Kinetically constrained models

Cristina Toninelli
Ceremade, Univ. Paris Dauphine

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Overview of Lecture 1

- Our first KCM: the FA-2f model
- More examples: the most popular KCM
- Motivations from physics: the liquids/glass transition
- A related deterministic dynamics: Bootstrap Percolation
- Ergodicity and mixing for KCM
- Spectral gap, persistence and mean infection time
- A first tool to upper bound time-scales: the BC technique
Overview of Lecture 2

• East model: scaling for \( q \downarrow 0 \)
• FA-1f model: scaling for \( q \downarrow 0 \)
• The general definition of KCM
• Universality results in \( d = 2 \):
  • universality classes and results for BP
  • universality classes and results for KCM
  • open issue: the case of sub-critical KCM and BP models
Overview of Lecture 3

- Sharp threshold for FA-2f
  - results
  - heuristics
  - sketch of the proof
- Out of equilibrium
  - key questions
  - results for East model
  - partial results for FA-1f
  - open issues
  - more on East model: aging
Fredrickson Andersen 2 spin facilitated model (FA-2f)

An interacting particle system on \( \{0, 1\}^{\mathbb{Z}^d} \), \( d \geq 2 \).

0=empty, 1=occupied.

Dynamics: birth and death of particles

- Fix a parameter \( q \in [0, 1] \)
- at rate 1 each site gets a proposal to update its state to empty at rate \( q \) and to occupied at rate \( 1 - q \).
- the proposal is accepted iff the site has at least 2 empty nearest neighbours = iff the kinetic constraint is satisfied
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  - injecting more vacancies can help filling more sites
  - coupling and censoring arguments fail
- There exist blocked configurations
  - ergodicity issues, several invariant measures
  - relaxation is not uniform on the initial condition
  - worst case analysis is too rough and coercive inequalities fail
- cooperative dynamics
  - finite empty regions cannot expand
  - subtle relaxation mechanism
  - sharp slowdown for $q \downarrow 0$
- Several IPS tools fail
  - new tools needed!
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Several IPS tools fail $\rightarrow$ new tools needed!
Let's change the constraint: other popular KCM

$x$ can be updated iff . . .

- **FA-$j_f$:**
  
  ... there are at least $j$ empty sites in \( \{x \pm \vec{e}_1, \ldots, x \pm \vec{e}_d\} \)

- **East model:**
  
  ... there is at least 1 empty site in \( \{x + \vec{e}_1, \ldots, x + \vec{e}_d\} \)

- **North-East model (d = 2):**
  
  ... both \( x + \vec{e}_1 \) and \( x + \vec{e}_2 \) are empty

- **Duarte model (d = 2):**
  
  ... there are at least 2 empty sites in \( \{x + \vec{e}_2, x - \vec{e}_1, x - \vec{e}_2\} \)
**KCM: motivations from physics**

Introduced in the ’80’s to model the *liquid/glass transition*

- major open problem in condensed matter physics;
- sharp divergence of timescales;
- no significant structural changes.

⇒ KCM dynamics mimic cage effect: if temperature is lowered free volume shrinks ($q \leftrightarrow e^{-1/T}$) ⇒ trivial equilibrium and yet sharp divergence of timescales when $q \downarrow 0$, aging, heterogeneities, . . . → glassy dynamics

⇒ Key question: how do KCM time-scales diverge for $q \downarrow 0$? ⇒ Sharp divergence → numerical simulations do not give clear-cut answers, some of the conjectures were wrong!
KCM: motivations from physics

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⇒ trivial equilibrium and yet sharp divergence of timescales when \( q \downarrow 0 \), aging, heterogeneities, \ldots \rightarrow glassy dynamics

⇒ Key question: how do KCM time-scales diverge for \( q \downarrow 0 \) ?

⇒ Sharp divergence \rightarrow numerical simulations do not give clear-cut answers, some of the conjectures were wrong!
Some notation

- $\Omega := \{0, 1\}^{\mathbb{Z}^d}$ is the configuration space
- for $\sigma \in \Omega$ and $x \in \mathbb{Z}^d$, $\sigma_x$ is the occupation variable at $x$
- for $\Lambda \subset \mathbb{Z}^d$, $\Omega_\Lambda := \{0, 1\}^\Lambda$ and $\sigma_\Lambda$ is the restriction of $\sigma$ to $\Lambda$
- $\mu := \mu^{(q)} = \text{Bernoulli} (1 - q)$ product measure on $\mathbb{Z}^d$
- $\mu_\Lambda := \mu^{(q)}_\Lambda = \text{Bernoulli} (1 - q)$ product measure on $\Lambda$
- for $f : \Omega \to \mathbb{R}$, we let $\mu_\Lambda(f) : \Omega_{\mathbb{Z}^d \setminus \Lambda} \to \mathbb{R}$ be the mean of $f$ w.r.t. $\mu_\Lambda$ with the other variable held fixed
- analogous definition for $\text{Var}_\Lambda(f)$
Formal definition of the Markov process

The generator acts on local functions $f : \Omega \rightarrow \mathbb{R}$ as

$$
\mathcal{L}f(\sigma) := \sum_{x \in \mathbb{Z}^d} c_x(\sigma)(\mu_x(f) - f(\sigma)) =
$$

$$
= \sum_{x \in \mathbb{Z}^d} c_x(\sigma)(q\sigma_x + (1 - q)(1 - \sigma_x))(f(\sigma^x) - f(\sigma))
$$

with

$$
\sigma^x(y) := \begin{cases} 
\sigma(y) & \text{if } y \neq x \\
1 - \sigma(x) & \text{if } y = x
\end{cases}
$$

$$
c_x(\sigma) := \begin{cases} 
1 & \text{if there constraint is satisfied at } x \\
0 & \text{otherwise}
\end{cases}
$$

The corresponding Dirichlet form is:

$$
\mathcal{D}(f) := -\mu(f \cdot \mathcal{L}f) = \sum_{x \in \mathbb{Z}^d} \mu(c_x \text{Var}_x(f)).
$$
Focus questions

Q. Is $\mu$ ergodic for the infinite volume process? Is it also mixing? And if so, how fast does converge to equilibrium in $L^2(\mu)$ occur?

Recall that, if we denote by $P_t$ is the Markov semigroup,

- $\mu$ is ergodic if for all $f \in L^2(\mu)$ the condition $P_t f = f \ \forall t \geq 0$ implies $f$ constant a.s. in $\mu$
- $\mu$ is mixing if $\forall f, g \in L^2(\mu)$ it holds $\lim_{t \to \infty} \mu(f P_t g) = \mu(f) \mu(g)$.

Thus mixing is stronger than ergodicity.
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  $\lim_{t \to \infty} \mu(f P_t g) = \mu(f) \mu(g)$.

Thus mixing is stronger than ergodicity.

$\Rightarrow$ To answer the above questions we should first introduce a related model: Bootstrap Percolation (BP)
2-neighbour Bootstrap Percolation

- At time $t = 0$ sites are i.i.d., empty with probability $q$, occupied with probability $1 - q$
- At time $t = 1$
  - each empty site remains empty
  - each occupied site is emptied iff it has at least 2 empty n.n.
- Iterate

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2-neighbour Bootstrap Percolation

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$\Rightarrow$ Final configuration: completely empty or $\exists$ clusters of mutually blocked particles

- BP blocked clusters
- BP is a discrete time deterministic monotone dynamics $\rightarrow$ easier to study

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2-neighbour Bootstrap Percolation

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$\Rightarrow$ Final configuration: completely empty or $\exists$ clusters of mutually blocked particles

- BP blocked clusters $\leftrightarrow$ blocked particles under FA-2f
- BP is a discrete time deterministic monotone dynamics $\rightarrow$ easier to study
Critical density and Infection time

- Will the whole lattice become empty?

- \( q_c := \inf\{q \in [0, 1] : \mu_q(\text{origin is emptied eventually}) = 1\} \)

- How many steps are needed to empty the origin?

- \( \tau^{BP}(q) := \mu_q(\text{first time at which origin is empty}) \)
• **Will the whole lattice become empty?**
  → Yes $\forall q > 0$ (Van Enter ’87)

• $q_c := \inf\{q \in [0, 1] : \mu_q(\text{origin is emptied eventually}) = 1\}$
  → $q_c = 0$

• **How many steps are needed to empty the origin?**

• $\tau^{BP}(q) := \mu_q(\text{first time at which origin is empty})$

  for $q \downarrow 0$ w.h.p. $\tau_0^{BP} = \exp\left(\frac{\lambda(d)}{q^{1/(d-1)}} (1 - o(1))\right)$

  • scaling (Aizenmann, Lebowitz ’88)
  • $\lambda(2) = \pi^2/18$ (Holroyd ’08)
  • $\lambda(d) = \ldots$ $d > 2$ (Balogh, Bollobas, Duminil-Copin, Morris ’12)
Bootstrap percolation

- Define analogously the BP processes corresponding to the constraints of FA-$j_f$, Duarte, North-East and East
- $q_c = 0$ for East, Duarte and FA-$j_f \forall j \in [1, d]$
- $q_c = 1 - p^{OP}_c$ for North-East
- for $q \downarrow 0$, w.h.p. $\tau^{BP}_0 \sim q^{-1/d}$ for FA-1f and East
- the scalings for FA-$j_f$ with $j > 1$ and for Duarte model are more complicate and diverge more rapidly as $q \downarrow 0$ (see Lecture 2)
Theorem (Cancrini, Martinelli, Roberto, C.T. ’08)
(i) ∀q > q_c, μ is mixing (and therefore ergodic);
(ii) ∀q < q_c, μ is not ergodic (and therefore not mixing).

Sketch of the proof
• for q > q_c we prove that 0 is a simple eigenvalue of L. Key ingredient: fix x ∈ Z^d and σ ∼ μ, then μ-a.s. there exists a legal path from σ to σ^x;
• for q < q_c blocked structures percolate → f := 1_ℰ is left invariant by the dynamics and it is not constant a.s. w.r.t. μ where

ℰ := {η : the origin cannot be emptied by BP}. 
Spectral gap and relaxation time

\[
gap := \inf_{\substack{f \in \text{Dom}(\mathcal{L}) \\ \text{Var}(f) \neq 0}} \frac{\mathcal{D}(f)}{\text{Var}(f)}
\]

i.e. \( \gap = T^{-1}_{\text{rel}} \), where \( T_{\text{rel}} \) is the smallest constant such that

\[
\text{Var}(f) \leq T_{\text{rel}} \mathcal{D}(f) \quad \forall f
\]

Thus, if \( \gap > 0 \), it holds

\[
\text{Var}(P_t f) = \mu(f P_t f) - \mu(f)^2 \leq \exp(-2t \gap) \text{Var}(f) \quad \forall f \in L^2(\mu)
\]

Theorem (Cancrini, Martinelli, Roberto, C.T. '08)

For FA-\( j \)f, East, North-East and Duarte models it holds \( \gap > 0 \) \( \forall q > q_c \)
**Spectral gap and relaxation time**

\[
gap := \inf_{f \in \text{Dom}(\mathcal{L}) \setminus \text{Var}(f) \neq 0} \frac{\mathcal{D}(f)}{\text{Var}(f)}
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i.e. \( \gap = T_{\text{rel}}^{-1} \), where \( T_{\text{rel}} \) is the smallest constant such that

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For FA-\( jf \), East, North-East and Duarte models it holds

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\gap > 0 \quad \forall q > q_c
\]
Persistence function

\[ F(t) := \int d\mu(\sigma) \mathbb{P}_\sigma(\sigma_s(0) = \sigma(0) \forall s \leq t) \]
Persistence function

\[ F(t) := \int d\mu(\sigma) \mathbb{P}_\sigma(\sigma_s(0) = \sigma(0) \forall s \leq t) \]

Recall that \( \forall A \subset \Omega \) it holds

\[ \mathbb{P}_\mu(\tau_A > t) \leq \exp(-t\lambda_A) \]

with \( \tau_A \) the hitting time of \( A \), and

\[ \lambda_A := \inf\left\{ D(f) : \mu(f^2) = 1, f \equiv 0 \text{ on } A \right\}. \]

Thus

\[ F(t) = \mathbb{P}(\tau_{\{\sigma(0)=1\}} > t) + \mathbb{P}(\tau_{\{\sigma(0)=0\}} > t) \leq e^{-\left(1-q\right)t \text{ gap}} + e^{-qt \text{ gap}} \]
Mean infection time

$\mathbb{E}_\mu(\tau_0)$ with $\tau_0 =$ hitting time of $\{\sigma(0) = 0\}$.

The upper bound on $F(t)$ implies

$$\mathbb{E}_\mu(\tau_0) \leq (1 + o(q)) \frac{T_{\text{rel}}}{q}$$
Mean infection time

\( \mathbb{E}_\mu(\tau_0) \) with \( \tau_0 = \) hitting time of \( \{\sigma(0) = 0\} \).

The upper bound on \( F(t) \) implies

\[
\mathbb{E}_\mu(\tau_0) \leq (1 + o(q)) \frac{T_{rel}}{q}
\]

**An easy lower bound**

For any KCM there exists \( \delta > 0 \) s.t. for \( q \) small enough it holds

\[
\mathbb{E}_\mu(\tau_0) \geq \delta \tau^{\text{BP}}(q)
\]

General but usually very far from the correct scaling.

**Key idea:** BP features only infecting moves while KCM has both infecting and healing moves \( \rightarrow \) BP infects the origin at least as fast as the corresponding KCM.
East model in $d = 1$: gap $> 0$ via the BC technique

- gap:=spectral gap for East on $\mathbb{Z}$
- for $\Lambda \subset \mathbb{Z}$, $\text{gap}_\Lambda :=$ spectral gap for East on $\Lambda$ with 0 b.c.
- $\Lambda_k := [0, 2^k]$ and $\gamma_k := 1/\text{gap}_{\Lambda_k}$
- $\text{gap} \geq \inf_k \text{gap}_{\Lambda_k}$
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- gap $\geq \inf_k \text{gap}_{\Lambda_k}$

$\rightarrow$ if we prove

$$\gamma_k \leq a_k \gamma_{k-1} \quad \text{with} \quad \prod_{k_0}^{\infty} a_k < \infty \quad \text{for a finite} \quad k_0$$

we have proven gap $> 0$
First idea

- Devide $\Lambda_k$ into two $B_1, B_2$ each of the form $\Lambda_{k-1}$
- define an auxiliary block dynamics: ...
- $T_{rel,\Lambda_k} \leq T_{rel}^{block} \max T_{rel,B_1} T_{rel,B_2}$
- $T_{rel}^{block} = 1$ (product measure) $\rightarrow \gamma_k \leq \gamma_{k-1}$
- ...so easy?!
- $\max T_{rel,B_1} T_{rel,B_2} = \infty$!
A general two-site constrained Poincaré inequality

Lemma

- \((X_1, \nu_1)\) and \((X_2, \nu_2)\) = finite probability spaces
- \((X, \nu)\) = the associated product space
- \(\mathcal{H} \subset X_2\) with \(\nu_2(\mathcal{H}) > 0\).

Then, for any \(f : X \to \mathbb{R}\), it holds

\[
\text{Var}_\nu(f) \leq \left(1 - \sqrt{1 - \nu_2(\mathcal{H})}\right)^{-1} \nu \left(1_{\{X_2 \in \mathcal{H}\}} \text{Var}_{\nu_1}(f) + \text{Var}_{\nu_2}(f)\right).
\]

- Let \(\vec{X} := (X_1, X_2)\). The inequality is an upper bound on \(T_{\text{rel}}\) for the Markov process reversible w.r.t. \(\nu\) with generator:

\[
\mathcal{L} f(\vec{X}) = 1_{\{X_2 \in \mathcal{H}\}} \left[\nu_1(f) - f(\vec{X})\right] + \left[\nu_2(f) - f(\vec{X})\right]
\]

- easy proof via direct calculation on eigenvectors
East model in \( d = 1 \): gap \( > 0 \) via the BC technique

\[ \Lambda_k = [0, 2^k] = B_1 \sqcup B_2, \quad B_1 := [0, 2^{k-1} - 1], \quad B_2 := [2^{k-1}, 2^k], \]

\[ I := [2^{k-1}, 2^{k-1} + 2^{k/3}], \quad \mathcal{H} \subset \Omega_{B_2} := \{ \eta : \exists \text{ at least one zero in } I \} \]

Use the two-site constrained Poincaré inequality to get

\[ \text{Var}_{\Lambda_k}(f) \leq \epsilon_k \mu_{\Lambda_k} \left( 1_{\mathcal{H}} \text{Var}_{B_1}(f) + \text{Var}_{B_2}(f) \right) \]

with \( \epsilon_k = \left( 1 - \sqrt{(1 - q)^{2^{k/3}}} \right)^{-1} \)
East model in $d = 1$: gap $> 0$ via the BC technique

Our goal: upper bound the r.h.s with $a_k(\text{gap}_{[0,2^k-1]})^{-1} D_{\Lambda_k}(f)$

$$r.h.s. := \epsilon_k \mu_{\Lambda_k} (1_H \text{Var}_{B_1}(f) + \text{Var}_{B_2}(f))$$

Via the Poincaré inequality (i.e. the definition of gap) for East:

$$\mu_{\Lambda_k}(\text{Var}_{B_i}(f)) \leq (\text{gap}(\mathcal{L}_{B_i}))^{-1} \sum_{x \in B_i} \mu_{\Lambda_k}(c_{x,B_i} \text{Var}_x(f))$$

$$c_{x,B_i}(\sigma) := \begin{cases} 1 - \sigma_{x+1} & \text{if } x \neq \text{rightmost site of } B_i \\ 1 & \text{otherwise} \end{cases}$$
East model in $d = 1$: gap > 0 via the BC technique

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$$c_{x,B_2}(\sigma) = x, \Lambda_k(\sigma), \text{ but } c_{x,B_1}(\sigma) \geq c_{x,\Lambda_k}(\sigma)!$$

$\rightarrow$ r.h.s. $\neq \epsilon_k \gamma_{k-1} \mu(D_{B_1}(f) + D_{B_2}(f)) = \epsilon_k \gamma_{k-1} D_{\Lambda_k}(f)$
East model in $d = 1$: gap $> 0$ via the BC technique

"Enlarge" $\text{Var}_{B_1}$ by convexity to the random interval $C := \ldots$

$$c_{x, C}(\sigma) 1_{\mathcal{H}} = c_{x, \Lambda_k}(\sigma) 1_{\mathcal{H}} \quad \forall x \in C, \sigma \in \Omega_{\Lambda_k}$$
East model in $d = 1$: gap $> 0$ via the BC technique

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$$c_{x,C}(\sigma)1_{\mathcal{H}} = c_{x,\Lambda_k}(\sigma)1_{\mathcal{H}} \quad \forall x \in C, \sigma \in \Omega_{\Lambda_k}$$

$$\Rightarrow \quad \mu_{\Lambda_k} (1_{\mathcal{H}} \text{Var}_{B_1} \leq \text{gap}_{\Lambda_1}^{-1} \sum_{x \in B_1 \cup I} \mu_{\Lambda_k} (c_{x,\Lambda_k} \text{Var}_x(f)))$$
East model in $d = 1$: gap $> 0$ via the BC technique

"Enlarge" $\text{Var}_{B_1}$ by convexity to the random interval $C := \ldots$

$$c_{x,C}(\sigma)1_{H} = c_{x,\Lambda_{k}}(\sigma)1_{H} \quad \forall x \in C, \sigma \in \Omega_{\Lambda_{k}}$$

$$\Rightarrow \mu_{\Lambda_{k}}(1_{H}\text{Var}_{B_1} \leq \text{gap}_{1}^{-1} \sum_{x \in B_1 \cup I} \mu_{\Lambda_{k}}(c_{x,\Lambda_{k}}\text{Var}_{x}(f))$$

$$\Rightarrow \text{Var}_{\Lambda_{k}} \leq \epsilon_{k} \gamma_{k-1}D_{\Lambda_{k}} + \sum_{x \in I} \mu_{\Lambda_{k}}(c_{x,\Lambda_{k}}\text{Var}_{x}(f))$$

- Technical point: "move" I and average over its positions
- use the variational definition of the spectral gap
East model in $d = 1$: gap $> 0$ via the BC technique

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- Technical point: "move" $I$ and average over its positions
- use the variational definition of the spectral gap

$$\Rightarrow \gamma_k \leq a_k \gamma_{k-1} \quad \text{with} \quad \prod_{k=0}^{\infty} a_k < \infty \quad \text{for a finite} \quad k_0$$
East model in $d = 1$: $\text{gap} > 0$ via the BC technique

**Theorem (Cancrini, Martinelli, Roberto, C.T ’08)**

For all $\delta > 0$ there exists $C_\delta$ s.t.

$$T_{\text{rel}} = \frac{1}{\text{gap}} \leq C_\delta \exp \left( \frac{|\log q|^2}{(2 - \delta) \log 2} \right)$$

**Remark**

- The conjectures in physics were incorrect, claiming $T_{\text{rel}} \sim \exp \left( \frac{|\log q|^2}{(2 - \delta) \log 2} \right)$
- The additional factor $1/2$ is due to the fact that (unexpectedly!) energy and entropy contributions are of the same order (more on Lecture 2)
**East model in d = 1: gap > 0 via the BC technique**

**Theorem (Cancrini, Martinelli, Roberto, C.T ’08)**

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  $$T_{rel} \sim \exp \left( \frac{|\log q|^2}{\log 2} \right)$$

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Can BC be used to upper bound $T_{\text{rel}}$ for all KCM?

Consider FA-2f on $\mathbb{Z}^2$

$$\Lambda_k := [1, 2^k] \times [1, 2^k]$$

$$I := [2^{k-1}, 2^{k-1} + 2^{k/3}] \times [1, 2^k]$$

$$\mathcal{H} := \{ \eta : \exists \text{top-bottom empty crossing in } I \}$$

Following the lines of the proof for East on $\mathbb{Z}$ we get

$$\gamma_k := \text{gap}^{-1}_{\Lambda_k} \leq a_k \gamma_{k-1}$$

with

$$a_k := \left(1 - \sqrt{1 - \mu(\mathcal{H})}\right)^{-1} \rightarrow \prod_{k_0}^\infty a_k < 1 \text{ iff } q > q_c^{\text{site perc.}}$$
Can BC be used to upper bound $T_{rel}$ for all KCM?

Consider FA-2f on $\mathbb{Z}^2$

$\Lambda_k := [1, 2^k] \times [1, 2^k]$

$I := [2^{k-1}, 2^{k-1} + 2^{k/3}] \times [1, 2^k]$

$\mathcal{H} := \{\eta : \exists \text{ top-bottom empty crossing in } I\}$

Following the lines of the proof for East on $\mathbb{Z}$ we get

$\gamma_k := \text{gap}^{-1}_{\Lambda_k} \leq a_k \gamma_{k-1}$

with $a_k := \left(1 - \sqrt{1 - \mu(\mathcal{H})}\right)^{-1} \rightarrow \prod_{k_0}^{\infty} a_k < 1$ iff $q > q_c^{\text{site perc.}}$

$\rightarrow$ BC cannot be used (alone) to upper bound $T_{rel}$ for all $q > 0$!

Renormalisation + other tools...
East model $d = 1$: combinatorics

Constraint $=$ to update a site we need its right neighbour empty
East model $d = 1$: combinatorics

Constraint = to update a site we need its right neighbour empty

- If we start from a single vacancy, and we can create 1 zero we reach only

  \[ \bullet \bullet \bullet \circ \circ \circ \]

- If we can create up to \( n \) simultaneous additional zeros, we reach also:

  \[ \bullet \bullet \bullet \circ \circ \circ \circ \]

- If we can create up to \( n \) simultaneous additional zeros, one of the configurations that we can reach has its leftmost vacancy at \( x - (2^n - 1) \);

  \[ \bullet \bullet \circ \circ \circ \circ \]

  All the others have leftmost vacancy in \( [x, x - (2^n - 1)] \) \implies the East model has logarithmic energy barriers

[Evans Sollich '99, see also Chung Diaconis Graham '01]

C. Toninelli Kinetically constrained models
**East model** $d=1$: *combinatorics*

**Constraint** = to update a site we need its right neighbour empty

- If we start from a single vacancy and we can create 1 zero we reach only
  
  - if we can create up to 2 simultaneous additional zeros we reach also:

  ![Diagram](image-url)
East model $d = 1$: combinatorics

**Constraint** = to update a site we need its right neighbour empty

- If we start from a single vacancy and we can create 1 zero we reach only

- If we can create up to 2 simultaneous additional zeros we reach also:

- If we can create up to $n$ simultaneous additional zeros
  - one of the configurations that we can reach has its leftmost vacancy at $x - (2^n - 1)$;
  - all the others have leftmost vacancy in $[x, x - (2^n - 1)]$
East model $d = 1$: combinatorics

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  - one of the configurations that we can reach has its leftmost vacancy at $x - (2^n - 1)$;
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⇒ the East model has logarithmic energy barriers
[Evans Sollich ’99, see also Chung Diaconis Graham ’01]
East model $d = 1$: scaling for $q \downarrow 0$

- The first vacancy at the left of origin is at $\ell \sim 1/q$
- Trivially, $\tau_{0}^{BP}(q) \sim 1/q$
- $\mathbb{E}_{\mu_{q}}(\tau_{0}) \sim \text{time to create } \log_{2}(\ell) \text{ empty sites}$
- $\rightarrow \mathbb{E}_{\mu_{q}}(\tau_{0}) = 1/q^{\Theta(1) |\log q|}$ \[Aldous, Diaconis JSP ’02\]
East model $d = 1$: scaling for $q \downarrow 0$

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- $E_{\mu_{q}}(\tau_{0}) \sim \text{time to create } \log_{2}(\ell) \text{ empty sites}$
- $\rightarrow E_{\mu_{q}}(\tau_{0}) = 1/q^{\Theta(1)|\log q|}$ [Aldous, Diaconis JSP ’02]
- Sharp result (taking entropy into account) in $d \geq 1$

$$\lim_{q \to 0} \frac{\log E_{\mu_{q}}(\tau_{0})}{|\log q|^{2}} = (2d \log 2)^{-1}$$

[Cancrini, Martinelli, Roberto, C.T. PTRF ’08] for $d = 1$
[Chleboun, Faggionato, Martinelli AoP ’16] for $d \geq 2$
**FA-1f: scaling for** $q \downarrow 0$

**Constraint** = to be update we need an empty nearest neighbour

- a vacancy can move of one step by creating one additional vacancy $\rightarrow \sim$ r.w. of rate $q^{-1} \rightarrow$ non-cooperative!

$\rightarrow \ d = 1 \quad \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-1}(1/q)^2 = q^{-3};$

$\rightarrow \ d = 2 \quad q^{-2} \leq \mathbb{E}_{\mu_q}(\tau_0) \leq q^{-2} |\log q|;$

$\rightarrow \ d \geq 3 \quad \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-1}(1/q^{1/d})^d = q^{-2}$

[Cancrini, Martinelli, Roberto, C.T. PTRF ’08 + Shapira JSP ’20]
**KCM: the general definition**

Configurations: \[ \eta \in \Omega := \{0, 1\}^\mathbb{Z}^d, \ 0 = \text{empty}, \ 1 = \text{occupied} \]

Fix a density parameter \( q \in [0, 1] \) and an update family \( \mathcal{U} \) with

\[ \mathcal{U} = \{U_1, \ldots, U_m\}, \ U_i \subset \mathbb{Z}^d \setminus \{0\}, \ |U_i| < \infty, \ m < \infty \]

i.e. \( \mathcal{U} \) is a finite collection of local neighbourhoods of the origin

Fix \( \eta \in \Omega \) and \( x \in \mathbb{Z}^d \): ”the constraint is satisfied at \( x \)” iff at least one of the translated sets \( U_i + x \) is completely empty

**Dynamics**: each site with the constraint satisfied is updated to empty at rate \( q \) and to occupied at rate \( 1 - q \)
Our examples

- **FA-\( j \)f model:** \( \mathcal{U} = \) all sets containing \( j \) nearest neighbours of the origin

- **East model:** \( \mathcal{U} = \{ \mathbf{e}_1, \ldots, \mathbf{e}_d \} \)

- **North-East model:** \( \mathcal{U} = \{ U_1 \} \) with \( U_1 = \{ (0, 1), (1, 0) \} \)

- **Duarte model:** \( \mathcal{U} = \{ U_1, U_2, U_3 \} \) with

\[
\begin{array}{ccc}
\begin{array}{c}
\bigcirc \quad x \\
U_1
\end{array} & \begin{array}{c}
\bigcirc \quad x \\
U_2
\end{array} & \begin{array}{c}
\bigcirc \quad x \\
U_3
\end{array}
\end{array}
\]
Universality class of $\mathcal{U}$ in $d = 2$

We need the notion of **stable** and **unstable directions**

- Fix a direction $\vec{u}$
- Start from a configuration which is
  - completely empty on the half plane perpendicular to $\vec{u}$ in the negative direction ($H_u$)
  - filled otherwise
- Run the bootstrap dynamics

\[ \vec{u} \text{ is } \begin{cases} 
\text{stable} & \text{if no other site can be emptied} \\
\text{unstable} & \text{otherwise}
\end{cases} \]
Stable and unstable directions: examples

Of course, the stability of a direction depends on $\mathcal{U}$

Ex. East model:

$\vec{u} = -\vec{e}_1$ is stable; $\vec{u} = \vec{e}_1 + \vec{e}_2$ is unstable

Instead:

- both directions are unstable for 1-neighbour bootstrap
- both directions are stable for North East
How to easily identify all stable and unstable directions

Draw the half planes $H_u$ and $\mathbb{Z}^2 \setminus H_u$ so that the separation line contains the origin. $\vec{u}$ is unstable iff $U_i \subset H_u$ for at least one $i$.

Easy-to-use criterion to determine the class of any $U$ (m simple geometric checks, $m =$ # of rules)
Supercritical universality class

\( \mathcal{U} \) is supercritical iff there exists an open semicircle \( \mathcal{C} \) which does not contain stable directions.

A supercritical model is

\[
\begin{cases}
\text{rooted} & \text{if it has at least 2 non opposite stable directions} \\
\text{unrooted} & \text{otherwise}
\end{cases}
\]

FA-1f
Unrooted

East
Rooted
$\mathcal{U}$ is critical if it is notsupercritical and there exists an open semicircle $\mathcal{C}$ with only a finite number of stable directions.

A critical model is

\[
\begin{cases}
  \text{finitely critical} & \text{if it has a finite number of stable directions} \\
  \text{infinitely critical} & \text{otherwise}
\end{cases}
\]
Subcritical universality class

Two equivalent definitions

\( \mathcal{U} \) is subcritical iff it is neither supercritical nor critical

or

\( \mathcal{U} \) is subcritical iff each open semicircle has infinite stable directions

\[ \Rightarrow q_c > 0: \text{blocked clusters percolate at } q < q_c \]

Example: North East model
BP universality results in \(d = 2\)

**Theorem** [Bollobás, Smith, Uzzell ’15 + Balister, Bollobás, Przykucki, Smith ’16 + Bollobás, Duminil-Copin, Morris, Smith ’16]

- **Supercritical**: \(q_c = 0\), \(\tau_{BP}^0(q) = 1/q^\Theta(1)\) w.h.p. as \(q \downarrow 0\)
- **Critical**: \(q_c = 0\), \(\tau_{BP}^0(q) = \exp(|\log q|^{O(1)}/q^\alpha)\) w.h.p. as \(q \downarrow 0\)
- **Subcritical**: \(q_c > 0\)

**Definition of the ”difficulty”, \(\alpha\)**

\[ \alpha = \min_C \max_{\vec{u} \in C} d(\vec{u}) \quad \text{with} \]

\[ d(\vec{u}) = \text{minimal number of empty sites to unstabilize } \vec{u} \]
**Difficulty: examples**

**FA-2f:**

- $\vec{e}_1$ is stable and $d(\vec{e}_1) = 1$
- same for $-\vec{e}_1$ and $\pm \vec{e}_2$
- all other directions are unstable

Exercice: check that Duarte model also has difficulty $\alpha = 1$ for FA-2f.

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Kinetically constrained models
Difficulty: examples

**FA-2f:**

- $\vec{e}_1$ is stable and $d(\vec{e}_1) = 1$
- same for $-\vec{e}_1$ and $\pm \vec{e}_2$
- all other directions are unstable

→ $\alpha = 1$ for FA-2f

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Difficulty: examples

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- $\vec{e}_1$ is stable and $d(\vec{e}_1) = 1$
- same for $-\vec{e}_1$ and $\pm \vec{e}_2$
- all other directions are unstable

$\rightarrow \alpha = 1$ for FA-2f

Exercice: check that Duarte model also has difficulty $\alpha = 1$
**KCM: universality results in \( d = 2 \)**

**Theorem [Martinelli, Morris, C.T. ’19; Marêché, Martinelli, C.T. ’20, Marêché, Hartarsky, Toninelli ’20, Hartarsky, Martinelli, C.T.’21]**

1. **Supercritical unrooted:** \( \tau(q) = \frac{1}{q^{\Theta(1)}} \) and \( \tau_0^{\text{BP}} \sim \frac{1}{q^{\Theta(1)}} \) (FA-1f)

2. **Supercritical rooted:** \( \tau_0(q) = \frac{1}{q^{\Theta(1)|\log q|}} \gg \tau_0^{\text{BP}} = \frac{1}{q^{\Theta(1)}} \) (East)

3. **Finitely critical:** \( \tau_0(q) \) and \( \tau_0^{\text{BP}} \sim \exp \left( \frac{\Theta(1)(\log q)^{\Theta(1)}}{q^\nu} \right) \) (FA-2f)

4. **Infinitely critical:**
   \[
   \tau(q) = \exp \left( \frac{(\log q)^c}{q^{2\nu}} \right) \gg \tau_0^{\text{BP}} = \exp \left( \frac{(\log q)^c}{q^\nu} \right) \) (Duarte)

\[ \rightarrow \text{For supercritical rooted and infinitely critical models} \]

\[
\mathbb{E}_{\mu_q}(\tau_0) \gg \tau_0^{\text{BP}}(q)^\nu \text{ for all } \nu.
\]

Hartarsky Marêché ’21+: log corrections!
**Supercritical models**

- **Unrooted**: large empty droplet can move back and forth
  
  \[ \rightarrow \text{renormalise to an FA-1f with effective density } q_{\text{eff}} = q^{\Theta(1)} \]
  \[ \rightarrow \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-\Theta(1)} \]

- **Rooted**: any empty droplet can move only inside a cone
  
  \[ \rightarrow \log \text{energy barriers as for East } \]
  \[ \rightarrow \text{renormalise to an East with effective density } q_{\text{eff}} = q^{\Theta(1)} \]
  \[ \rightarrow E_{\mu_q}(\tau_0) \sim q^{-\Theta\left(\log q_{\text{eff}}\right)} \]
  \[ \text{eff} = e^{\Theta(\log q_{\text{eff}})} 2 \]
Supercritical models

• **Unrooted**: large empty droplet can move back and forth

  \[ \rightarrow \text{renormalise to an FA-1f with effective density } q_{\text{eff}} = q^{\Theta(1)} \]

  \[ \rightarrow \mathbb{E}_{\mu_q}(\tau_0) \sim q^{-\Theta(1)} \]

• **Rooted**: any empty droplet can move only inside a cone

  \[ \Rightarrow \text{logarithmic energy barriers as for East} \]

  [Marêché ’20]

  \[ \rightarrow \text{renormalise to an East with effective density } q_{\text{eff}} = q^{\Theta(1)} \]

  \[ \rightarrow \mathbb{E}_{\mu_q}(\tau_0) \sim q_{\text{eff}}^{\Theta(|\log q_{\text{eff}}|)} \approx e^{\Theta(\log q)^2} \]
Duarte model: heuristics

Constraint at $x$: at least 2 vacancies in $\{x - \vec{e}_1, x + \vec{e}_2, x - \vec{e}_2\}$

An empty segment of length $\ell = \frac{1}{q |\log q|}$ can (typically) create an empty segment to its right, but never to its left!

→ it is a mobile droplet with East-like dynamics and

$\text{density } q_{\text{eff}} = q^\ell = e^{-\Theta(\log q)^2/q}$
Duarte model: heuristics

- nearest empty droplet to the origin is at distance $L \sim q_{\text{eff}}^{-1}$

$\rightarrow T^\text{BP} \sim L = \exp \left( \frac{\Theta(1) \log q^2}{q} \right)$

[Mountford '95]

$\gg T^\text{BP}$

[Martinelli, Morris, C.T. '19 + Maréchal, Martinelli, C.T. '20]
Duarte model: heuristics

- nearest empty droplet to the origin is at distance $L \sim q_{\text{eff}}^{-1}$

  $\rightarrow T^{\text{BP}} \sim L = \exp \left( \frac{\Theta(1) |\log q|^2}{q} \right)$

  [Mountford ’95]

- Duarte droplets move East like → to empty the origin we have to create $\log(L)$ simultaneous droplets

  $\rightarrow \mathbb{E}_{\mu_q}(\tau_0) \sim q_{\text{eff}}^{-\log L} \sim \exp \left( \frac{\Theta(1) |\log q|^4}{q^2} \right) \gg T^{\text{BP}}$

The general critical case

- Droplets are empty regions with model dependent shape of size $\ell = q^{-\alpha|\log q|}$ and density $q_{\text{eff}} = q^\ell$.
The general critical case

- **Droplets** are empty regions with model dependent shape of size $\ell = q^{-\alpha} |\log q|$ and density $q_{\text{eff}} = q^\ell$

- For **infinitely critical KCM** the droplet motion is East like

  $$\rightarrow \tau_0 \sim q_{\text{eff}}^{|\log q_{\text{eff}}|} = \exp \left( \frac{|\log q|^{O(1)}}{q^{2\alpha}} \right)$$
The general critical case

- Droplets are empty regions with model dependent shape of size $\ell = q^{-\alpha}|\log q|$ and density $q_{\text{eff}} = q^\ell$.

- For infinitely critical KCM the droplet motion is East like

$$\rightarrow \tau_0 \sim q_{\text{eff}}^{\Theta(|\log q_{\text{eff}}|)} = \exp \left( \frac{|\log q|^{O(1)}}{q^{2\alpha}} \right)$$

- For finitely critical KCM the droplet motion is a subtle combination of East on mesoscopic scales ($L \sim q^{-\Theta(1)}$) and FA-1f on macroscopic scales ($\sim q_{\text{eff}}^{-1}$)

$$\rightarrow \tau_0 \sim q_{\text{eff}}^{\Theta(\log L)} = \exp \left( \frac{|\log q|^{O(1)}}{q^\alpha} \right)$$
Upper bound: main obstacles and tools

- droplets move only on a good environment
- the environment evolves and can ”lose its goodness”
- no monotonicity → we cannot ”freeze” the environment
- the motion of droplets is not random walk like
- it is very difficult to use canonical path arguments
Upper bound: main obstacles and tools

- droplets move only on a good environment
- the environment evolves and can ”lose its goodness”
- no monotonicity → we cannot ”freeze” the environment
- the motion of droplets is not random walk like
- it is very difficult to use canonical path arguments

→ a very flexible long range Poincaré inequality
  [Martinelli, C.T. ’19]
→ renormalisation
→ Matryoshka Dolls: a new technique to compare Dirichlet forms avoiding canonical paths
  [Martinelli, Morris, C.T. ’19]
Lower bound

- **Key idea:** construct a bottleneck involving $\log(L)$ droplets
- **Main difficulty:** droplets are not "rigid objects"!
- **Solution:** an algorithmic identification of droplets and of an efficient cut-set...
More on the upper bound: the case of FA-2f

- renormalise on $\ell \times \ell$ boxes, $\ell = 1/q \log(1/q)$

- auxiliary long range block dynamics:
  - put equilibrium on box $B_x$ at rate 1 iff it belongs to a good cluster with two droplets at distance at most $L = \exp(1/q \log(1/q)^2)$

- establish a general long range Poincaré inequality that yields $T_{rel}^{aux} = O(1)$

- use canonical paths for reversible Markov chains or better repeat the same game inside the path on a smaller scale: now the renormalised sites are the columns of the box . . . Matryoshka Dolls!
Theorem [Hartarsky, Martinelli, C.T. ’20]

As $q \downarrow 0$, w.h.p. for the stationary FA-2f model on $\mathbb{Z}^d$ it holds

$$
\tau_0 = \exp \left( \frac{d \times \lambda(d)}{q^{1/(d-1)}} (1 - o(1)) \right), \quad d \geq 2
$$

the same result holds for $\mathbb{E}_{\mu_q}(\tau_0)$. Thus, w.h.p. $\tau_0 = (\tau_0^{BP})^{d + o(1)}$.
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the same result holds for $\mathbb{E}_{\mu_q}(\tau_0)$. Thus, w.h.p. $\tau_0 = (\tau_0^{BP})^{d+o(1)}$.

Remark

- This is not a corollary of the BP result: the emptying/occupying mechanism of FA-2f has no counterpart in BP!
- We settle contrasting conjectures in physics literature
High level ideas

- Relaxation is driven by the motion of unlikely large patches of empty sites, the mobile droplets

- droplet density $\rho_D := \exp\left(-\frac{d \times \lambda(d)}{q^{1/d-1}} (1 + o(1))\right)$
  
droplet length $L_D := \text{poly}(q)$

- Mobile droplets move in any direction . . .
High level ideas

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  contradiction with ”finite empty regions cannot expand”?!
**High level ideas**

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- Droplet length $L_D := \text{poly}(q)$
- Mobile droplets move in any direction . . .
  contradiction with "finite empty regions cannot expand"?!

  ![Diagram showing droplet movement](image)

  ...1 adjacent $\circ$ allows expansion!

- Motion requires few additional empty sites $\rightarrow$ this good environment is very likely for large droplets ($q \downarrow 0$)
High level ideas

• $\tau_0 \sim$ time for the mobile droplet to arrive near the origin
High level ideas

- $\tau_0 \sim$ time for the mobile droplet to arrive near the origin
- motion of droplets $\sim$ coalescing + branching + SSEP

$\rightarrow \tau_0 \sim 1/\rho_D$
High level ideas

- $\tau_0 \sim$ time for the mobile droplet to arrive near the origin
- motion of droplets $\sim$ coalescing + branching + SSEP

$$\rightarrow \tau_0 \sim 1/\rho_D$$

- $\tau_{0}^{\text{BP}} \sim$ distance of mobile droplet to origin

$$\rightarrow \tau_{0}^{\text{BP}} \sim 1/\rho_D^{1/d} \sim \tau_0^{1/d}$$
How do droplets look like? the $d=2$ case

- Multi-scale construction: empty core of size $\frac{1}{\sqrt{q}} +$ empty sites that allow to move the core anywhere inside without creating a larger empty interval.
Multi-scale construction: empty core of size $1/\sqrt{q}$ + empty sites that allow to move the core anywhere inside without creating a larger empty interval

- the black square has no double rows fully occupied and one row with no consecutive filled sites $\rightarrow$ it is emptiable
- vertical arrow = no double rows fully occupied
- horizontal arrow = no double columns fully occupied
- $\ell_n := e^{n\sqrt{q}}/\sqrt{q}$, $N = 8|\log q|/\sqrt{q} \rightarrow \ell_N = L_D = \text{poly}(q)$
More precisely...

A multi-scale definition

- \( \ell_n := e^n \sqrt{q} / \sqrt{q} \), \( N = 8 \left| \log q \right| / \sqrt{q} \) \( \rightarrow \) \( \ell_N = L_D = \text{poly}(q) \)
More precisely...

A multi-scale definition

- $\ell_n := e^n \sqrt{q} / \sqrt{q}$, $N = 8 \log q / \sqrt{q} \rightarrow \ell_N = L_D = \text{poly}(q)$

- a rectangle $R$ is of class $n$ if
  - $R$ is a single site for $n = 0$;
  - $R = \ell_m \times h$ with $h \in (\ell_{m-1}, \ell_m]$ for $n = 2m$;
  - $R = w \times \ell_m$ with $w \in (\ell_m, \ell_{m+1}]$ for $n = 2m + 1$
More precisely...

A multi-scale definition

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  - \( R = w \times \ell_m \) with \( w \in (\ell_m, \ell_{m+1}] \) for \( n = 2m + 1 \)

- Super-good (SG) rectangles:
  - a rectangle of class 0 is SG if it is empty;
  - a rectangle of class \( n \) is SG if it contains a SG rectangle \( R' \) of class \( n - 1 \) (the core) AND it satisfies traversability conditions elsewhere, i.e. no double column/row fully occupied.
More precisely...

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  - \( R = w \times \ell_m \) with \( w \in (\ell_m, \ell_{m+1}] \) for \( n = 2m + 1 \)

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Droplets are defined as \( \ell_N \times \ell_N \) SG rectangles
How does droplet motion look like?

A droplet can

- **coalesce** with a nearby droplet on time scale of order 1
How does droplet motion look like?

A droplet can

- **coalesce** with a nearby droplet on time scale of order $1$
- **create** a new droplet nearby on time scale $\rho_D^{-1}$
How does droplet motion look like?

A droplet can

- **coalesce** with a nearby droplet on time scale of order 1
- **create** a new droplet nearby on time scale $\rho_D^{-1}$
- **swap** its position with a neighboring box on time scale

$$T \sim \exp \left( \frac{|\log q|^3}{q^{1/(2d-2)}} \right) \ll \rho_D^{-1} \sim \exp \left( \frac{d \times \lambda(d)}{q^{1/(d-1)}} \right)$$
A droplet can

- **coalesce** with a nearby droplet on time scale of order 1
- **create** a new droplet nearby on time scale $\rho_D^{-1}$
- **swap** its position with a neighboring box on time scale $T \sim \exp\left(\frac{|\log q|^3}{q^{1/(2d-2)}}\right) \ll \rho_D^{-1} \sim \exp\left(\frac{d \times \lambda(d)}{q^{1/(d-1)}}\right)$

Coalescing + Branching + Simple Exclusion $\rightarrow g$-CBSEP

$g$ for ”generalized” (not just 0/1)
From heuristics to proof: hints for the upper bound

- Hitting times $\leftrightarrow$ Dirichlet eigenvalues
- renormalize on the droplet size
From heuristics to proof: hints for the upper bound

- **Hitting times ↔ Dirichlet eigenvalues**
- renormalize on the droplet size

\[
\tau_0 \leq T_{rel}^{FA-2f,D} T_{rel}^{g-CBSEP}
\]

- \( T_{rel}^{FA-2f,D} \) = relaxation time of the FA-2f chain inside a droplet
- \( T_{rel}^{g-CBSEP} \) = relaxation time of the g-CBSEP chain
From heuristics to proof: hints for the upper bound

- **Hitting times ↔ Dirichlet eigenvalues**
- renormalize on the droplet size

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\[ T_{\text{rel}}^{\text{FA-2f,D}} = \text{relaxation time of the FA-2f chain inside a droplet} \]
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- establish the following Poincaré inequalities

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From heuristics to proof: hints for the upper bound

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\[ \rightarrow \tau_0 \leq \exp \left( \frac{d \times \lambda(d)}{q^{1/(d-1)}} (1 - o(1)) \right) \]
Thanks for your attention!


• **Advertisement** *Kinetically Constrained Models* C.Toninelli, to appear in *Springer Briefs in Mathematical Physics*
The $g$-CBSEP chain

- $G = (V, E)$: finite connected graph
- $(S, \pi)$: finite probability space
- $S = S_0 \sqcup S_1$ and $\rho = \pi(S_1)$
- given $\sigma \in S^V$, $x \in V$ is occupied iff $\sigma_x \in S_1$
- $g$-CBSEP is defined on $\Omega_+ := \{\sigma \text{ with at least one particle}\}$
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**Theorem [Hartarsky, Martinelli, C.T. '20]**

As \( \rho \downarrow 0 \), \( T_{\text{rel}}^{g\text{-CBSEP}} \leq O(\rho^{-1} \log(1/\rho)) \)
FA-$jf$ model

- for $jn$-BP for all $d \geq j \geq 2$, w.h.p. it holds

$$
\tau_0^{BP} \sim \exp^{j-1} \left( \frac{\tilde{\lambda}(d, j)}{q^{1/(d-j+1)}} \right)
$$

$\exp^k = \text{exponential iterated } k \text{ times}$ (Balogh, Bollobas, Duminil-Copin, Morris ’12)

**Same scaling for $\tau_0$** (Hartarsky, Martinelli, C.T. in progress)

- $j = 1$: $\tau_0^{BP} = 1/q^{1/d}$, $\tau_0 = 1/q^{\nu(d)}$, $\nu(1) = 3$, $\nu(d) = 2 \text{ for } d \geq 2$ (log corrections in $d = 2$) (Cancrini, Roberto, Martinelli, C.T. ’08 + Shapira ’20)

- $d < j$: $\tau_0 = \tau_0^{BP} = \infty$ w.h.p. for $q \to 0$
Finitely critical $\mathcal{U}$: an example

To move of one step towards $\vec{e}_2$ the droplet has to move East-like to the right till reaching the first infected pair of empty sites.
The move of one step in the $-\vec{e}_1$ direction the droplet has to move in the direction $\vec{e}_2$ until reaching the first infected pair of empty sites. A subtle hierarchical combination of East paths. . .
Start with a single vacancy at the origin.
East model: why a log barrier?

Start with a single vacancy at the origin

\[ S = \text{configs reachable via paths with } \leq n \text{ simultaneous } 0\text{'s; } \]

\[ L(n) = \text{distance from the origin of leftmost } 0 \text{ maximized on } S \]
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1. optimal path proceeds via stepping-stones: create isolated vacancy at \(-L_1(n)\); restart from it to create an isolated vacancy at \(-L_1(n) - L_1(n - 1)\); \ldots

\[ \rightarrow L(n) = L_1(n) + L_1(n - 1) + \ldots L_1(1) \]

2. to put an isolated 0 at \(-L_1(n)\) we should have a 0 at \(-L_1(n) + 1\) and remove it using at most \(n - 1\) vacancies

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\[ \Rightarrow L(n) = 2^n - 1 \]
A general long-range constrained Poincaré inequality

Lemma [Martinelli, C.T. ’19]

• $(\mathbb{X}, \nu) =$ finite probability space
• $(\tilde{\Omega}, \tilde{\mu}) = (\mathbb{X}^{\mathbb{Z}^d}, \otimes_{x \in \mathbb{Z}^d} \nu_x)$
• for $x \in \mathbb{Z}^d$ let
  • $\mathbb{Z}^{d}_{x,\uparrow} := x + \{y : y_1 + \ldots y_d > 0\}$
  • $\Delta_x \subset \mathbb{Z}^{d}_{x,\uparrow}$ be a finite set
  • $A_x$ be an event depending only on $\{\omega(y)\}_{y \in \Delta_x}$

Assume $\sup_{z} \sum_{x \in \mathbb{Z}^d} (1 - \mu(A_x)) < 1/4$, then

$$\text{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^d} \mu(1_{A_x} \text{Var}_x(f)) \quad \forall f \text{ local}$$

More general oriented neighborhood and product of characteristic functions possible . . .