Scaling limits for symmetric exclusion with open boundary

Patrícia Gonçalves

YEP - Eindhoven
August 2021
Lecture 2: Long jumps
Exclusion in contact with infinitely many reservoirs
The finite variance case
If jumps are arbitrarily big?

Let $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \begin{cases} \frac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where $c_\gamma$ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and take (by now) $\gamma > 2$ so that its variance is finite

$$\sigma^2_\gamma = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$
The infinitesimal generator:

\[ \mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,r} + \mathcal{L}_{N,\ell} \] where

\[
(\mathcal{L}_{N,0} f)(\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x - y)[f(\eta^{x,y}) - f(\eta)],
\]

\[
(\mathcal{L}_{N,\ell} f)(\eta) = \frac{\kappa}{N\theta} \sum_{\substack{x \in \Lambda_N \\ y \leq 0}} p(x - y)c_x(\eta; \alpha)[f(\eta^x) - f(\eta)],
\]

\[
(\mathcal{L}_{N,r} f)(\eta) = \frac{\kappa}{N\theta} \sum_{\substack{x \in \Lambda_N \\ y \geq N}} p(x - y)c_x(\eta; \beta)[f(\eta^x) - f(\eta)]
\]

where

\[ c_x(\eta; \alpha) := (1 - \eta_x)\alpha + (1 - \alpha)\eta_x. \]

\[ c_x(\eta; \beta) := (1 - \eta_x)\beta + (1 - \beta)\eta_x. \]
Hydrodynamic Limit:

Theorem: Let $g : [0, 1] \to [0, 1]$ be a measurable function and let $\{\mu_N\}_{N \geq 1}$ be a sequence of probability measures in $\Omega_N$ associated with $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\mathbb{P}_{\mu_N}\left(\left| \frac{1}{N-1} \sum_{x \in \Lambda_N} H\left(\frac{x}{N}\right) \eta_{t\Theta(N)}(x) - \int_0^1 H(q) \rho_t(q) dq \right| > \delta \right) \to 0,$$

where the time scale is diffusive if $\theta \geq 2 - \gamma$ and $N^{\gamma+\theta}$ otherwise and $\rho_t(\cdot)$ is the UNIQUE weak solution of the corresponding hydrodynamic equation with initial condition $g(\cdot)$. 
Heat equation:
\[ \partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial^2_q \rho_t(q) \]

\[ \theta = 1 \text{ Robin b.c.:} \]
\[ \partial_q \rho_t(0) = \frac{2 m \kappa}{\sigma^2} (\rho_t(0) - \alpha), \]
\[ \partial_q \rho_t(1) = \frac{2 m \kappa}{\sigma^2} (\beta - \rho_t(1)), \]

Reaction-diffusion eq.:
\[ \partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial^2_q \rho_t(q) \]
\[ + \kappa (V_0(q) - V_1(q) \rho_t(q)) \]

Reaction equation:
\[ \partial_t \rho_t(q) = \kappa (V_0(q) - V_1(q) \rho_t(q)) \]

Above
\[ V_1(q) = \frac{c_{\gamma}}{\gamma} \left( \frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma} \right) \]
\[ V_0(q) = \frac{c_{\gamma}}{\gamma} \left( \frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma} \right). \]
Stationary solutions:

\[
\begin{align*}
\theta &> 1 \\
\theta &\geq 1 \\
2 - \gamma &< \theta < 1 \\
\theta &\geq 2 - \gamma \\
\theta &< 2 - \gamma
\end{align*}
\]
A simple computation shows that

\[
\Theta(N) \mathcal{L}_N(\langle \pi_s^N, H \rangle) = \frac{\Theta(N)}{N} \sum_{x,y \in \Lambda_N} p(y - x) \left[ H \left( \frac{y}{N} \right) - H \left( \frac{x}{N} \right) \right] \eta_s(x) \\
+ \frac{\kappa \Theta(N)}{N^{1+\theta}} \sum_{x \in \Lambda_N} (Hr^-_N)(\frac{x}{N})(\alpha - \eta_s(x)) \\
+ \frac{\kappa \Theta(N)}{N^{1+\theta}} \sum_{x \in \Lambda_N} (Hr^+_N)(\frac{x}{N})(\beta - \eta_s(x)),
\]

where for all \( x \in \Lambda_N \)

\[
r^-_N\left( \frac{x}{N} \right) = \sum_{y \geq x} p(y), \quad r^+_N\left( \frac{x}{N} \right) = \sum_{y \leq x-N} p(y).
\]

Extend \( H \) to \( \mathbb{R} \) in such a way that it remains two times continuously differentiable, and the first term at the RHS is
Let $H \in C^2_c(\mathbb{R})$, we have

$$\limsup_{N \to \infty} \sup_{x \in \Lambda_N} \left| N^2 K_N H\left(\frac{x}{N}\right) - \frac{\sigma^2}{2} \Delta H\left(\frac{x}{N}\right) \right| = 0.$$ 

For $\Theta(N) = N^{\theta+\gamma}$ and $\theta < 2 - \gamma$ the first term above vanishes as $N \to \infty$. 
The infinite variance case
What about $\gamma \in (1, 2)$?

For any $\kappa > 0$ and $\theta = 0$, we get a collection of fractional reaction-diffusion equations with Dirichlet boundary conditions given by

$$\partial_t \rho_t(q) = \mathbb{L}_\kappa \rho_t(q) + \kappa V_0(q).$$

The operator $\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1$, where $\mathbb{L}$ is the regional fractional laplacian and

$$V_1(q) = \frac{c_\gamma}{\gamma} \left( \frac{1}{q^\gamma} + \frac{1}{(1 - q)^\gamma} \right)$$

and

$$V_0(q) = \frac{c_\gamma}{\gamma} \left( \frac{\alpha}{q^\gamma} + \frac{\beta}{(1 - q)^\gamma} \right).$$
The operator $\mathbb{L}$:

Let $(-\Delta)^{\gamma/2}$ be the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1 + |q|)^{1+\gamma}} du < \infty$$

by (provided the limit exists)

$$(-\Delta)^{\gamma/2}H(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} 1_{|u-q| \geq \varepsilon} \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$  

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0, 1]$, whose action on functions $H \in C_c^\infty(0, 1)$ is given by

$$(\mathbb{L}H)(q) = -(-\Delta)^{\gamma/2}H(q) + V_1(q)H(q)$$

$$= c_\gamma \lim_{\varepsilon \to 0} \int_{0}^{1} 1_{|u-q| \geq \varepsilon} \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} dy, \quad q \in (0, 1).$$
Fractional Sobolev space:

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $g : (0, 1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$, with

$$\|g\|_{\gamma/2} := \langle g, g \rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint_{[0,1]^2} \frac{(g(u) - g(q))^2}{|u - q|^{1+\gamma}} \, du \, dq.$$

The space $L^2(0, T; \mathcal{H}^{\gamma/2})$ is the set of measurable functions $f : [0, T] \to \mathcal{H}^{\gamma/2}$ such that $\int_0^T \|f_t\|_{\mathcal{H}^{\gamma/2}}^2 \, dt < \infty$ where $\|f_t\|_{\mathcal{H}^{\gamma/2}}^2 := \|f_t\|^2 + \|f_t\|_{\gamma/2}^2$. 
The notion of weak solution:

Let $g : [0, 1] \to [0, 1]$ be a measurable function. We say that $ho : [0, T] \times [0, 1] \to [0, 1]$ is a weak solution of the PDE above if:

1. $\rho \in L^2(0, T; \mathcal{H}^{\gamma/2})$ and
   $$\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t(q))^2}{q^\gamma} + \frac{(\beta - \rho_t(q))^2}{(1-q)^\gamma} \right\} dq \, dt < \infty,$$

2. For all $t \in [0, T]$ and any function $H \in C_c^{1,\infty}([0, T] \times (0, 1))$:

$$\int_0^1 \rho_t(q) H_t(q) \, dq - \int_0^1 g(q) H_0(q) \, dq$$

$$- \int_0^t \int_0^1 \rho_s(q) \left( \partial_s + \mathbb{L}_\kappa \right) H_s(q) \, dq \, ds$$

$$- \kappa \int_0^t \int_0^1 V_0(q) H_s(q) \, dq \, ds = 0,$$
Characterizing limit points:

\[ N^\gamma \mathcal{L}_N(\langle \pi_s^N, H \rangle) = \frac{N^\gamma}{N} \sum_{x,y \in \Lambda_N} p(y - x) \left[ H\left(\frac{y}{N}\right) - H\left(\frac{x}{N}\right) \right] \eta_s(x) \]
\[ + \frac{\kappa N^\gamma}{N} \sum_{x \in \Lambda_N} (H r_N^{-})(\frac{x}{N}) (\alpha - \eta_s(x)) + \frac{\kappa N^\gamma}{N} \sum_{x \in \Lambda_N} (H r_N^{+})(\frac{x}{N}) (\beta - \eta_s(x)). \]

For \( H \) with compact support in \([a, 1 - a]\) for \( a \in (0, 1) \) we have

\[ \lim_{N \to \infty} \left| N^\gamma \sum_{y \in \Lambda_N} p(y - x) \left[ H\left(\frac{y}{N}\right) - H\left(\frac{x}{N}\right) \right] - (\mathcal{L}H)(\frac{x}{N}) \right| = 0, \]
\[ \lim_{N \to \infty} \left| N^\gamma (r_N^{-})(\frac{x}{N}) - r^{-}(\frac{x}{N}) \right| = 0, \]
\[ \lim_{N \to \infty} \left| N^\gamma (r_N^{+})(\frac{x}{N}) - r^{+}(\frac{x}{N}) \right| = 0 \]

uniformly in \([a, 1 - a]\).
Characterizing limit points:

Thus, the first term on the right hand side above can be replaced by

$$\langle \pi_t^N, \mathbb{L}H \rangle \rightarrow \int_0^1 (\mathbb{L}H)(q) \rho_t(q) dq,$$

as $N$ goes to $\infty$.

The other terms can be replaced by

$$\kappa \langle \alpha - \pi_t^N, Hr^- \rangle + \kappa \langle \beta - \pi_t^N, Hr^+ \rangle$$

which converges to

$$\kappa \int_0^1 H(q)r^-(q)(\alpha - \rho_t(q)) dq + \kappa \int_0^1 H(q)r^+(q)(\beta - \rho_t(q)) dq$$

$$= \kappa \int_0^1 H(q)V_0(q) dq - \kappa \int_0^1 H(q)V_1(q) \rho_t(q) dq,$$

as $N$ goes to $\infty$. 
Uniqueness of weak solution:

To prove it we do the following. Let $\bar{\rho} = \rho^1 - \rho^2$, where $\rho^1$ and $\rho^2$ are two weak solutions starting from $g$. We have $\bar{\rho}_t(0) = \bar{\rho}_t(1) = 0$. Then,

$$\langle \bar{\rho}_t, H_t \rangle - \int_0^t \langle \bar{\rho}_s, (\partial_s + \mathbb{L}) H_s \rangle ds + \kappa \int_0^t \langle V_1 H_s, \bar{\rho}_s \rangle ds = 0.$$ 

Take now $H_N(s, q) = \int_s^t G_N(r, q) \, dr$ where $(G_N)_{N \geq 0}$ is a sequence of functions in $C^{1,\infty}_c([0, T] \times (0, 1))$ converging to $\bar{\rho}$. Plug $H_N$ in the equation and take $N \rightarrow \infty$ to get

$$\int_0^t \int_0^1 \bar{\rho}_s^2(q) \, dq \, ds + \frac{1}{2} \left\| \int_0^t \bar{\rho}_s \, ds \right\|_{\gamma/2}^2 + \frac{\kappa}{2} \left\| \int_0^t \bar{\rho}_s \, ds \right\|_{V_1}^2 = 0.$$ 

From this we conclude the uniqueness.
Open problems:

- Heat eq. & Neumann b.c.
- Heat eq. & Robin b.c.
- Heat eq. & Neumann b.c.
- Reaction eq. & Dirichlet b.c.

\[ \theta = 2 - \gamma \]
Conjecture:

For $\theta > 0$ small and $\gamma \in (1, 2)$ the solution should correspond to the solution when $\kappa = 0$. Supported by the result:

Let $g : [0, 1] \to [0, 1]$ be a measurable function and let $\rho^\kappa$ be the weak solution of

$$\partial_t \rho_t(q) = \mathbb{I}_\kappa \rho_t(q) + \kappa V_0(q),$$

with Dirichlet boundary conditions and initial condition $g(\cdot)$. Then $\rho^\kappa$ converges strongly to $\rho^0$ in $L^2(0, T; H^{\gamma/2})$ as $\kappa$ goes to 0, where $\rho^0$ is the weak solution of the equation with $\kappa = 0$ and initial condition $g(\cdot)$. 
Solved problem:

\[ \gamma = 2 \]

\[ \gamma = 1 \]

\[ \gamma = 0 \]

\[ \theta = 2 - \gamma \]
Stationary solutions:
Lecture 3: Fluctuations
The space of test functions: Let $S_{\theta}$ denote the set of functions $H \in C^\infty([0, 1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

1. for $\theta < 1$: $\partial^2_k H(0) = \partial^2_k H(1) = 0$;
2. for $\theta = 1$: $\partial^{2k+1} H(0) = \partial^2_k H(0)$ and $\partial^{2k+1} H(1) = -\partial^2_k H(1)$;
3. for $\theta > 1$: $\partial^{2k+1} H(0) = \partial^{2k+1} H(1) = 0$.

The density fluctuation field $\mathcal{Y}^N_t$ is the time-trajectory of linear functionals acting on functions $H \in S_{\theta}$ as

$$\mathcal{Y}^N_t(H) = \frac{1}{\sqrt{N-1}} \sum_{x=1}^{N-1} H\left(\frac{x}{N}\right) \left(\eta_{tN^2}(x) - \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\right).$$
**Operators:**

For $\theta \geq 0$, let $-\Delta_\theta$ be the positive self-adjoint operator on $L^2[0, 1]$, defined on $H \in \mathcal{S}_\theta$ by

$$\Delta_\theta H(u) = \begin{cases} \partial^2_u H(u), & \text{if } u \in (0, 1), \\ \partial^2_u H(0^+), & \text{if } u = 0, \\ \partial^2_u H(1^-), & \text{if } u = 1. \end{cases}$$

Let $\nabla_\theta : \mathcal{S}_\theta \to C^\infty([0, 1])$ be the operator given by

$$\nabla_\theta H(u) = \begin{cases} \partial_u H(u), & \text{if } u \in (0, 1), \\ \partial_u H(0^+), & \text{if } u = 0, \\ \partial_u H(1^-), & \text{if } u = 1. \end{cases}$$

Let $T_t^\theta : \mathcal{S}_\theta \to \mathcal{S}_\theta$ be the semigroup associated to the PDE with the corresponding boundary conditions with $\alpha = \beta = 0$. 
Fluctuations: $\theta = 1$
For each $N \in \mathbb{N}$, the measure $\mu_N$ is associated to a measurable profile $\rho_0 : [0, 1] \rightarrow [0, 1]$ (This is the same condition for hydrodynamics!).

For $\rho_0^N (x) = \mathbb{E}_{\mu_N} [\eta_0(x)]$

$$\max_{x \in \Lambda_N} | \rho_0^N (x) - \rho_0 (\frac{x}{N}) | \lesssim \frac{1}{N}.$$ 

For

$$\varphi_0^N (x, y) = \mathbb{E}_{\mu_N} [\eta(x)\eta(y)] - \rho_0^N (x)\rho_0^N (y)$$

it holds that

$$\max_{1 \leq x < y \leq N-1} | \varphi_0^N (x, y) | \lesssim \frac{1}{N}.$$ 

Examples - initial measures:

If for a given measurable profile $\rho_0 : [0, 1] \to [0, 1]$, we take $\mu_N$ as the Bernoulli product measure given by

$$\mu_N\{\eta : \eta(x) = 1\} = \rho_0\left(\frac{x}{N}\right)$$

then all the conditions above are true.

If $\mu_{ss}$ is the stationary measure, then all the conditions above are true, by choosing the profile $\rho_0$ as the stationary profile $\bar{\rho}$ given above.
\theta = 1:

For each \( N \geq 1 \), let \( \mathcal{Q}_N \) be the probability measure on \( \mathcal{D}([0, T], S'_\theta) \) induced by \( Y^N \) and \( \mu_N \).

The sequence of measures \( \{\mathcal{Q}_N\}_{N \in \mathbb{N}} \) is tight on \( \mathcal{D}([0, T], S'_\theta) \) and all limit points \( \mathcal{Q} \) are p.m. concentrated on paths \( Y \) satisfying

\[
Y_t(H) = Y_0(T^1_t H) + \mathcal{W}_t(H),
\]

for any \( H \in S_\theta \). Above \( \mathcal{W}_t(H) \) is a mean zero Gaussian variable of variance \( \int_0^t \| \nabla_1 T^1_{t-r} H \|_{L^2,1(\rho_r)}^2 \, dr \), where \( \rho(t, u) \) is the solution of the hydrodynamic equation. Moreover,

\[
\mathbb{E}_\mathcal{Q} \left[ Y_0(H) \mathcal{W}_t(G) \right] = 0 \quad \text{for all} \quad H, G \in S_\theta.
\]
If \( \{Y_0^N\}_{N \in \mathbb{N}} \) converges, as \( N \to \infty \), to a mean-zero Gaussian field \( Y_0 \) with covariance given on \( H, G \in S_\theta \) by

\[
\mathbb{E} \left[ Y_0(H) Y_0(G) \right] := \sigma(H, G),
\]

then, the sequence \( \{Q_N\}_{N \in \mathbb{N}} \) converges, as \( N \to \infty \), to a generalized Ornstein-Uhlenbeck process, which is the formal solution of: \( \partial_t Y_t = \Delta_1 Y_t dt + \sqrt{2 \chi(\rho_t)} \nabla_1 W_t \), where \( W_t \) is a space-time white noise of unit variance. As a consequence, the covariance of the limit field \( Y_t \) is given on \( H, G \in S_\theta \) by

\[
\mathbb{E} \left[ Y_t(H) Y_s(G) \right] = \sigma(T^1_t H, T^1_s G)
\]

\[
+ \int_0^s \langle \nabla_1 T^1_{t-r} H, \nabla_1 T^1_{s-r} G \rangle_{L^2,1(\rho_r)} dr.
\]
Suppose to start the process from $\mu_{ss}$ with $\alpha \neq \beta$. Then, $\mathcal{Y}^N$ converges to the centered Gaussian field $\mathcal{Y}$ with covariance given on $H, G \in \mathcal{S}_\theta$ by:

$$
E_{\mu_{ss}}[\mathcal{Y}(H)\mathcal{Y}(G)] = \int_0^1 \chi(\overline{\rho}(u)) H(u) G(u) \, du
$$

$$
- \left( \frac{\beta - \alpha}{3} \right)^2 \int_0^1 \left[ (-\Delta_1)^{-1} H(u) \right] G(u) \, du
$$

where $\overline{\rho}(\cdot)$ is the stationary solution of the PDE.
**Associated martingales:**

Let \( H : [0, 1] \rightarrow \mathbb{R} \) be a test function and note that

\[
\mathcal{M}_t^N(H) := y_t^N(H) - y_0^N(H) - \int_0^t N^2 \mathcal{Q}_N y_s^N(H) \, ds
\]

is a martingale where

\[
N^2 \mathcal{Q}_N y_s^N(H) = \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H \left( \frac{x}{N} \right) \bar{\eta}_{sN^2}(x)
\]

\[
+ \sqrt{N} \left[ \nabla^+_N H(0) - H \left( \frac{1}{N} \right) \right] \bar{\eta}_{sN^2}(1)
\]

\[
+ \sqrt{N} \left[ H \left( \frac{N-1}{N} \right) + \nabla^-_N H(1) \right] \bar{\eta}_{sN^2}(N - 1).
\]

Note that the second term at the right hand side of the previous expression is \( y_s^N(\Delta_N H) \). Above, we have used the notation

\[
\nabla^+_N H(x) = N \left[ H \left( \frac{x+1}{N} \right) - H \left( \frac{x}{N} \right) \right], \quad \nabla^-_N H(x) = N \left[ H \left( \frac{x}{N} \right) - H \left( \frac{x-1}{N} \right) \right].
\]
The correlation estimate:

For each \( x, y \in V_N = \{(x, y); x, y \in \mathbb{N}, 0 < x < y < N\} \) and \( t \in [0, T] \), let

\[
\varphi_t^N(x, y) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)\eta_{tN^2}(y)] - \rho_t^N(x)\rho_t^N(y),
\]

and set \( \varphi_t^N(x, y) = 0 \), for \( x = 0 \) or \( y = N \), we set

Proposition:

If

\[
\max_{x, y \in V_N} |\varphi_0^N(x, y)| \lesssim \frac{1}{N},
\]

then

\[
\sup_{t \geq 0} \max_{(x, y) \in V_N} |\varphi_t^N(x, y)| \lesssim \frac{1}{N}.
\]
Fluctuations : \( \theta \neq 1 \)
\( \theta \neq 1: \)

\[
N^2 L_N \mathcal{Y}_s^N(H) = \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H \left( \frac{x}{N} \right) \left( \eta_s N^2(x) - \rho_s^N(x) \right) \\
+ \sqrt{N} \nabla_N^+ H(0) \bar{\eta}_s N^2(1) - \sqrt{N} \nabla_N^- H(1) \bar{\eta}_s N^2(N - 1) \\
- \frac{N^{3/2}}{N^{\theta}} H \left( \frac{1}{N} \right) \bar{\eta}_s N^2(1) - \frac{N^{3/2}}{N^{\theta}} H \left( \frac{N-1}{N} \right) \bar{\eta}_s N^2(N - 1).
\]

For \( x \in \{1, N - 1\} \) and \( t \in [0, T] \) it holds

\[
\mathbb{E}_{\mu_N} \left[ \left( \int_0^t C^\theta_N (\eta_s N^2(x) - \rho_s^N(x)) \, ds \right)^2 \right] \lesssim (C^\theta_N)^2 \frac{N^{\theta}}{N^2}.
\]

Apply last result with \( C^\theta_N = \sqrt{N} \mathbf{1}_{\{\theta < 1\}} + N^{3/2-\theta} \mathbf{1}_{\{\theta > 1\}} \).
The initial measures:

We fix an initial profile $\rho_0 : [0, 1] \to [0, 1]$ which is measurable and of class $C^6$, and we assume that

$$\max_{x \in \Lambda_N} \left| \rho_0^N(x) - \rho_0 \left( \frac{x}{N} \right) \right| \lesssim \frac{1}{N}.$$ 

Moreover, we also assume that

$$\max_{y \in \Lambda_N} |\varphi_0^N(x, y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, & \theta \leq 1, \\ \frac{1}{N}, & \theta \geq 1, \end{cases} \quad \text{for } x = 1, N - 1,$$

and that

$$\max_{(x, y) \in V_N} |\varphi_0^N(x, y)| \lesssim \frac{1}{N}.$$
The correlation estimate:

Proposition:

If

\[ \max_{y \in \Lambda_N} |\varphi_0^N(x, y)| \lessapprox \begin{cases} \frac{N^\theta}{N^2}, & \theta \leq 1, \\ \frac{1}{N}, & \theta \geq 1, \end{cases} \text{ for } x = 1, N - 1, \]

then,

\[ \max_{(x,y) \in V_N} |\varphi_0^N(x, y)| \lessapprox \frac{1}{N}, \]

\[ \sup_{t \geq 0} \max_{y \in \Lambda_N} |\varphi_t^N(x, y)| \lessapprox \begin{cases} \frac{N^\theta}{N^2}, & \theta \leq 1, \\ \frac{1}{N}, & \theta \geq 1, \end{cases} \text{ for } x = 1, N - 1, \]

\[ \sup_{t \geq 0} \max_{(x,y) \in V_N} |\varphi_t^N(x, y)| \lessapprox \frac{1}{N}. \]
Ingredients for correlations

Show that \( \varphi^N_t(x, y) \) is solution of

\[
\begin{align*}
    \partial_t \varphi^N_t(x, y) &= N^2 \mathcal{A}_N^\theta \varphi^N_t(x, y) - (\nabla^+_N \rho^N_t(x))^2 \delta_{y=x+1}, (x, y) \in V_N, \\
    \varphi^N_t(x, y) &= 0, (x, y) \in \partial V_N, \\
    \varphi^N_0(x, y) &= \mathbb{E}_{\mu_N}[\eta_0(x)\eta_0(y)] - \rho^N_0(x)\rho^N_0(y), (x, y) \in V_N \cup \partial V_N,
\end{align*}
\]

where \( \mathcal{A}_N^\theta \) acts on \( f : V_N \cup \partial V_N \to \mathbb{R} \) as

\[
(\mathcal{A}_N^\theta f)(u) = \sum_{v \in V_N} c^\theta_N(u, v) [f(v) - f(u)],
\]

and it is the infinitesimal generator of the RW in \( V_N \cup \partial V_N \) which is absorbed at \( \partial V_N \). Above,

\[
c^\theta_N(u, v) = \begin{cases} 
    1, & \text{if } \|u - v\| = 1 \text{ and } u, v \in V_N, \\
    N^{-\theta}, & \text{if } \|u - v\| = 1 \text{ and } u \in V_N, v \in \partial V_N, \\
    0, & \text{otherwise}.
\end{cases}
\]
Show that \( \rho_t^N (\cdot) \) is a solution of

\[
\left\{ \begin{array}{l}
\partial_t \rho_t^N (x) = (N^2 \mathfrak{B}_N^\theta \rho_t^N) (x) , \ x \in \Lambda_N , \ t \geq 0 , \\
\rho_t^N (0) = \alpha , \rho_t^N (N) = \beta , \ t \geq 0 ,
\end{array} \right.
\]

where \( \mathfrak{B}_N^\theta \) acts on \( f : \Lambda_N \cup \{0, N\} \to \mathbb{R} \) as

\[
(\mathfrak{B}_N^\theta f)(x) = \sum_{y=0}^{N} \xi_{x,y}^{N,\theta} (f(y) - f(x)), \quad \text{for} \ x \in \Lambda_N
\]

and it is the infinitesimal generator of the RW in \( \bar{\Lambda}_N \) which is absorbed at the points \( \{0, N\} \). Above

\[
\xi_{x,y}^{N,\theta} = \begin{cases} 
1 , & \text{if } |y - x| = 1 \text{ and } x, y \in \Lambda_N , \\
N^{-\theta} , & \text{if } x = 1, y = 0 \text{ and } x = N - 1 , y = N , \\
0 , & \text{otherwise}.
\end{cases}
\]
**Ingredients for correlations:**

The stationary solutions of the equations above are given by

\[
\varphi_{ss}^N(x, y) = -\frac{(\alpha - \beta)^2(x + N^\theta - 1)(N - y + N^\theta - 1)}{(2N^\theta + N - 2)^2(2N^\theta + N - 3)}
\]

and

\[
\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_t N^2(x)] = a_N x + b_N,
\]

where

\[
a_N = \frac{\beta - \alpha}{2N^\theta + (N - 2)} \quad \text{and} \quad b_N = a_N (N^\theta - 1) + \alpha.
\]

The time spent by the 1-d RW at the points \(x = 1\) and \(x = N - 1\) is of order \(O\left(\frac{N^\theta}{N^2}\right)\) (good bound when \(\theta < 1\) but not when \(\theta > 1\)). When \(\theta > 1\) we compare with the reflected RW and we prove that the time now is of order \(O\left(\frac{1}{N}\right)\). We need the same estimates in the 2-d setting for the time spent by the RW on the diagonal.
For the future:

- What about hydrostatics, for the long jumps case?
- Fluctuations?
- Other boundary conditions?
References:


Thank you and...
References:


Thank you and...