^{-\log L} \sim \exp\left(\frac{\Theta(1)|\log q|^4}{q^2}\right) \gg T^{\text{BP}}$

[Martinelli, Morris, C.T. '19 + Marêché, Martinelli, C.T. '20]

The general critical case

- Droplets are empty regions with model dependent shape of size $\ell = q^{-\alpha}|\log q|$ and density $q_{\text{eff}} = q^{\ell}$

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- For infinitely critical KCM the droplet motion is East like

$$\rightarrow \tau_0 \sim q_{\text{eff}}^{\Theta(|\log q_{\text{eff}}|)} = \exp\left(\frac{|\log q|^{O(1)}}{q^{2\alpha}}\right)$$

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- Droplets are empty regions with model dependent shape of size $\ell = q^{-\alpha}|\log q|$ and density $q_{\text{eff}} = q^{\ell}$
- For infinitely critical KCM the droplet motion is East like

$$\to \tau_0 \sim q_{\text{eff}}^{\Theta(|\log q_{\text{eff}}|)} = \exp\left(\frac{|\log q|^{O(1)}}{q^{2\alpha}}\right)$$

- For finitely critical KCM the droplet motion is a subtle combination of East on mesoscopic scales $\left(L \sim q^{-\Theta(1)}\right)$ and FA-1f on macroscopic scales ($\sim q_{\text{eff}}^{-1}$)

$$\to \tau_0 \sim q_{eff}^{\Theta(\log L)} = \exp\left(\frac{|\log q|^{O(1)}}{q^{\alpha}}\right)$$

Upper bound: main obstacles and tools

- droplets move only on a good environment
- the environment evolves and can ”lose its goodness”
- no monotonicity $\rightarrow$ we cannot ”freeze” the environment
- the motion of droplets is not random walk like
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Upper bound: main obstacles and tools

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→ a very flexible long range Poincaré inequality
[Martinelli, C.T. ’19]

→ renormalisation

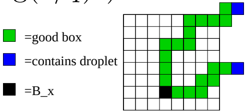
→ Matryoshka Dolls: a new technique to compare Dirichlet forms avoiding canonical paths
[Martinelli, Morris, C.T. ’19]

Lower bound

- **Key idea**: construct a bottleneck involving $\log(L)$ droplets
- **Main difficulty**: droplets are not "rigid objects"!
- **Solution**: an algorithmic identification of droplets and of an efficient cut-set...

More on the upper bound: the case of FA-2f

- renormalise on $\ell \times \ell$ boxes, $\ell = 1/q \log(1/q)$
- auxiliary long range *block* dynamics:
  put equilibrium on box B_x at rate 1 iff it belongs to a good cluster with two droplets at distance at most
  $L = \exp(1/q \log(1/q)^2)$



- establish a general long range Poincaré inequality that yields $T_{rel}^{aux} = O(1)$
- use canonical paths for reversible Markov chains or better repeat the same game inside the path on a smaller scale: now the renormalised sites are the columns of the box
  …Matryoshka Dolls!

FA2f: sharp treshold

Theorem [Hartarsky, Martinelli, C.T. '20]

As $q \downarrow 0$, w.h.p. for the stationary FA-2f model on $\mathbb{Z}^d$ it holds

$$\tau_0 = \exp\left(\frac{d \times \lambda(d)}{q^{1/(d-1)}}(1 - o(1))\right), \quad d \geq 2$$

the same result holds for $\mathbb{E}_{\mu_q}(\tau_0)$. Thus, w.h.p. $\tau_0 = (\tau_0^{\mathrm{BP}})^{d+o(1)}$.

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Remark

- This is not a corollary of the BP result:
  the emptying/occupying mechanism of FA-2f has no counterpart in BP!
- We settle contrasting conjectures in physics literature

High level ideas

- Relaxation is driven by the motion of unlikely large patches of empty sites, the mobile droplets
- droplet density $\rho_D := \exp\left(-\frac{d\times\lambda(d)}{q^{1/d-1}}(1+o(1))\right)$

  droplet length $L_D := \text{poly}(q)$
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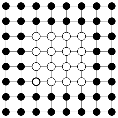

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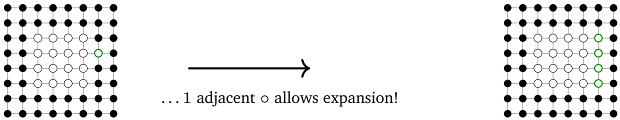

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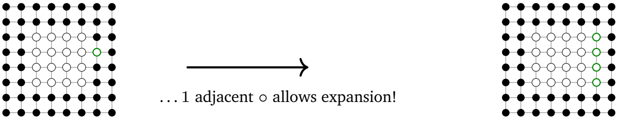

- Motion requires few additional empty sites → this *good environment* is very likely for large droplets ($q \downarrow 0$)

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- $\tau_0^{\mathrm{BP}} \sim$ distance of mobile droplet to origin

$$\rightarrow \tau_0^{\mathrm{BP}} \sim 1/\rho_D^{1/d} \sim {\tau_0}^{1/d}$$

How do droplets look like? the d=2 case

- Multi-scale construction: empty core of size $1/\sqrt{q}$ + empty sites that allow to move the core anywhere inside without creating a larger empty interval

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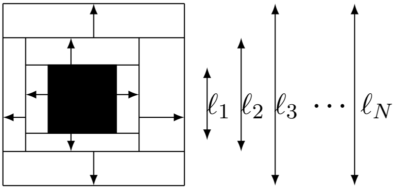

- the black square has no double raws fully occupied and one raw with no consecutive filled sites $\to$ it is emptiable
- vertical arrow = no double raws fully occupied
- horizontal arrow = no double columns fully occupied
- $\ell_n := e^{n\sqrt{q}}/\sqrt{q}, \quad N = 8|\log q|/\sqrt{q} \ \to \ \ell_N = L_D = \text{poly}(q)$

More precisely...

A multi-scale definition

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  - R is a single site for $n = 0$;
  - $R = \ell_m \times h$ with $h \in (\ell_{m-1}, \ell_m]$ for $n = 2m$;
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Droplets are defined as $\ell_N \times \ell_N$ SG rectangles

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Coalescing + Branching + Simple Exclusion $\rightarrow$ g-CBSEP

g for ”generalized” (not just $0/1$)

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- *Hitting times* $\leftrightarrow$ *Dirichlet eigenvalues*
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$$\to \tau_0 \le \exp\left(\frac{d \times \lambda(d)}{q^{1/(d-1)}} (1 - o(1)) \right)$$

Thanks for your attention!

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The g-CBSEP chain

- $G = (V, E)$: finite connected graph
- $(\mathcal{S}, \pi)$: finite probability space
- $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ and $\rho = \pi(\mathcal{S}_1)$
- given $\sigma \in \mathcal{S}^V$, $x \in V$ is *occupied* iff $\sigma_x \in \mathcal{S}_1$
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Theorem [Hartarsky, Martinelli, C.T. '20]

As $\rho \downarrow 0$, $\;T_{\text{rel}}^{\text{g-CBSEP}} \le O(\rho^{-1} \log(1/\rho))$

FA-jf model

- for jn-BP for all $d \geq j \geq 2$, w.h.p. it holds

$$\tau_0^{\text{BP}} \sim \exp^{j-1}\left(\frac{\tilde{\lambda}(d,j)}{q^{1/(d-j+1)}}\right)$$

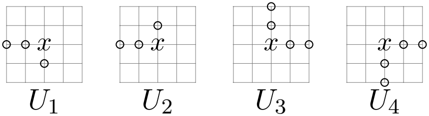
$\exp^k$ = exponential iterated k times (Balogh, Bollobas, Duminil-Copin, Morris '12)

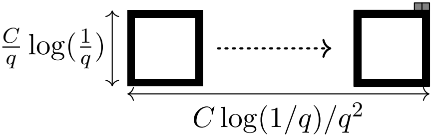
Same scaling for τ_0 (Hartarsky, Martinelli, C.T. in progress)

- $j = 1$: $\tau_0^{\text{BP}} = 1/q^{1/d}$, $\tau_0 = 1/q^{\nu(d)}$,
  $\nu(1) = 3$, $\nu(d) = 2$ $d \geq 2$ (log corrections in $d = 2$)
  (Cancrini, Roberto, Martinelli, C.T. '08 + Shapira '20)

- $d < j$: $\tau_0 = \tau_0^{\text{BP}} = \infty$ w.h.p. for $q \to 0$

Finitely critical $\mathcal{U}$: an example





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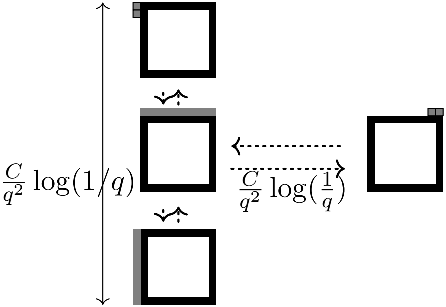

The move of one step in the $-\vec{e}_1$ direction the droplet has to move in the direction $\vec{e}_2$ until reaching the first infected pair of empty sites. A subtle hierarchical combination of East paths...

East model: why a log barrier?

Start with a single vacancy at the origin

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1. optimal path proceeds via *stepping-stones*: create isolated vacancy at $-L_1(n)$; restart from it to create an isolated vacancy at $-L_1(n) - L_1(n-1)$; ...
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2. to put an isolated 0 at $-L_1(n)$ we should have a 0 at $-L_1(n) + 1$ and remove it using at most $n - 1$ vacancies
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$$\Rightarrow \mathbf{L(n) = 2^n - 1}$$

A general long-range constrained Poincaré inequality

Lemma [Martinelli, C.T. '19]

- $(\mathbb{X}, \nu)$= finite probability space
- $(\tilde{\Omega}, \tilde{\mu}) = (\mathbb{X}^{\mathbb{Z}^d}, \otimes_{x \in \mathbb{Z}^d} \nu_x)$
- for $x \in \mathbb{Z}^d$ let
    - $\mathbb{Z}^d_{x,\uparrow} := x + \{y : y_1 + \dots y_d > 0\}$
    - $\Delta_x \subset \mathbb{Z}^d_{x,\uparrow}$ be a finite set
    - $\mathcal{A}_x$ be an event depending only on $\{\omega(y)\}_{y \in \Delta_x}$

Assume $\sup_z \sum_{\substack{x \in \mathbb{Z}^d \\ x \cup \Delta_x \ni z}} (1 - \mu(\mathcal{A}_x)) < 1/4$, then

$$\mathrm{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^d} \mu\left(1_{\mathcal{A}_x} \mathrm{Var}_x(f)\right) \quad \forall f \ \text{ local}$$

More general oriented neighborhood and product of characteristic functions possible ...