Scaling limits for symmetric exclusion with open boundary

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Joint with

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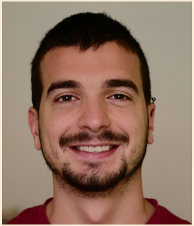

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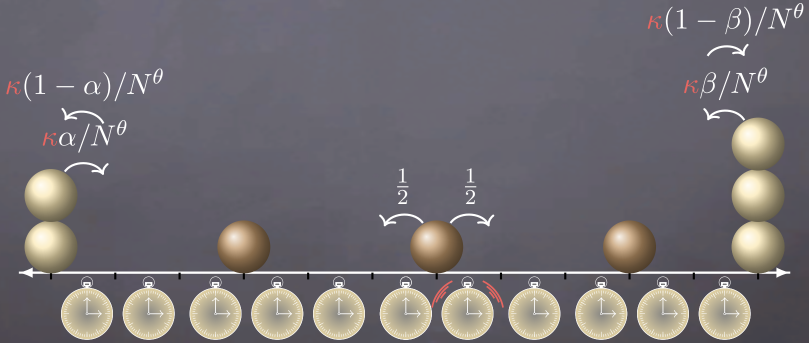
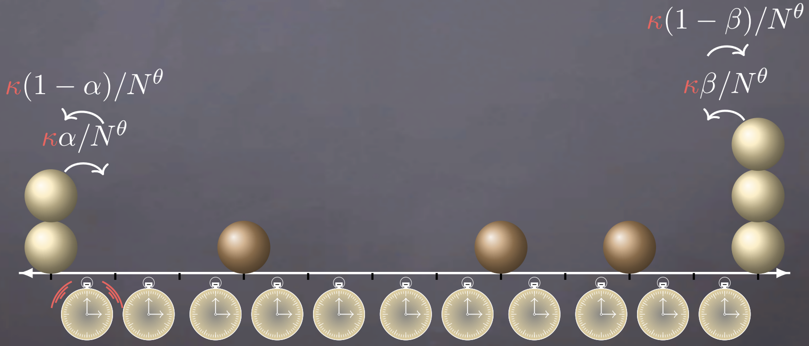
Lecture 1: SSEP in contact with reservoirs

SSEP in contact with reservoirs



SSEP in contact with reservoirs

SSEP in contact with reservoirs



SSEP in contact with reservoirs

SSEP in contact with reservoirs

SSEP in contact with reservoirs

The dynamics:

- For $N \geq 1$ let $\Lambda_N = \{1, \dots, N-1\}$.
- We denote the process by $\{\eta_t : t \geq 0\}$ which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$.
- The infinitesimal generator $\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,b}$ is given on $f : \Omega_N \to \mathbb{R}$, by

$$(\mathcal{L}_{N,0} f)(\eta) = \sum_{x=1}^{N-2} \frac{1}{2}\Big(f(\eta^{x,x+1}) - f(\eta)\Big),$$

$$(\mathcal{L}_{N,b} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{x \in \{1, N-1\}} c_{r_x}(\eta(x))\Big(f(\eta^x) - f(\eta)\Big),$$

where for $x = 1$ and $x = N-1$, $c_{r_x}(\eta(x)) = r_x(1-\eta(x)) + (1-r_x)\eta(x)$, $r_1 = \alpha$ and $r_{N-1} = \beta$.

Goal: analyse the impact of changing the strength of the reservoirs (by changing θ) on the macroscopic behavior of the system.

Invariant measures:

If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant (equilibrium measures): $\nu_\rho(\eta : \eta(x) = 1) = \rho$.

If $\alpha \neq \beta$ the Bernoulli product measure is no longer invariant, but since we have a finite state irreducible Markov process there exists a UNIQUE invariant measure: the stationary measure (non-equilibrium) denoted by μ_{ss}.

By the matrix ansatz method one can get information about this measure. (Not in the long jumps case.)

Reversibility:

For $\alpha = \beta = \rho$ the Bernoulli product measures ν_ρ are reversible. To show it, fix two functions $f, g : \Omega_N \to \mathbb{R}$. We need to show that

$$\int_{\Omega_N} g(\eta)\mathcal{L}_N f(\eta) d\nu_\rho = \int_{\Omega_N} f(\eta)\mathcal{L}_N g(\eta) d\nu_\rho.$$

Starting with the exchange dynamics we see that for fixed $x \in \Lambda_N$, by a change of variables $\xi = \eta^{x,x+1}$, we have

$$\sum_{\eta\in\Omega_N} g(\eta) f(\eta^{x,x+1})\nu_\rho(\eta) = \sum_{\xi\in\Omega_N} g(\xi^{x,x+1}) f(\xi)\frac{\nu_\rho(\xi^{x,x+1})}{\nu_\rho(\xi)}\nu_\rho(\xi).$$

Since

$$\nu_\rho(\xi) = \prod_{x\in\Lambda_N} \rho^{\xi(x)}(1-\rho)^{1-\xi(x)}.$$

Reversibility:

- if $\xi(x) = 1$ and $\xi(x+1) = 0$, denoting by $\tilde{\xi}$ the configuration ξ removing its values at x and $x+1$ so that $\xi = (\tilde{\xi}, \xi(x), \xi(x+1))$, then $\nu_\rho(\xi) = \nu_\rho(\tilde{\xi})\rho(1-\rho)$ and $\nu_\rho(\xi^{x,x+1}) = \nu_\rho(\tilde{\xi})(1-\rho)\rho$, so that
$$\frac{\nu_\rho(\xi^{x,x+1})}{\nu_\rho(\xi)} = 1. \tag{1}$$
- if $\xi(x) = 0$ and $\xi(x+1) = 1$, then $\nu_\rho(\xi) = \nu_\rho(\tilde{\xi})(1-\rho)\rho$ and $\nu_\rho(\xi^{x,x+1}) = \nu_\rho(\tilde{\xi})\rho(1-\rho)$, so that (1) is also true.

From this we get
$$\int_{\Omega_N} g(\eta) f(\eta^{x,x+1}) d\nu_\rho = \sum_{\xi \in \Omega_N} g(\xi^{x,x+1}) f(\xi) \nu_\rho(\xi).$$
which proves the result for $\mathcal{L}_{N,0}$. For the flip dynamics it is analogous.

Hydrodynamic Limit:

♣ **For** $\eta \in \Omega_N$**, let**

$$\pi_t^N(\eta, dq) = \frac{1}{N-1}\sum_{x=1}^{N-1} \eta_{tN^2}(x)\delta_{x/N}(dq),$$

be the *empirical measure. (Diffusive time scaling!)*
♣ **Assumption: fix** $g : [0,1] \to [0,1]$ **measurable and a sequence of probability measures** $\{\mu_N\}_{N\geq 1}$ **such that for every** $H \in C([0,1])$,

$$\tfrac{1}{N-1}\sum_{x=1}^{N-1} H(\tfrac{x}{N})\,\eta(x) \to_{N\to+\infty} \int_0^1 H(q)\,g(q)dq,$$

wrt μ_N**.** (μ_N **is associated with** $g(\cdot)$)

Hydrodynamic Limit:

♣ **Assumption: fix $g : [0,1] \to [0,1]$ measurable and a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $H \in C([0,1])$,**

$$\frac{1}{N-1}\sum_{x=1}^{N-1} H\left(\tfrac{x}{N}\right)\eta(x) \to_{N\to+\infty} \int_0^1 H(q)\, g(q) dq,$$

wrt μ_N. (i.e. $\pi_0^N(\eta, dq) \to_{N\to+\infty} g(q)dq$)

♣ **Then: for any $t > 0$,**

$$\pi_t^N(\eta, dq) \to_{N\to+\infty} \rho(t,q)dq,$$

wrt $\mu_N(t)$, where $\rho(t,q)$ evolves according to a PDE, the hydrodynamic equation.

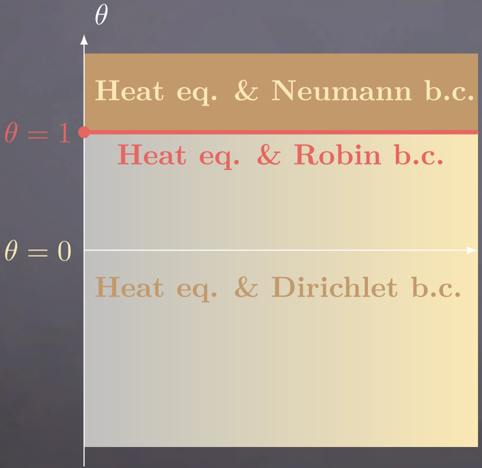
Hydrodynamic Limit:

Theorem: Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated with $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\mathbb{P}_{\mu_N}\Big(\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta_{tN^2}(x) - \int_0^1 H(q)\rho(t,q)dq\Big| > \delta\Big) \to_N 0,$$

and $\rho_t(\cdot)$ is the UNIQUE weak solution of the heat equation with different types of boundary conditions depending on the range of the parameter θ and with initial condition $g(\cdot)$.

Hydrodynamic equations:



Heat equation:
$\partial_t \rho_t(q) = \frac{1}{2}\partial_q^2 \rho_t(q)$.

- $\theta > 1$ **Neumann b.c.:**
  $\partial_q \rho_t(0) = \partial_q \rho_t(1) = 0$.
- $\theta = 1$ **Robin b.c.:**
  $\partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha)$,
  $\partial_q \rho_t(1) = \kappa(\beta - \rho_t(1))$.
- $\theta < 1$ **Dirichlet b.c.:**
  $\rho_t(0) = \alpha$, $\rho_t(1) = \beta$.

Hydrostatic Limit:

Theorem: Let μ_{ss} be the stationary measure for the process $\{\eta_t\}_{t\geq 0}$. Then, μ_{ss} is associated to $\bar{\rho} : [0,1] \to [0,1]$ given on $q \in (0,1)$ by

$$\bar{\rho}(q) = \begin{cases} (\beta-\alpha)q+\alpha\,;\,\theta<1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\,\theta=1, \\ \frac{\beta+\alpha}{2}\,;\,\theta>1. \end{cases}$$

$\bar{\rho}(\cdot)$ is a stationary solution of the hydrodynamic equation.

The proof:

How do we prove the results?

Two things to do:

- ♣ Tightness of $\mathbb{Q}_N$, where $\mathbb{Q}_N$ is induced by $\mathbb{P}_{\mu_N}$ and the map

$$\pi^N_\cdot : \mathcal{D}([0,T],\Omega_N) \longrightarrow \mathcal{D}([0,T],\mathcal{M}_+)$$

- ♣ Characterization of limit points: limit points are concentrated on trajectories of measures that are absolutely continuous wrt the Lebesgue measure and the density is a weak solution of the corresponding PDE:

$$\mathbb{Q}(\pi_\cdot : \pi_t(dq) = \rho(t,q)dq \text{ and } \rho_t(q) \text{ is solution to the PDE}) = 1.$$

Let us focus on last item.

The notion of weak solution:

Let $g:[0,1]\to[0,1]$ be a measurable function. We say that $\rho:[0,T]\times[0,1]\to[0,1]$ is a weak solution of the HEDBC if:

♣ $\rho\in L^2(0,T;\mathcal{H}^1)$;

♣ ρ satisfies the weak formulation:

$$\begin{aligned}&\int_0^1 \rho_t(q)H_t(q)-g(q)H_0(q)\,dq\\&-\int_0^t\int_0^1 \rho_s(q)\Big(\frac{1}{2}\partial_q^2+\partial_s\Big)H_s(q)\,ds\,dq\\&+\frac{1}{2}\int_0^t \beta\partial_q H_s(1)-\alpha\partial_q H_s(0)\,ds=0,\end{aligned}$$

for all $t\in[0,T]$ and any function $H\in C_0^{1,2}([0,T]\times[0,1])$.

Other notion of solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the HEDBC if:

- ♣ $\rho \in L^2(0,T;\mathcal{H}^1)$;
- ♣ ρ satisfies the weak formulation:

$$\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq \\ - \int_0^t \int_0^1 \rho_s(q) \Big(\frac{1}{2} \partial_q^2 + \partial_s \Big) H_s(q)\, ds\, dq = 0,$$

  for all $t \in [0,T]$ and any function $H \in C_c^{1,2}([0,T] \times [0,1])$;
- ♣ $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$, for $t \in (0,T]$.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Robin b.c. if:

♣ $\rho \in L^2(0,T;\mathcal{H}^1)$,

♣ ρ satisfies the weak formulation:

$$\begin{aligned}
&\int_0^1 \rho_t(q)H_t(q)dq - \int_0^1 g(q)H_0(q)\,dq \\
&\qquad - \int_0^t \int_0^1 \rho_s(q)\Big(\frac{1}{2}\partial_q^2 + \partial_s\Big)H_s(q)\,ds\,dq \\
&\qquad + \frac{1}{2}\int_0^t \{\rho_s(1)\partial_q H_s(1) - \rho_s(0)\partial_q H_s(0)\}\,ds \\
&\qquad - \frac{\kappa}{2}\int_0^t \{H_s(0)(\alpha - \rho_s(0)) + H_s(1)(\beta - \rho_s(1))\}\,ds = 0,
\end{aligned}$$

for all $t \in [0,T]$ and any function $H \in C^{1,2}([0,T] \times [0,1])$.

Characterizing limit points:

Dynkin's formula: Let $\{\eta_t\}_{t\geq 0}$ be a Markov process with generator $\mathcal{L}$ and with countable state space E. Let $F : \mathbb{R}^+ \times E \to \mathbb{R}$ be a bounded function such that

- $\forall \eta \in E, F(\cdot, \eta) \in C^2(\mathbb{R}^+)$,
- there exists a finite constant C, such that $\sup_{(s,\eta)} |\partial_s^j F(s,\eta)| \leq C$, for $j = 1, 2$.

For $t \geq 0$, let

$$M_t^F = F(t, \eta_t) - F(0, \eta_0) - \int_0^t (\partial_s + \mathcal{L}) F(s, \eta_s) ds.$$

Then, $\{M_t^F\}_{t\geq 0}$ is a martingale wrt $\mathcal{F}_s = \sigma(\eta_s; s \leq t)$.

Characterizing limit points:

Let us fix a test function $H : [0,1] \to \mathbb{R}$ and apply Dynkin's formula with

$$F(t, \eta_t) = \langle \pi_t^N, H \rangle = \frac{1}{N-1} \sum_{x=1}^{N-1} \eta_{tN^2}(x) H\Big(\frac{x}{N}\Big).$$

Note that F does not depend on time only through the process $\eta_\cdot$. A simple computation shows that

$$\begin{aligned} N^2 \mathcal{L}_N \langle \pi_s^N, H \rangle = \langle \pi_s^N, \frac{1}{2} \Delta_N H \rangle \\ + \frac{1}{2} \nabla_N^+ H(0) \eta_{sN^2}(1) - \frac{1}{2} \nabla_N^- H(1) \eta_{sN^2}(N-1) \\ + \kappa N^{1-\theta} H\Big(\frac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) \\ + \kappa N^{1-\theta} H\Big(\frac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) \end{aligned}$$

$\theta \in [0,1)$:

Take a function $H : [0,1] \to \mathbb{R}$ such that $H(0) = H(1) = 0$ and then we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)ds + O(N^{-\theta}).$$

If we can replace $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β (this will be made rigorous ahead but only works for $\theta < 1$!) then above we have

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

Compare with the PDE (note that H does not depend on time).

Still $\theta \in [0,1)$:

Take the expectation above to get

$$\frac{1}{N}\sum_{x=1}^{N-1} H\Big(\frac{x}{N}\Big)\Big(\rho_t^N(x)-\rho_0^N(x)\Big) - \int_0^t \frac{1}{N}\sum_{x=1}^{N-1}\frac{1}{2}\Delta_N H\Big(\frac{x}{N}\Big)\rho_s^N(x)ds$$
$$-\frac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}) = 0.$$

Assume that $\rho_t^N(x) \sim \rho_t(x/N)$ and take the limit in N to get

$$\int_0^1 \rho_t(q)H(q) - \rho_0(q)H(q)dq - \int_0^t \int_0^1 \frac{1}{2}\partial_q^2 H(q)\rho_s(q)dqds$$
$$-\frac{1}{2}\int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0$$

Compare with the PDE (note that H does not depend on time).

$\theta < 0$:

Recall that the previous error blows up when $N \to \infty$. So now, we take a function $H : [0,1] \to \mathbb{R}$ with compact support and then we get

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H \rangle ds.$$

Again compare with the PDE but note that H does not depend on time.
In this case we do not see the Dirichlet boundary conditions and we need extra results to conclude.

$\theta = 1$:

Now, we take a function $H : [0,1] \to \mathbb{R}$ and we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1) ds$$
$$- \frac{\kappa}{2}\int_0^t H\Big(\tfrac{1}{N}\Big)(\alpha - \eta_{sN^2}(1)) + H\Big(\tfrac{N-1}{N}\Big)(\beta - \eta_{sN^2}(N-1)) ds.$$

If we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by its average in a box around 1 (resp. $N-1$) (this works for any $\theta \geq 1$):

$$\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) := \frac{1}{\epsilon N}\sum_{x=1}^{1+\epsilon N} \eta_{sN^2}(x), \quad \overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) := \frac{1}{\epsilon N}\sum_{x=N-1}^{N-1-\epsilon N} \eta_{sN^2}(x)$$

and noting that $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

$\theta > 1$:

Again we take a function $H : [0,1] \to \mathbb{R}$ and in this case the terms from the boundary vanish. So we get

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \frac{1}{2}\Delta_N H\rangle ds$$
$$- \frac{1}{2}\int_0^t \nabla_N^+ H(0)\eta_{sN^2}(1) - \nabla_N^- H(1)\eta_{sN^2}(N-1)ds + O(N^{1-\theta}).$$

As above, if we can replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by its average in a box around 1 (resp. $N-1$) and noting that $\overrightarrow{\eta}^{\epsilon N}_{sn^2}(1) \sim \rho_s(0)$ (resp. $\overrightarrow{\eta}^{\epsilon N}_{sN^2}(N-1) \sim \rho_s(1)$) we would get the terms in the PDE (compare).

Keystone ingredients: replacement lemmas

Recall that we need to prove that

For any $t > 0$, we have that:

- for $\theta < 1$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\eta_{sN^2}(1) - \alpha)\, ds\Big|\right] = 0;$$

- for $\theta \geq 1$

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\Big|\int_0^t (\eta_{sN^2}(1) - \overrightarrow{\eta}^{\epsilon N}_{sN^2}(1))\, ds\Big|\right] = 0;$$

and the similar result for $N-1$.

Replacing by α:

From entropy's and Jensen's inequality, the expectation is bounded from above by

$$\frac{H(\mu_N|\nu^N_{h(\cdot)})}{BN} + \frac{1}{BN}\log \mathbb{E}_{\nu^N_{h(\cdot)}}\left[e^{BN|\int_0^t(\eta_{sN^2}(1)-\alpha)ds|}\right].$$

Above B is a positive constant and $h(\cdot)$ is a profile to choose later on. We remove the absolute value inside the exponential since $e^{|x|} \leq e^x + e^{-x}$ and

$$\limsup_{N\to\infty} \log(a_N + b_N) \leq \max\left\{\limsup_{N\to\infty}\log(a_N), \limsup_{N\to\infty}\log(b_N)\right\}.$$

Note that if $\alpha \leq h(\cdot) \leq \beta$, then:

$$H(\mu_N|\nu^N_{h(\cdot)}) \leq NC(\alpha,\beta).$$

Apply FK formula:

Feynman-Kac's formula: Let $\mathcal{L}$ be the generator of a Markov process $\{\eta_t\}_{t\geq 0}$ on a countable state space E. Let ν be a p.m. on E and $V : [0,\infty) \times E \to \mathbb{R}$ bounded. Let

$$\Gamma_t = \sup_{\{f:||f||_2=1\}} \{\langle V_t, f^2\rangle_\nu + \langle \mathcal{L}f, f\rangle_\nu\}.$$

Then $\mathbb{E}_\nu\left[e^{\int_0^t V(r,\eta_r)dr}\right] \leq e^{\{\int_0^t \Gamma_s ds\}}$.

Then we have to estimate:

$$\sup_f \left\{ \langle \eta(1) - \alpha, f\rangle_{\nu^N_{h(\cdot)}} + \frac{N}{B}\langle L_N\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{h(\cdot)}} \right\},$$

where the supremum is over densities f with respect to $\nu^N_{h(\cdot)}$.

Controlling Dirichlet forms:

For a probability measure μ on Ω_N, we define

$$D_N(\sqrt{f},\mu) \quad := \quad (D_{N,0}+D_{N,b})(\sqrt{f},\mu)$$

where $D_{N,0}(\sqrt{f},\mu) := \dfrac{1}{2}\sum_{x=1}^{N-2} I_{x,x+1}(\sqrt{f},\mu)$, with
$I_{x,x+1}(\sqrt{f},\mu) = \int \left(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)}\right)^2 d\mu$ and

$$D_{N,b}(\sqrt{f},\mu) : = \frac{\kappa}{2N^\theta}\Big(I_1^{r_1}(\sqrt{f},\mu) + I_{N-1}^{r_{N-1}}(\sqrt{f},\mu)\Big)$$

with $I_x^{r_x}(\sqrt{f},\mu) := \int c_{r_x}(\eta(x))\left(\sqrt{f(\eta^x)} - \sqrt{f(\eta)}\right)^2 d\mu.$

We claim that for any positive constant B if $h(\cdot)$ is a Lipschitz function with $h(0) = \alpha$, $h(1) = \beta$ and locally constant at 0 and 1, then, there exists a constant $C_{\alpha,\beta,h} > 0$ such that

$$\frac{N}{B}\langle L_N\sqrt{f}, \sqrt{f}\rangle_{\nu^N_{h(\cdot)}} \le -\frac{N}{4B} D_N(\sqrt{f}, \nu^N_{h(\cdot)}) + \frac{C_{\alpha,\beta,h}}{B}.$$

Let $T : \eta \in \Omega_N \to T(\eta) \in \Omega_N$ be a transformation and $c : \eta \to c(\eta)$ a positive local function. Let f be a density with respect to a p.m. μ on Ω_N. Then:

$$\left\langle c(\eta)[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}], \sqrt{f(\eta)} \right\rangle_\mu$$

$$\le -\frac{1}{4}\int c(\eta)\left(\left[\sqrt{f(T(\eta))}\right] - \left[\sqrt{f(\eta)}\right]\right)^2 d\mu$$

$$+ \frac{1}{16}\int \frac{1}{c(\eta)}\left[c(\eta) - c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2 \left(\left[\sqrt{f(T(\eta))}\right] + \left[\sqrt{f(\eta)}\right]\right)^2 d\mu.$$

So far we have to bound

$$\sup_f \Big\{ \langle \eta(1) - \alpha, f \rangle_{\nu^N_{h(\cdot)}} - \frac{N}{4B} D_N(\sqrt{f}, \nu^N_{h(\cdot)}) + \frac{C_{\alpha,\beta,h}}{B} \Big\},$$

where the supremum is carried over densities f with respect to $\nu^N_{h(\cdot)}$. To finish we use

For any density f with respect to $\nu^N_{h(\cdot)}$ and any positive constant A, it holds

$$\Big| \langle \eta(1) - \alpha, f \rangle_{\nu^N_{h(\cdot)}} \Big| \lesssim \frac{1}{A} I_1^{r_1}(\sqrt{f}, \nu^N_{h(\cdot)}) + A + [h(\tfrac{1}{N}) - \alpha].$$

The same result holds if α is replaced by β and $\eta(1)$ with $\eta(N-1)$.

Now take $A = BCN^{\theta-1}\kappa^{-1}$, which is the final error and note that it vanishes, as $N \to \infty$, if $\theta < 1$.

The empirical profile:

Fix an initial measure μ_N in Ω_N. For $x \in \Lambda_N$ and $t \geq 0$, let

$$\rho_t^N(x) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^N(0) \;=\; \alpha \text{ and } \rho_t^N(N) \;=\; \beta\,, \text{ for all } t \geq 0\,.$$

A simple computation shows that $\rho_t^N(\cdot)$ is a solution of

$$\partial_t \rho_t^N(x) \;=\; N^2(\mathcal{B}_N \rho_t^N)(x)\,, \quad x \in \Lambda_N\,, \quad t \geq 0$$

where the operator $\mathcal{B}_N$ acts on functions $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$\begin{aligned}
&N^2(\mathcal{B}_N f)(x) \;=\; \Delta_N f(x), \quad \text{for } x \in \{2, \cdots, N-2\},\\
&N^2(\mathcal{B}_N f)(1) = N^2(f(2) - f(1)) + \tfrac{\kappa N^2}{N^\theta}(f(0) - f(1)),\\
&N^2(\mathcal{B}_N f)(N{-}1) = N^2(f(N{-}2) - f(N{-}1)) + \tfrac{\kappa N^2}{N^\theta}(f(N) - f(N{-}1)).
\end{aligned}$$

Stationary empirical profile:

The stationary solution of the previous equation is given by

$$\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$$

where $a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta+\kappa(N-2)}$ and $b_N = a_N(\frac{N^\theta}{\kappa} - 1) + \alpha$, so that

$$\lim_{N\to\infty} \max_{x\in\Lambda_N} \left|\rho_{ss}^N(x) - \bar{\rho}(\tfrac{x}{N})\right| = 0$$

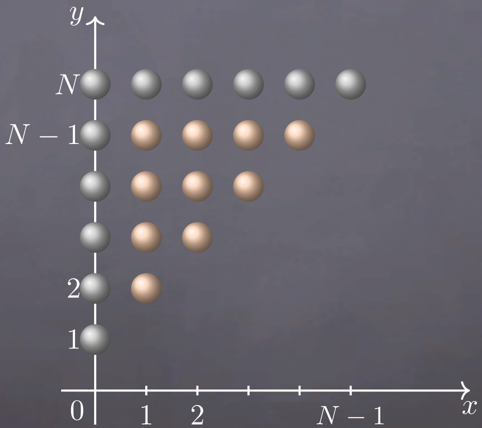
where

$$\bar{\rho}(q) = \begin{cases} (\beta-\alpha)q + \alpha\,;\, \theta < 1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q + \alpha + \frac{\beta-\alpha}{2+\kappa}\,;\, \theta = 1, \\ \frac{\beta+\alpha}{2}\,;\, \theta > 1, \end{cases}$$

is a stationary solution of the hydrodynamic equation.

Stationary correlations:

Let $V_N = \{(x, y) \in \{0, \cdots, N\}^2 : 0 < x < y < N\}$, and its boundary $\partial V_N = \{(x, y) \in \{0, \cdots, N\}^2 : x = 0 \text{ or } y = N\}$.



Stationary correlations:

For $x < y \in V_N$, let $\varphi_t^N(x,y)$ the two point correlation function between the occupation sites at $x < y \in V_N$ is defined by

$$\varphi_t^N(x,y) = \mathbb{E}_{\mu_N}[(\eta_{tN^2}(x) - \rho_t^N(x))(\eta_{tN^2}(y) - \rho_t^N(y))].$$

Doing some simple, but long, computations we see that φ_t^N is a solution of

$$\begin{cases} \partial_s \varphi_s(x,y) = \Delta_V^N \varphi_s(x,y) + g_s^N(x,y) + f_s^N(x,y)\,, & (x,y) \in V_N, \\ \varphi_s(x,y) = 0\,, & (x,y) \in \partial V_N, \end{cases}$$

where the discrete laplacian $\Delta_{V_N}^N : V_N \cup \partial V_N \to \mathbb{R}$ is defined by

$$\begin{cases} (\Delta_V^N f)(x,y) \;=\; N^2(f(x+1,y) + f(x-1,y) + f(x,y-1) \\ \qquad\qquad\qquad +f(x,y+1) - 4f(x,y))\,, \quad \text{for } |x-y| > 1\,, \\ (\Delta_V^N f)(x,x+1) = N^2(f(x-1,x+1) + f(x,x+2) - 2f(x,x+1)) \\ (\Delta_V^N f)(x,y) = 0, \quad \text{if } (x,y) \in \partial V_N\,. \end{cases}$$

Stationary correlations:

Above

$$g_t^N(x,y) = -(\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1},$$
$$\nabla_N^+ \rho_t^N(x) = N(\rho_t^N(x+1) - \rho_t^N(x))$$
$$f_s^N(x,y) = \Big(N^2 - \frac{N^2}{N^\theta}\Big)\varphi_t^N(x,y)\delta_{\{|y-x|=1,\ x=1 \text{ or } y=N-1\}}.$$

From simple, but long, computations we conclude that

$$\varphi_{ss}^N(x,y) = -\frac{(\alpha-\beta)^2(x+N^\theta-1)(N-y+N^\theta-1)}{(2N^\theta+N-2)^2(2N^\theta+N-3)}. \tag{2}$$

from where it follows that

$$\max_{x<y} |\varphi_{ss}^N(x,y)| = \begin{cases} O\Big(\frac{N^\theta}{N^2}\Big),\ \theta<1, \\ O\Big(\frac{1}{N}\Big),\ \theta=1, \\ O\Big(\frac{1}{N^\theta}\Big),\ \theta>1, \end{cases} \rightarrow_{N\to\infty} 0. \tag{3}$$

Hydrostatics:

If μ_{ss} is the stationary measure for $\{\eta_t : t \geq 0\}$ then we just need to show that it is associated to the stationary profile $\bar{\rho} : [0,1] \to [0,1]$. That is, for any G continuous

$$\mu_{ss}\left(\eta \in \Omega_N : \left| \frac{1}{N-1} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) \eta(x) - \int_0^1 G(q)\rho_0(q)dq \right| > \delta \right) \to 0.$$

By triangular and Markov's inequalities, we bound the previous probability from above by $1/\delta$ times

$$\mathbb{E}_{\mu_{ss}}\Big[\Big| \frac{1}{N-1} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) \Big(\eta(x) - \rho^N_{ss}(x)\Big) \Big|\Big]$$

$$+ \Big| \frac{1}{N-1} \sum_{x \in \Lambda_N} G\left(\tfrac{x}{N}\right) \rho^N_{ss}(x) - \int_0^1 G(q)\bar{\rho}(q)dq \Big|$$

The last term can be bounded from above by

$$\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\left(\rho_{ss}^N(x)-\bar{\rho}\Big(\tfrac{x}{N}\Big)\right)\Big|$$

$$+\Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\bar{\rho}\Big(\tfrac{x}{N}\Big)-\int_0^1 G(q)\bar{\rho}(q)dq\Big|.$$

The 1st term in last expression is bounded by

$$\frac{1}{N-1}\sum_{x\in\Lambda_N}\Big|G\left(\tfrac{x}{N}\right)\Big|\Big|\rho_{ss}^N(x)-\bar{\rho}\Big(\tfrac{x}{N}\Big)\Big|\leq\|G\|_\infty\max_{x\in\Lambda_N}\Big|\rho_{ss}^N(x)-\bar{\rho}\Big(\tfrac{x}{N}\Big)\Big|$$

Applying Cauchy-Schwarz, the remaining term is bounded by

$$\Big(\Big|\frac{1}{N^2}\sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right)\mathbb{E}_{\mu_{ss}}\Big[(\eta(x)-\rho_{ss}^N(x))^2\Big]$$

$$+\frac{2}{N}\sum_{x<y} G\left(\tfrac{x}{N}\right)G\left(\tfrac{y}{N}\right)\varphi_{ss}^N(x,y)\Big)^{\frac{1}{2}}$$

$$\leq\Big(\frac{C\|G\|_\infty}{N}+2\|G\|_\infty\max_{x<y}\varphi_{ss}^N(x,y)\Big)^{\frac{1}{2}}.$$

All the terms vanish as $N\to\infty$ from our previous bounds.