The algebraic approach to duality: an overview

Frank Redig
DIAM,
Delft University of Technology

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Collaborators
Mario Ayala (Delft, Avignon)
Gioia Carinci (Modena)
Simone Floreani (Delft)
Chiara Franceschini (Lisbon)
Cristian Giardinà (Modena)
Wolter Groenevelt (Delft)
Erik Koelink (Nijmegen)
Jorge Kurchan (Paris)
Federico Sau (Vienna)
Tomohiro Sasamoto (Tokio)

Nutshell introduction

- Models of interacting particle systems (IPS) are used to understand systems of non-equilibrium statistical physics.
- Some of these IPS have extra structure which allows to produce exact formulas for quantities such as density or temperature profile, correlation functions.
- This extra structure comes (often) from a special property called "duality" or "self-duality", which enables to connect the model of interest to another, simpler, dual one, via a duality function.

- The existence of and relation between a model and its dual turn out to be two different representations of an abstract element of a Lie-algebra.
- In this way, we can classify and constructively produce such systems, starting from the underlying Lie-algebras. As a consequence, IPS with duality properties come in families, associated to Lie-algebras.
- This constructive Lie-algebraic approach is robust and enables to find all the duality functions, including orthogonal ones.
- In the meanwhile this method has been applied to many more Lie-algebras than the ones discussed in this talk, including e.g. construction of multi-type asymmetric exclusion processes (Jeffrey Kuan and collaborators).

- For symmetric (=detailed balance in the bulk) systems, these algebras generating families of models with duality are classical Lie-algebras such as $SU(2), SU(n), SU(1,1)$, Heisenberg.
- The "correct" asymmetric companion model of the symmetric models, can be found via q-deformation of these algebras, i.e., the corresponding quantum Lie algebras, where $0 < q < 1$, models the asymmetry.
- Duality and self-duality is strictly weaker than integrability, the relation between these two concepts is not entirely clear.

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1. Basics of duality

- We consider two Markov processes $\{\eta(t), t \geq 0\}$, $\{\xi(t) : t \geq 0\}$ on the state spaces Ω, resp. $\widehat{\Omega}$.
- We denote their semigroups

$$S_t f(\eta) = \mathbb{E}_\eta f(\eta(t)), \widehat{S}_t f(\xi) = \widehat{\mathbb{E}}_\xi (f(\xi(t))$$

- Generators

$$Lf(\eta) = \lim_{t \to 0} \frac{1}{t}(S_t f(\eta) - f(\eta))$$

- Relation between S_t and L is "exponentiation" $S_t = e^{tL}$ where exponential should be defined appropriately. In finite state space case it is simply the matrix exponential

$$e^{tL} = \sum_{n=0}^{\infty} \frac{t^n L^n}{n!}$$

In more general cases it is defined via the Hille-Yosida theorem via the resolvents.

A measurable function $D : \widehat{\Omega} \times \Omega \to \mathbb{R}$ is called a duality function for duality between the processes $\{\xi(t) : t \geq 0\}$ and $\{\eta(t), t \geq 0\}$ if for all $t > 0$, for all $\eta \in \Omega, \xi \in \widehat{\Omega}$ we have the duality relation

$$\mathbb{E}_\eta D(\xi, \eta(t)) = \widehat{\mathbb{E}}_\xi D(\xi(t), \eta)$$

Equivalently, for all $t \geq 0$, for all $\eta \in \Omega, \xi \in \widehat{\Omega}$

$$S_t D(\xi, \cdot)(\eta) = \widehat{S}_t D(\cdot, \eta)(\xi)$$

In many cases also equivalently, for all $\eta \in \Omega, \xi \in \widehat{\Omega}$

$$LD(\xi, \cdot)(\eta) = \widehat{L} D(\cdot, \eta)(\xi)$$

(generator duality) we write $\hat{L} \longrightarrow^D L$, thinking of it as a "relation" between $\hat{L}, L$ parametrized by D.

Remarks

1. Semigroup duality can be defined in more general context, allowing e.g. $\hat{S}_t$ to be a general (not necessarily Markov) semigroup. A simple example of this is S_t the semigroup of Brownian motion and $D(\xi, \eta) = e^{i\xi\cdot\eta}$ then

$$S_t D(\xi, \cdot)(\eta) = \mathbb{E} e^{i\xi(\eta + B(t))} = e^{i\xi\eta} e^{-\frac{1}{2}\xi^2 t} = \hat{S}_t D(\cdot, \eta)(\xi)$$

2. The notation $\hat{L} \longrightarrow^D L$ can be used for general operators, and then satisfies

$$\hat{A} \longrightarrow^D A, \hat{B} \longrightarrow^D B \text{ implies } \hat{A}\hat{B} \longrightarrow^D BA$$

i.e., order of multiplication is reversed.

2. Examples

2.1 Wright Fisher diffusion

$$
\begin{aligned}
Lf(\eta) &= \eta(1-\eta)f''(\eta), \eta \in [0,1] \\
\hat{L}f(\xi) &= \xi(\xi-1)(f(\xi-1)-f(\xi)), \xi \in \mathbb{N}
\end{aligned}
$$

$\{\eta(t), t \geq 0\}$ is a diffusion process on $[0,1]$, and $\{\xi(t), t \geq 0\}$ is a jump process on $\mathbb{N}$. Then with $D(n,x) = \eta^\xi$ we have

$$LD(\xi,\cdot)(\eta) = \xi(\xi-1)(\eta^{\xi-1}-\eta^\xi) = \hat{L}D(\cdot,\eta)(\xi)$$

Time dependent moments in the diffusion process can be computed using the jump process, i.e.,

$$\mathbb{E}_\eta(\eta(t)^\xi) = \hat{\mathbb{E}}_\xi(\eta^{\xi(t)})$$

2.2 Independent random walkers

Let $\{\mathcal{X}(t), t \geq 0\} := \{X_i(t) : t \geq 0, i \in I\}$ denote independent random walkers on a graph with vertex set V. Assume symmetry, i.e., $p_t(x,y) = p_t(y,x)$ for all $x, y \in V$. Then the associated configuration process

$$\eta(t) = \sum_{i \in I} \delta_{X_i(t)}$$

is a Markov process. For $x \in V$, $t \geq 0$ denote $\eta_x(t)$ the number of particles at x at time t. Consider

$$D(x, \eta) = \eta_x$$

then we prove that

$$\mathbb{E}_\eta D(x, \eta(t)) = \mathbb{E}_x^{RW} D(x(t), \eta)$$

which is called "self-duality with a single dual particle".

$$
\begin{aligned}
\mathbb{E}_\eta D(x,\eta(t)) &= \mathbb{E}_{\mathcal{X}}(\sum_i I(X_i(t)=x)) \\
&= \sum_i \mathbb{E}_{\mathcal{X}}(I(X_i(t)=x)) \\
&= \sum_i \mathbb{E}^{RW}_{X_i(0)}(I(X_i(t)=x)) \\
&= \sum_i p_t(X_i(0),x) \\
&= \sum_i p_t(x,X_i(0)) \\
&= \mathbb{E}^{RW}_x(\sum_i I(X_i(0)=X(t))) \\
&= \mathbb{E}^{RW}_x(\eta_{X(t)}(0))
\end{aligned}
$$

We used "consistency" in the third equality to pass from $\mathbb{E}_{\mathcal{X}}$ to $\mathbb{E}_{X_i(0)}$, and we used symmetry in fifth equality (which can be generalized to reversibility).

2.3 Symmetric exclusion

For V a finite set, let the configuration space be $\Omega = \{0,1\}^V$. Let $p(x,y) = p(y,x)$ denote a symmetric function $p : V \times V \to \mathbb{R}^+$. Denote e_x the configuration with one particle at x and no particles anywhere else. Then the generator writes as follows

$$Lf(\eta) = \sum_{x,y \in V} p(x,y) L_{x,y} f(\eta)$$

with $L_{x,y}$ the "single edge generator"

$$L_{x,y} f(\eta) = \eta_x(1-\eta_y)(f(\eta - e_x + e_y) - f(\eta))$$

Compute now for $D(x,\eta) = \eta_x$

$$\begin{aligned} LD(x,\eta) &= \sum_y p(y,x)\eta_y(1-\eta_x) - \sum_y p(x,y)\eta_x(1-\eta_y) \\ &= \sum_y p(x,y)(\eta_y - \eta_x) \end{aligned}$$

then we see with $\hat{L}f(x) = \sum_y p(x,y)(f(y) - f(x))$

$$\hat{L}D(\cdot,\eta)(x) = LD(x,\cdot)(\eta)$$

This implies

$$\mathbb{E}_\eta(\eta_x(t)) = \mathbb{E}_x^{RW}(\eta_{X(t)}(0))$$

just as in the case of independent random walkers! From the "graphical representation" one obtains consistency and one derives the more general self-duality

$$\mathbb{E}_\eta D(\xi,\eta(t)) = \mathbb{E}_\eta D(\xi(t),\eta)$$

for

$$D(\xi,\eta) = \prod_{x\in V} I(\xi_x \leq \eta_x)$$

The particular case is then recovered by putting $\xi = e_x$

2.4 General model

$$Lf(\eta) = \sum_{x,y \in V} p(x,y)\eta_x(\alpha + \sigma\eta_y)(f(\eta - e_x + e_y) - f(\eta))$$

where $\sigma \in \{-1, 0, 1\}$, p symmetric.

- For $\sigma = -1$, $\alpha \in \mathbb{N}$, this is SEP(α) with state space $\{0, 1, \dots, \alpha\}^V$.
- For $\sigma = 0$ this is a system of independent random walkers with state space $\mathbb{N}^V$.
- For $\sigma = 0$, $\alpha > 0$ this is SIP(α) with state space $\mathbb{N}^V$.

These are the three basic particle systems where self-duality holds, and where the algebraic formalism can be illustrated clearly and simply. From the previous computation, it is now clear that for these three models we already have

$$\mathbb{E}_\eta D(x, \eta(t)) = \mathbb{E}_x^{RW(\alpha)}(\eta_{X(t)}(0))$$

Basic properties of the general model

- Reversible product measures The reversible measures of these processes are products of:
  1. SIP: Negative binomials (discrete Gamma distributions): $\nu_p^{(\alpha)}(n) = \frac{p^n\Gamma(\alpha+n)}{n!\Gamma(\alpha)}(1-p)^\alpha$, $0 < p < 1$.
  2. SEP: Binomials: $\nu_p(n) = \binom{\alpha}{n}(1-p)^{\alpha-n}p^n$, $0 < p < 1$, $n \in \{0, \ldots, \alpha\}$.
  3. IRW: Poisson: $\nu_\theta = \frac{\theta^n}{n!}e^{-\theta}$.
- Factorized "classical" self-duality functions The processes are self-dual with $D(\xi, \eta) = \prod_{x\in V} d(\xi_x, \eta_x)$ where d is given by

$$d(k, n) = \frac{n!}{(n-k)!}\frac{1}{m(\alpha, k)}I(k \leq n)$$

with

$$m(\alpha, k) = \begin{cases} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, & \text{for SIP} \\ \frac{\alpha!}{(\alpha-k)!}, & \text{for SEP} \\ 1, & \text{for IRW} \end{cases}$$

- The "classical" self-duality functions and the reversible product measures have the relation

$$\int D(\xi,\eta)\nu(d\eta) = (\int D(\delta_0,\eta))^{|\xi|}$$

it is therefore natural to parametrize the measures such that

$$\int D(\xi,\eta)\nu_\theta(d\eta) = \theta^{|\xi|}$$

- Notice that this relation also shows the invariance of the measures ν_θ via

$$\begin{aligned}\int \mathbb{E}_\eta D(\xi,\eta(t))\nu_\theta(d\eta) &= \int \mathbb{E}_\xi D(\xi(t),\eta)\nu_\theta(d\eta) \\ &= \mathbb{E}_\xi(\theta^{|\xi(t)|}) = \theta^{|\xi|}\end{aligned}$$

- This relation also allows to "recover" all the duality functions from the "first one" $D(e_0,\eta)$.

The structure of the duality functions is

$$D(\xi, \eta) = \prod_x \binom{\eta_x}{\xi_x} \frac{1}{M(\xi_x)}$$

where

$$M(k) = \begin{cases} k! \text{ for IRW} \\ \binom{\alpha}{k} \text{ for SEP} \\ \frac{\Gamma(\alpha+k}{\Gamma(\alpha)k!} \text{ for SIP} \end{cases}$$

$M(\xi) = \prod_i M(\xi_i)$ is a reversible weight, and therefore (as we will see later) for the three models we have the same "intertwiner" $\prod_x \binom{\eta_x}{\xi_x}$ which expresses that choosing a random subconfiguration commutes with the dynamics (consistency).

2.5 Oldest example of duality

$\eta(t)$ Brownian motion on $[0,\infty)$ reflected at 0, $\xi(t)$ Brownian motion on $[0,\infty)$ absorbed at 0. The transition densities of these processes are explicit, with $g_t(x) = e^{-x^2/2t}/\sqrt{2\pi t}$

$$p_t^{\pm}(x,y) = g_t(x-y) \pm g_t(x+y)$$

From that one finds, by explicit computation

$$\mathbb{P}_\eta(\eta(t) \geq \xi) = \hat{\mathbb{P}}_\xi(\xi(t) \leq \eta)$$

in other words

$$\mathbb{E}_\eta D(\xi, \eta(t)) = \hat{\mathbb{E}}_\xi D(\xi(t), \eta)$$

with $D(\xi,\eta) = I(\xi \leq \eta)$. In this example the duality function is not in the domain of the generator. Neither does this example seem to fit in the algebraic formalism very well.

3. Duality and symmetries

Let us for simplicity assume that the processes are on finite state spaces, and thus the generators are matrices. The matrix form of duality reads as follows:

$$
\begin{aligned}
\hat{L}D(\cdot,\eta)(\xi) &= \sum_{\xi'} \hat{L}(\xi,\xi')D(\xi',\eta) \\
&= (\hat{L}D)_{\xi,\eta} \\
&= LD(\xi,\cdot)(\eta) \\
&= \sum_{\eta'} L(\eta,\eta')D(\xi,\eta') \\
&= \sum_{\eta'} D(\xi,\eta')L^T(\eta',\eta) \\
&= (DL^T)_{\xi,\eta}
\end{aligned}
$$

So duality in matrix notation reads

$$\hat{L}D = DL^T$$

Two generators L_1 and L_2 are intertwined with intertwiner Λ_{12} if

$$L_1\Lambda_{12} = \Lambda_{12}L_2$$

If $L_1 = L_2$ and $\Lambda_{12} = S$ then we say that S is a symmetry of L, i.e.,

$$SL - LS = [S, L] = 0$$

We then have the following basic, simple, and very useful properties.

1. **"Cheap self-duality" from a reversible weight.** If M is a reversible weight, i.e.,

$$M(\xi)L(\xi,\eta) = M(\eta)L(\eta,\xi)$$

for all η, ξ, then

$$D(\xi,\eta) = \frac{1}{M(\xi)}\delta_{\xi,\eta}$$

is a self-duality function.

2. **New dualities from combining dualities with intertwiners.** If L_1, L_2, L_3 are three generators and we have the intertwining

$$L_1 \Lambda_{12} = \Lambda_{12} L_2$$

and the duality

$$L_2 D_{23} = D_{23} L_3^T$$

then $D_{13} = \Lambda_{12} D_{23}$ is a duality function between L_1 and L_3, i.e.,

$$L_1 D_{13} = L_3 D_{13}^T$$

As a consequence if D is a self-duality for L and S commutes with L, then SD is a self-duality for L as well.

3. **Dualities from intertwiners.** If M_2 is a reversible measure for L_2 and the intertwiner Λ_{12} is in kernel operator form, i.e.,

$$\Lambda_{12} f(x_1) = \sum_{x_2} D_{12}(x_1, x_2) M(x_2) f(x_2)$$

then the kernel D_{12} is a duality function between L_1 and L_2, i.e.,

$$L_1 D_{12} = D_{12} L_2^T$$

As a consequence if S is a symmetry of L in kernel operator form $Sf(\eta) = \sum_\xi D(\xi, \eta) M(\eta) f(\eta)$ with M a reversible weight for L, then D is a self-duality function for L.

4. Algebraic properties of generators with dualities

4.1 Wright-Fisher example

Define the following operators on $f : \mathbb{N} \to \mathbb{R}$ (where we also define $f(-1) = 0$)

$$\begin{aligned} af(n) &= nf(n-1) \\ a^\dagger f(n) &= f(n+1) \end{aligned} \tag{1}$$

these operators satisfy the so called dual (or conjugate) Heisenberg commutation relation:

$$[a, a^\dagger] = -I$$

Consider on the other hand the operators working on functions $f : [0,1] \to \mathbb{R}$

$$\begin{aligned} Af(x) &= f'(x) \\ A^\dagger f(x) &= xf(x) \end{aligned} \tag{2}$$

these satisfy the Heisenberg commutation relations, i.e.,

$$[A, A^\dagger] = I$$

- We have the following dualities with $D(n,x) = x^n$

$$a \longrightarrow^D A, a^\dagger \longrightarrow^D A^\dagger \tag{3}$$

- Now the jump process generator $\hat{L}f(n) = n(n-1)(f(n-1) - f(n))$ is equal to $a^2 a^\dagger I(I - a^\dagger)$, whereas the Wright-Fisher diffusion generator is equal to $A^\dagger(I - A^\dagger)A^2$. Now we see that these are dual via (3).
- More generally, a "word" composed of $a, a^\dagger$ such as e.g. $aaa^\dagger a^\dagger a$ is dual to the "reversed" word $AA^\dagger A^\dagger A^2$ So we see that the duality between between the generators extends actually to a duality between two algebras.

4.2 Independent random walkers: self-duality

$$L_{12}f(\eta) = \eta_1[f(\eta - e_1 + e_2) - f(\eta)] + \eta_2[f(\eta - e_2 + e_1) - f(\eta)]$$

can be rewritten in terms of the operators from (1)

$$L_{12} = -(a_1 - a_2)(a_1^\dagger - a_2^\dagger)$$

From this form we infer symmetries $S = a_1 + a_2$, $S^\dagger = a_1^\dagger + a_2^\dagger$. Indeed,

$$\begin{aligned}[(a_1 - a_2)(a_1^\dagger - a_2^\dagger), a_1 + a_2] &= (a_1 - a_2)[a_1^\dagger - a_2^\dagger, a_1 + a_2] \\ &= (a_1 - a_2)(I - I) = 0\end{aligned}$$

We have the reversible weight

$$M(\eta) = \frac{1}{\eta_1!}\frac{1}{\eta_2!}$$

with corresponding cheap duality $D_{ch}(\xi, \eta) = \eta_1!\eta_2!\delta_{\xi_1,\eta_1}\delta_{\xi_2,\eta_2}$ Now we claim

$$e^{S^\dagger} D(\cdot, \eta)(\xi) = \frac{\eta_1!}{(\eta_1 - \xi_1)!}\frac{\eta_2!}{(\eta_2 - \xi_2)!}$$

$$
\begin{aligned}
e^{a^\dagger}\delta_{\cdot,n}(k) &= \sum_{r=0}^{\infty}\frac{(a^\dagger)^r}{r!}\delta_{\cdot,n}(k)\\
&= \frac{(a^\dagger)^{n-k}}{(n-k)!}\delta_{\cdot,n}(k)\\
&= \frac{1}{(n-k)!} Ik(\leq n)
\end{aligned}
$$

So we obtain the self-duality function
$D(\xi,\eta) = \prod_x \frac{\eta_x!}{(\eta_x-\xi_x)!} I(\xi_x \leq \eta_x)$

4.3 Independent random walkers: duality

Consider

$$\mathcal{L}_{12}f(x_1,x_2) = -(x_1 - x_2)(\partial_{x_1} - \partial_{x_2})$$

this is the generator of the deterministic process

$$\dot{x}_1(t) = x_2(t) - x_1(t) = -\dot{x}_2(t)$$

then by the duality between the representations (1) and (2) we know that this process with generator $-(A_1^\dagger - A_2^\dagger)(A_1 - A_2)$ is dual to the process with generator

$$-(a_1 - a_2)(a_1^\dagger - a_2^\dagger)$$

in other words

$$\mathbb{E}_\eta(x_1^{\eta_1(t)} x_2^{\eta_2(t)}) = x_1(t)^{\eta_1} x_2(t)^{\eta_2}$$

4.4 Algebraic structure of SIP

$$\begin{aligned} L_{12}f(\eta_1,\eta_2) &= \eta_1(\alpha+\eta_2)(f(\eta-e_1+e_2)-f(\eta)) \\ &+ \eta_2(\alpha+\eta_1)(f(\eta-e_2+e_1)-f(\eta)) \end{aligned}$$

Introduce

$$\begin{aligned} K^+f(n) &= (\alpha+n)f(n+1) \\ K^-f(n) &= nf(n-1) \\ K^0f(n) &= (\frac{\alpha}{2}+n)f(n) \end{aligned} \tag{4}$$

These operators satisfy the commutation relations of the dual (conjugate) algebra of $SU(1,1)$, i.e.,

$$\begin{aligned} \pm K^\pm &= [K^\pm, K^0] \\ 2K^0 &= [K^+, K^-] \end{aligned} \tag{5}$$

and we have

$$L_{12} = K_1^+K_2^- + K_1^-K_2^+ - 2K_1^0K_2^0 + \frac{\alpha^2}{2}$$

As a consequence, L_{12} commutes with $K_1^u + K_2^u$ with $u \in \{+, -, 0\}$ Let us do one commutator

$$
\begin{aligned}
& [K_1^+ K_2^- + K_1^- K_2^+ - 2K_1^0 K_2^0, K_1^+ + K_2^+] \\
= \;& [K_1^-, K_1^+] K_2^+ - 2[K_1^0, K_1^+] K_2^0 \\
+ \;& K_1^+ [K_2^-, K_2^+] - 2K_1^0 [K_2^0, K_2^+] \\
= \;& -2K_1^0 K_2^+ + 2K_1^+ K_2^0 - 2K_1^+ K_2^0 + 2K_1^0 K_2^+ = 0
\end{aligned}
$$

A cheap self-duality function derived from a reversible weight is given by

$$D_{ch}(\xi, \eta) = \prod_x \eta_x! \frac{\Gamma(\alpha)}{\Gamma(\alpha + \eta_x)} \delta_{\xi_x, \eta_x}$$

One then has

$$e^{K_1^+ + K_2^+} D_{ch}(\cdot, \eta)(\xi) = \prod_x \frac{\eta_x! \Gamma(\alpha)}{(\eta_x - \xi_x)! \Gamma(\alpha + \xi_x)}$$

which follows via $e^{K+} \delta_{\cdot, n}(k) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k)} I(k \leq n)$

4.4 Coproduct

We define

$$\Delta(K^u) = K_1^u + K_2^u$$

and extend this to an algebra homomorphism between $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{A}$. This is well defined via linearity and $\Delta(a)\Delta(b) =: \Delta(ab)$ because Δ preserves the commutation relations, i.e.,

$$[\Delta(K^u), \Delta(K^v)] = \Delta([K^u, K^v])$$

for $u, v \in \{+, -, 0\}$.

Let us verify this for $u = +, v = -$:

$$\begin{aligned}\Delta(K^+K^-) &= (K_1^+ + K_2^+)(K_1^- + K_2^-) \\ &= K_1^+K_1^- + K_2^+K_1^- + K_1^+K_2^- + K_2^+K_1^-\end{aligned}$$

$$\Delta(K^-K^+) = K_1^-K_1^+ + K_2^-K_2^+ + K_1^-K_2^+ + K_2^-K_1^+$$

so we see

$$\Delta([K^+, K^-]) = [K_1^+, K_1^-] + [K_2^+, K_2^-] = 2K_1^0 + 2K_2^0 = \Delta[2K^0]$$

whereas

$$\begin{aligned}[\Delta(K^+), \Delta(K^-)] &= [K_1^+ + K_2^+, K_1^- + K_2^-] \\ &= [K_1^+, K_1^-] + [K_2^+, K_2^-] = 2K_1^0 + 2K_2^0\end{aligned}$$

Coproduct of the Casimir

The operator

$$C = (K^0)^2 - \frac{1}{2}(K^+K^- + K^-K^+)$$

is called the Casimir and is central, i.e., commutes with K^+, K^-, K^0. We then have the following

$$\Delta(-C) = K_1^+K_2^+ + K_2^+K_1^- - 2K_1^0K_2^0 - C_1 - C_2$$

Because C_1, C_2 are central in $\mathcal{A} \otimes \mathcal{A}$ the symmetries of $\Delta(-C)$ are the same as those of $K_1^+K_2^+ + K_2^+K_1^- - 2K_1^0K_2^0$, which are $\Delta(K^u)$, $u \in \{+, -, 0\}$, because preserves commutators.
So we have understood that the generator of SIP(α) is up to central elements the coproduct of the Casimir, and therefore, we have that it commutes with $K_1^u + K_2^u$, $u \in \{+, -, 0\}$ automatically.

This outlines a general procedure to construct generators with "many" symmetries.

- Start from a central element C.
- Apply a coproduct $\Delta(C)$ to turn it into an operator working on two variables.
- If (in a representation) this operator is a generator with a reversible measure, then this generator has several self-dualities, coming from the symmetries $\Delta(A)$.

This procedure has been succesfully applied to construct several new processes with self-dualities such as $ASEP(q,j), ASIP(q,k)$ and multi species models (works of Jeffrey Kuan and collaborators).

5. Charlier polynomial duality

Remember the representation (1)

$$\begin{aligned} a^\dagger f(n) &= f(n+1) \\ af(n) &= nf(n-1) \end{aligned}$$

which satisfies the dual Heisenberg commutation relation $[a, a^\dagger] = -I$. Now assume that we have a pair $A, A^\dagger$ satisfying the Heisenberg commutation relation $[A, A^\dagger] = I$, and a duality function D such that

$$a^\dagger \longrightarrow^D A^\dagger, a \longrightarrow^D A \tag{6}$$

then we have

$$-(a_2 - a_1)(a_2^\dagger - a_1^\dagger) \longrightarrow^D -(A_2^\dagger - A_1^\dagger)(A_2 - A_1)$$

We will consider three such cases, which are moreover such that

$$(A_2^\dagger - A_1^\dagger)(A_2 - A_1) = (a_2 - a_1)(a_2^\dagger - a_1^\dagger) \tag{7}$$

- Case 1: cheap self-duality

$$A^\dagger f(n) = af(n), Af(n) = a^\dagger f(n)$$

with $D(k,n) = n!\delta_{k,n}$.

- Case 2: classical self-duality

$$A^\dagger f(n) = af(n) = nf(n-1), Af(n) = a^\dagger f(n) - f(n) = f(n+1) - f(n)$$

then we have (6) with this time $D(k,n) = \frac{n!}{(n-k)!} I(k \leq n)$.

- Case 3: orthogonal self-duality

$$A^\dagger f(n) = f(n) - \frac{n}{\lambda} f(n-1), Af(n) = \lambda f(n) - \lambda f(n+1)$$

This case is special because $A^* = \lambda A^\dagger$ in $L^2(\nu_\lambda)$ with ν_λ the Poisson measure, and additionally, $A1 = 0$. The duality function is $D_\lambda(k,n) = (A^\dagger)^k 1 = e^\lambda C_k(n)$ with C_k the k-th order Charlier polynomial.

Let us see that these are orthogonal in $L^2(\nu_\lambda)$. By the commutation relations we have

$$[A, (A^\dagger)^n] = n(A^\dagger)^{n-1}$$

so we get (put $\lambda = 1$ for simplicity)

$$\begin{aligned}
\langle (A^\dagger)^{k+1}1, (A^\dagger)^k 1\rangle &= \langle (A^\dagger)^k 1, A(A^\dagger)^k 1\rangle \\
&= \langle (A^\dagger)^k 1, (A^\dagger)^k A1\rangle + \langle (A^\dagger)^k 1, [A, (A^\dagger)^k]1\rangle \\
&= k\langle (A^\dagger)^k 1, \langle (A^\dagger)^{k-1} 1\rangle \\
&= \ldots = k(k-1)...\langle 1, A1\rangle = 0
\end{aligned}$$

This representation yields orthogonal polynomial self-duality (in $L^2(\nu_\lambda)$).

6. Derived models

Starting from the SIP(α) generator

$$\begin{aligned} L_{12} f(\eta_1, \eta_2) &= \eta_1(\alpha + \eta_2)(f(\eta - e_1 + e_2) - f(\eta)) \\ &+ \eta_2(\alpha + \eta_1)(f(\eta - e_2 + e_1) - f(\eta)) \end{aligned}$$

we can consider the following "derived" generators

- Diffusion limit. Put $\eta_i = \lfloor x_i N \rfloor$, $x_i \in (0, \infty)$ and let $N \to \infty$, then in the limit $N \to \infty$ the process $x_i(t)$ has generator

$$\mathcal{L}_{12} = x_1 x_2 (\partial_{x_1} - \partial_{x_2})^2 - \alpha(x_2 - x_1)(\partial_{x_2} - \partial_{x_1})$$

  From self-duality of SIP(α) one then finds duality between this generator and the SIP(α) generator.

- Thermalization

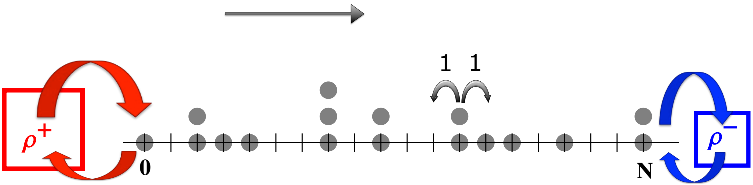
$$\mathcal{L}_{12} f(\eta) = \lim_{t\to\infty} (e^{tL_{12}} - I) f(\eta)$$

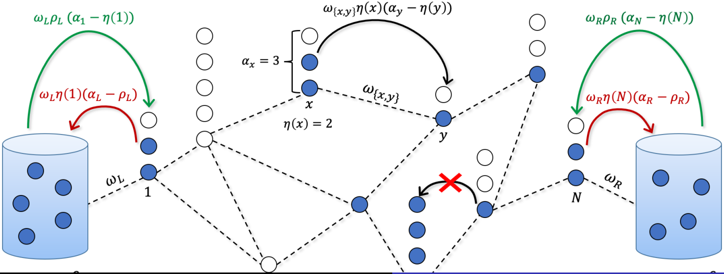
this is a discrete mass redistribution model where (at random times) the initial mass is redistributed as $(\eta_1, \eta_2) \to (U, \eta_1 + \eta_2 - U)$ with U a beta binomial random variable. This model is self-dual with the same self-duality function as SIP.

- Diffusion limit + thermalization. This yields continuum mass redistribution model of KMP type, i.e., initial mass (x_1, x_2) is redistributed as $(x_1 + x_2)U, (x_1 + x_2)(1 - U)$ with U Beta distributed. This model is dual to the thermalized SIP.

Non-equilibrium version

$$L = \sum_{i=1}^{N} L_{i,i+1} + L_1^{\rho_L} + L_N^{\rho_R}$$





$$L_{dual} = \sum_{i=1}^{N} L_{i,i+1} + \mathcal{L}_{1,0} + \mathcal{L}_{N,N+1}$$

where $\mathcal{L}_{1,0}, \mathcal{L}_{N,N+1}$ describe rate one hopping from 1 to 0 (resp. N to $N+1$), both 0 and $N+1$ are absorbing. Duality functions

$$\mathcal{D}(\xi,\eta) = \rho_L^{\xi_0} \rho_R^{\xi_{N+1}} \prod_{i=1}^{N} d(\xi_i,\eta_i)$$

The duality then allows to reduce computations of n-point correlations to computation of absorption probabilities for n particles. E.g. for the "profile" we only need one dual particle:

$$\mathbb{E}_{\mu_{\rho_L,\rho_R}}(D(e_x,\eta)) = \rho_L \mathbb{P}_x(X(\infty)=0) + \rho_R \mathbb{P}_x(X(\infty)=N+1)$$

7. Orthogonal Polynomial (self)-duality

7.1 Definition and easy consequences

$$\mathbb{E}_\eta D_\rho(\xi, \eta(t)) = \mathbb{E}_\xi D_\rho(\xi(t), \eta)$$

where $D_\rho(\xi, \cdot)$ is a collection of orthogonal polynomials (of degree $|\xi|$) in $L^2(\nu_\rho)$, where ν_ρ is a reversible (product measure). More precisely

$$\int D_\rho(\xi, \eta) D_\rho(\xi', \eta) \nu_\rho(d\eta) = \delta_{\xi,\xi'} a_\rho(\xi)$$

with $a_\rho(\xi) = \|D_\rho(\xi, \cdot)\|^2_{L^2(\nu_\rho)}$ We have the following basic easy properties

- Positivity and decay of time dependent stationary correlations

$$\int D_\rho(\xi', \eta) \mathbb{E}_\eta D(\xi, \eta(t)) \nu_\rho(d\eta) = p_t(\xi, \xi') a_\rho(\xi') \geq 0$$

- Decay of time dependent variance

$$\text{Var}_{\nu_\rho}(S_t D_\rho(\xi, \eta)) = a_\rho(\xi) p_{2t}(\xi, \xi)$$

- $a_\rho(\xi)$ satisfies detailed balance

Proof: use reversibility

$$\begin{aligned}\mathrm{Var}_{\nu_\rho}(S_t D_\rho(\xi,\eta)) &= \langle S_t D_\rho(\xi,\eta), S_t D_\rho(\xi,\eta)\rangle \\ &= \langle D_\rho(\xi,\eta), S_{2t} D_\rho(\xi,\eta)\rangle \\ &= p_{2t}(\xi,\xi) a_\rho(\xi)\end{aligned}$$

$$\begin{aligned}\int S_t D_\rho(\xi,\eta) D_\rho(\xi',\eta)\nu_\rho(d\eta) &= p_t(\xi,\xi') a_\rho(\xi') \\ &= \int D_\rho(\xi,\eta) S_t D_\rho(\xi',\eta)\nu_\rho(d\eta) \\ &= p_t(\xi',\xi) a_\rho(\xi)\end{aligned}$$

7.2 Obtaining orthogonal duality functions from classical ones

Let us call $D(\xi, \cdot)$ denote the classical duality functions, and fix a reversible measure ν_ρ. Let us call

$$V_n = cl(vct\{D(\xi, \cdot) : |\xi| \leq n\})$$

where the closure is in $L^2(\nu_\rho)$ then V_n is an increasing sequence of closed subspaces. Because of self-duality we have that elements of V_n are mapped to elements of V_n: indeed, if $|\xi| \leq n$ then, because of conservation of the number of particles

$$S_t D(\xi, \eta) = \sum_{\xi'} p_t(\xi, \xi') D(\xi', \eta) \in V_n$$

Then we have the following

- The semigroup commutes with P_{V_n}, the orthogonal projection on V_n.
- The semigroup commutes with the orthogonal projection on $V_{n+1} \cap V_n^{\perp}$.
- The Gramm-Schmidt orthogonalization of the classical self-duality functions are self-duality functions.

Proof (due to Stefan Wagner)
Let $f \in L^2(\nu_\rho)$. Because $S_t P_{V_n} f \in V_n$, we have $S_t P_{V_n} = P_{V_n} S_t P_{V_n} f$. Now decompose

$$S_t f = S_t(P_{V_n} f + P_{V_n^\perp} f)$$

then we show that

$$S_t P_{V_n^\perp} f \in V_n^\perp$$

which implies $P_{V_n} S_t P_{V_n^\perp} f = 0$. Let $g \in V_n$ then, using that $S_t g \in V_n$ and reversibility we have

$$\langle S_t P_{V_n^\perp} f, g \rangle = \langle P_{V_n^\perp} f, S_t g \rangle = 0$$

As a consequence

$$P_{V_n}(S_t f) = P_{V_n} S_t P_{V_n} f = S_t P_{V_n}$$

which shows the commutation property. To see point 2: $P_{V_{n+1} \cap V_n^\perp} = P_{V_{n+1}} - P_{V_n}$

Relation between classical dualities and orthogonal dualities

The abstract orthogonalisation takes a simple form for the three systems SIP, SEP, IRW. For $\xi = \sum_{i=1}^{n} \delta_{x_i}$ we have

$$\begin{aligned} \mathcal{D}_\theta(\xi, \eta) &= \sum_{\xi' \leq \xi} (-\theta)^{|\xi| - |\xi'|} \binom{\xi}{\xi'} D(\xi', \eta) \\ &= \sum_{I \subset [n]} (-\theta)^{n-|I|} D\left(\sum_{i \in I} \delta_{x_i}, \eta \right) \end{aligned}$$

With $[n] = \{1, \ldots, n\}$. Or in the alternative notation:

$$\mathcal{D}_\theta(x_1, \ldots, x_n; \eta) = \sum_{I \subset [n]} (-\theta)^{n-|I|} D\left((x_i)_{i \in I}; \eta \right)$$

8. Macroscopic limits

From now on we work on the vertex set $V = \mathbb{Z}^d$, and assume $p(x, y)$ translation invariant. For a local function f (i.e., a functions only depending on a finite number of occupancy numbers) we define its fields on scale N as follows

- Hydrodynamic field

$$X_N(f, \eta)[\phi] = \frac{1}{N^d} \sum_x \phi(\tfrac{x}{N}) \tau_x f(\eta)$$

  If $f(\eta) = q_0(\eta) = \eta_0$ this is called the density field.

- Fluctuation field

$$\frac{1}{N^{d/2}} \sum_x \phi(\tfrac{x}{N}) [\tau_x f(\eta) - \mathbb{E}_{\nu_\rho}(\tau_x f)]$$

  If $f(\eta) = q_0(\eta) = \eta_0$ this is called the density fluctuation field.

Expectation of density field: one dual particle

Notice that $D(e_x, \eta) = C\eta_x$, to looking at the density field is equivalent with looking at

$$\frac{1}{N^d}\sum_x \phi(\tfrac{x}{N})D(e_x, \eta)$$

Let us now rescale time diffusively, and assume that at time zero η is distributed according to a measure μ^N with $\mu_N(D(e_x, \eta) = \rho(x/N)$, and compute, using duality with a single dual particle, using also the notation

$$z(t) = \lim_{N\to\infty}\frac{X(N^2 t)}{N}$$

with $X(t)$ a single dual particle starting at 0.

$$
\begin{aligned}
& \mathbb{E}_{\mu_N}\left(\frac{1}{N^d}\sum_x \phi(\tfrac{x}{N}) D(e_x,\eta(N^2 t))\right) \\
= \quad & \frac{1}{N^d}\sum_{x,y} \phi(\tfrac{x}{N}) p_{tN^2}(x,y)\mathbb{E}_{\mu_N} D(e_y,\eta) \\
= \quad & \frac{1}{N^d}\sum_{x,y} \phi(\tfrac{x}{N}) p_{tN^2}(x,y)\rho(\tfrac{y}{N}) \\
= \quad & \frac{1}{N^d}\sum_{x,y} \phi(\tfrac{x}{N}) p_{tN^2}(0,z)\rho(\tfrac{x}{N}+\tfrac{z}{N}) \\
\longrightarrow \quad & \int \phi(x)\mathbb{E}\rho(x+z(t))dx
\end{aligned}
$$

We obtain that the expectation of the density field converges to $\int \phi(x)\rho(t,x)dx$ with $\rho(t,x)$ the solution of the PDE

$$\partial_t \rho(t,x) = \mathcal{L}\rho(t,x)$$

with $\mathcal{L}$ the generator of $z(t)$. This is called the hydrodynamic equation.
Usually, $z(t) = B(Dt)$ so this is the heat equation.
We can generalize this and look at the so-called higher order hydrodynamic fields (cf. Chen, Sau, MPRF to appear)

$$\frac{1}{N^{kd}} \sum_x \phi(\tfrac{x_1}{N}, \ldots, \tfrac{x_k}{N}) D(x_1, \ldots, x_k; \eta(N^2 t))$$

where $D(x_1 + \ldots x_k, \eta) = D(e_{x_1} + \ldots e_{x_k}, \eta)$ and one proves that these converge to $\int_{\mathbb{R}^{dk}} \phi(x)\rho(t,x)$ where $\rho(t,x)$ satisfies

$$\partial_t \rho(t,x) = (\otimes^k \mathcal{L})\rho(t,x)$$

8.2 Propagation of local equilibrium

For $\theta : \mathbb{Z}^d \to \mathbb{R}^+$ consider the inhomogeneous product measure

$$\nu_\theta = \otimes_{x \in \mathbb{Z}^d} \nu_{\theta(x)}$$

then we have

$$\int D(x_1, \ldots, x_k; \eta) \nu_\theta(d\eta) = \prod_{i=1}^{k} \theta(x_i)$$

and hence by self-duality at later times $t \geq 0$

$$\int \mathbb{E}_\eta \left(D(x_1, \ldots, x_k; \eta(t)) \right) \nu_\theta(d\eta) = \mathbb{E}_{x_1,\ldots,x_k} \left(\prod_{i=1}^{k} \theta(x_i(t)) \right)$$

Problem is that this does not factorize in general (unless $\sigma = 0$), and so these product measures are not reproduced in time. However, if θ varies slowly and time is rescaled diffusively, then we will have approximate factorization.

Let $\rho : \mathbb{R}^d \to [0,\infty)$ We call a family of measures μ_N a local equilibrium with profile ρ ($LEQ(\rho)$) if for all $x \in \mathbb{R}^d$, $x_1, \ldots, x_k \in \mathbb{Z}^d$ we have

$$\lim_{N\to\infty} \int D(\lfloor xN \rfloor + x_1, \ldots \lfloor xN \rfloor + x_k; \eta)\mu_N(d\eta) = \rho(x)^k$$

Important example

$$\mu_N = \otimes_{x\in\mathbb{Z}^d} \nu_{\rho(\frac{x}{N})} \tag{8}$$

Propagation of local equilibrium then means that

$$\mu_N = LEQ(\rho) \text{ implies } \mu_N S(N^2 t) = LEQ(\rho_t)$$

where ρ_t solves the hydrodynamic equation.

Let us see how this propagation arrises when we start from (8).

$$\begin{aligned}
&\int \mathbb{E}_\eta D\left(\lfloor xN\rfloor + x_1, \ldots \lfloor xN\rfloor + x_k;\right) \\
&= \mathbb{E}_{\lfloor xN\rfloor + x_1, \ldots \lfloor xN\rfloor + x_k}\left(\prod_{i=1}^k \rho\left(\frac{X_i(N^2 t)}{N}\right)\right) \\
&\approx \prod_{i=1}^k \mathbb{E}_{\lfloor xN\rfloor + x_i}\rho\left(\frac{X_i(N^2 t)}{N}\right) \\
&\approx \prod_{i=1}^k \mathbb{E}_{\lfloor xN\rfloor}\rho\left(\frac{X_i(N^2 t)}{N}\right) \\
&\approx \rho(t,x)^k
\end{aligned}$$

The most important step is the first approximation which comes from the fact that k dual particles $X_i(t)$ can be coupled to k independent random walkers $\tilde{X}_i(t)$ (starting at the same positions) such that

$$|X_i(t) - \tilde{X}_i(t)| = o(\sqrt{t})$$

this can be done due to the locality and symmetry of the interaction, combined with the fact that a random walk spend at most order $\sqrt{t}$ at a fixed location.

8.3 The Boltzmann-Gibbs principle and orthogonal polynomial duality

If one considers the fluctuation field of a local function f

$$Y_N(f,\eta) = \frac{1}{N^{d/2}} \sum_{x\in\mathbb{Z}^d} \phi(\tfrac{x}{N})(\tau_x f(\eta) - \mathbb{E}_{\nu_\rho}(f))$$

then the Boltmann Gibbs principle tells that this can be approximated by the density field times a constant (depending on f, ρ) in the following sense. There exists $C(f,\rho)$ such that

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\rho} \left(\int_0^T Y_N(f - C(f,\rho)q_0, \eta(N^2 s) ds \right)^2 = 0 \qquad (9)$$

which expresses that in the sense of (9)

$$Y_N(f,\eta(N^2 t)) \approx C(f,\rho) Y_N(q_0, \eta(N^2 t))$$

$C(f,\rho) = [\frac{d}{d\theta}\nu_\theta(f)]_{\theta=\rho}$.

- The density fluctuation field corresponds to the field of the orthogonal duality polynomial $D_\rho(e_0, \cdot)$.
- Therefore, another way of seeing the Boltzmann-Gibbs principle is to say that the fluctuation fields of all higher order orthogonal duality polynomials are negligeable in the sense of (9).
- Let us understand this fact by starting with a simple example of the fluctuation field of the orthogonal duality polynomial $D_\rho(2e_0, \cdot)$, and showing that is indeed negligible in the sense (9).

So we want to estimate

$$\frac{1}{N^d}\sum_x\sum_y \phi(x/N)\phi(y/N)\int_0^T dt\int_0^T ds\,($$

$$\mathbb{E}_{\nu_\rho}(D_\rho(x,x;\eta(N^2t)D(y,y;\eta(N^2s)))$$

$$\frac{1}{N^d}\sum_x\sum_y \phi(x/N)\phi(y/N)\int_0^T dt\int_0^t ds\,($$

$$\mathbb{E}_{\nu_\rho}(D_\rho(x,x;\eta(N^2(t-s)D(y,y;\eta(0)))$$

$$= \frac{2C(\rho)}{N^d}\sum_x\sum_y \phi(x/N)\phi(y/N)\int_0^T dt\int_0^t ds p_{N^2(t-s)}(x,x;y,y)$$

$$\leq \frac{2C(\rho)}{N^d}\sum_x |\phi(x/N)|\int_0^T dt\int_0^t ds\,($$

$$\mathbb{E}_{x,x}(|\phi|(X(N^2(t-s)))I(X(N^2(t-s)=Y(N^2(t-s))))$$

Putting $Z(t) = X(t) - Y(t)$, and changing to $\tau = N^2(t-s)$ this can be estimated further by

$$\frac{2C(\rho)T\|\phi\|_\infty}{N^d} \sum_x |\phi(x/N)| \frac{1}{N^2} \int_0^{N^2 T} \mathbb{E}_0^Z(I(Z(\tau) = 0)) d\tau$$

Now the order of $\int_0^{N^2 T} \mathbb{E}_0^Z(I(Z(s) = 0)) ds$ is dimension dependent and this gives that the quantity of interest is of order $1/N$ in $d = 1$, $\log(N)/N$ in $d = 2$, $1/N^2$ in $d \geq 3$.

For the more general case

$$\mathbb{E}_{\nu_\rho}\left(\int_0^T \frac{1}{N^{d/2}}\sum_x \phi(x/N) D_\rho(x+x_1,\ldots,x+x_n,\eta(N^2 s)\right)^2$$

we find an upperbound of the form

2

$$\sum_{\sigma\in S_n}\int_0^T \mathbb{E}_{x+x_1,\ldots,x+x_n} I(X_1(N^2 r)-x_{\sigma(1)}=\ldots=X_n(N^2 r)-x_{\sigma(n)})dr$$

leading to similar estimates.

Open problems

- Perturbation theory around systems with duality
- Duality beyond Markov processes
- Link integrability duality