Scaling limits for symmetric exclusion with open boundary

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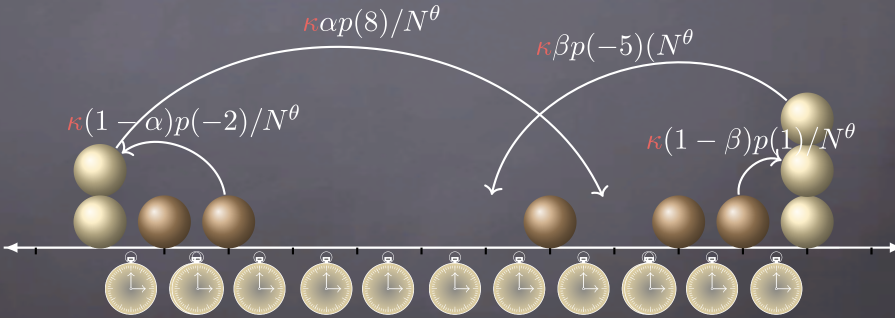
Lecture 2: Long jumps

Exclusion in contact with infinitely many reservoirs



The finite variance case

If jumps are arbitrarily big?

Let $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \begin{cases} \frac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where c_γ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and take (by now) $\gamma > 2$ so that its variance is finite

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$

The infinitesimal generator:

$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,r} + \mathcal{L}_{N,\ell}$ where

$$(\mathcal{L}_{N,0} f)(\eta) = \frac{1}{2} \sum_{x,y \in \Lambda_N} p(x-y)[f(\eta^{x,y}) - f(\eta)],$$

$$(\mathcal{L}_{N,\ell} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \le 0}} p(x-y) c_x(\eta;\alpha)[f(\eta^x) - f(\eta)],$$

$$(\mathcal{L}_{N,r} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{\substack{x \in \Lambda_N \\ y \ge N}} p(x-y) c_x(\eta;\beta)[f(\eta^x) - f(\eta)]$$

where

$$c_x(\eta;\alpha) := (1-\eta_x)\alpha + (1-\alpha)\eta_x.$$

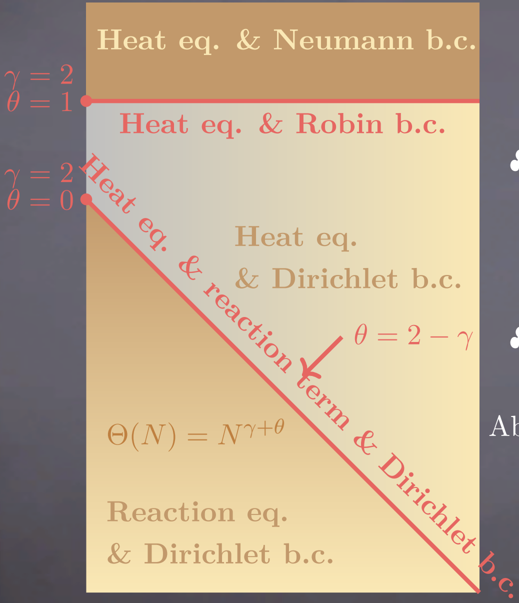
$$c_x(\eta;\beta) := (1-\eta_x)\beta + (1-\beta)\eta_x.$$

Hydrodynamic Limit:

Theorem: Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated with $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\mathbb{P}_{\mu_N}\left(\left|\frac{1}{N-1}\sum_{x\in\Lambda_N} H\left(\tfrac{x}{N}\right)\eta_{t\Theta(N)}(x) - \int_0^1 H(q)\rho_t(q)dq\right| > \delta\right) \to 0,$$

where the time scale is diffusive if $\theta \geq 2-\gamma$ and $N^{\gamma+\theta}$ otherwise and $\rho_t(\cdot)$ is the UNIQUE weak solution of the corresponding hydrodynamic equation with initial condition $g(\cdot)$.



- **Heat equation:**

$$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q)$$

  - $\theta = 1$ **Robin b.c.:**

$$\partial_q \rho_t(0) = \frac{2m\kappa}{\sigma^2}(\rho_t(0) - \alpha),$$
$$\partial_q \rho_t(1) = \frac{2m\kappa}{\sigma^2}(\beta - \rho_t(1)),$$

- **Reaction-diffusion eq.:**

$$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q) + \kappa (V_0(q) - V_1(q) \rho_t(q))$$

- **Reaction equation:**

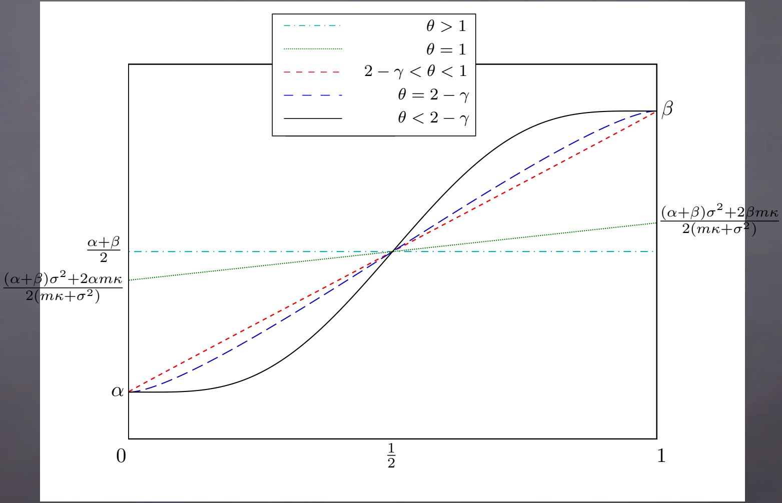
$$\partial_t \rho_t(q) = \kappa (V_0(q) - V_1(q) \rho_t(q))$$

Above

$$V_1(q) = \frac{c_\gamma}{\gamma} \Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma} \Big)$$

$$V_0(q) = \frac{c_\gamma}{\gamma} \Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma} \Big).$$

Stationary solutions:



Characterizing limit points:

A simple computation shows that

$$\Theta(N)\mathcal{L}_N(\langle \pi_s^N, H\rangle) = \frac{\Theta(N)}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)$$
$$+\frac{\kappa\Theta(N)}{N^{1+\theta}} \sum_{x\in\Lambda_N} (Hr_N^-)(\tfrac{x}{N})(\alpha - \eta_s(x))$$
$$+\frac{\kappa\Theta(N)}{N^{1+\theta}} \sum_{x\in\Lambda_N} (Hr_N^+)(\tfrac{x}{N})(\beta - \eta_s(x)),$$

where for all $x \in \Lambda_N$

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y).$$

Extend H to $\mathbb{R}$ in such a way that it remains two times continuously differentiable, and the first term at the RHS is

$$\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}(K_N H)(\tfrac{x}{N})\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\leq 0}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

$$-\frac{\Theta(N)}{N}\sum_{x\in\Lambda_N}\sum_{y\geq N}\left[H(\tfrac{y}{N})-H(\tfrac{x}{N})\right]p(x-y)\eta_s(x)$$

where $(K_N H)(\frac{x}{N}) = \sum_{y\in\mathbb{Z}} p(y-x)\left[H(\frac{y}{N})-H(\frac{x}{N})\right].$

Let $H \in C_c^2(\mathbb{R})$, we have

$$\limsup_{N\to\infty}\sup_{x\in\Lambda_N}\left|N^2 K_N H\left(\tfrac{x}{N}\right)-\frac{\sigma^2}{2}\Delta H(\tfrac{x}{N})\right| = 0.$$

For $\Theta(N) = N^{\theta+\gamma}$ and $\theta < 2-\gamma$ the first term above vanishes as $N\to\infty$.

The infinite variance case

What about $\gamma \in (1,2)$?

For any $\kappa > 0$ and $\theta = 0$, we get a collection of fractional reaction-diffusion equations with Dirichlet boundary conditions given by

$$\partial_t \rho_t(q) = \mathbb{L}_\kappa \rho_t(q) + \kappa V_0(q).$$

The operator $\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1$, where $\mathbb{L}$ is the regional fractional laplacian and

$$V_1(q) = \frac{c_\gamma}{\gamma}\Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\Big)$$

$$V_0(q) = \frac{c_\gamma}{\gamma}\Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\Big).$$

The operator $\mathbb{L}$:

Let $(-\Delta)^{\gamma/2}$ be the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1+|q|)^{1+\gamma}} du < \infty$$

by (provided the limit exists)

$$(-\Delta)^{\gamma/2} H\,(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H\,(q) + V_1(q) H(q) \\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} dy, \quad q \in (0,1).\end{aligned}$$

Fractional Sobolev space:

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $g:(0,1)\to\mathbb{R}$ such that $\|g\|_{\gamma/2}<\infty$, with

$$\|g\|_{\gamma/2} := \langle g,g\rangle_{\gamma/2} = \frac{c_\gamma}{2}\iint_{[0,1]^2}\frac{(g(u)-g(q))^2}{|u-q|^{1+\gamma}}dudq.$$

The space $L^2(0,T;\mathcal{H}^{\gamma/2})$ is the set of measurable functions $f:[0,T]\to\mathcal{H}^{\gamma/2}$ such that $\int_0^T\|f_t\|^2_{\mathcal{H}^{\gamma/2}}dt<\infty$ where $\|f_t\|^2_{\mathcal{H}^{\gamma/2}} := \|f_t\|^2+\|f_t\|^2_{\gamma/2}$.

The notion of weak solution:

Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the PDE above if:

- ♣ $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$ and $\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t(q))^2}{q^\gamma} + \frac{(\beta-\rho_t(q))^2}{(1-q)^\gamma} \right\} dq\, dt < \infty$,
- ♣ For all $t \in [0,T]$ and any function $H \in C_c^{1,\infty}([0,T] \times (0,1))$:

$$\int_0^1 \rho_t(q) H_t(q)\, dq - \int_0^1 g(q) H_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\partial_s + \mathbb{L}_\kappa \Big) H_s(q)\, dq ds - \kappa \int_0^t \int_0^1 V_0(q) H_s(q) dq\, ds = 0,$$

Characterizing limit points:

$$N^\gamma \mathcal{L}_N(\langle \pi_s^N, H\rangle) = \frac{N^\gamma}{N} \sum_{x,y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right]\eta_s(x)$$

$$+\frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^-)(\tfrac{x}{N})(\alpha-\eta_s(x)) + \frac{\kappa N^\gamma}{N}\sum_{x\in\Lambda_N}(Hr_N^+)(\tfrac{x}{N})(\beta-\eta_s(x)).$$

For H with compact support in $[a, 1-a]$ for $a \in (0,1)$ we have

$$\lim_{N\to\infty}\left| N^\gamma \sum_{y\in\Lambda_N} p(y-x)\left[H(\tfrac{y}{N}) - H(\tfrac{x}{N})\right] - (\mathbb{L}H)(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left| N^\gamma (r_N^-)(\tfrac{x}{N}) - r^-(\tfrac{x}{N})\right| = 0,$$

$$\lim_{N\to\infty}\left| N^\gamma (r_N^+)(\tfrac{x}{N}) - r^+(\tfrac{x}{N})\right| = 0$$

uniformly in $[a, 1-a]$.

Characterizing limit points:

Thus, the first term on the right hand side above can be replaced by

$$\langle \pi_t^N, \mathbb{L}H \rangle \to \int_0^1 (\mathbb{L}H)(q)\rho_t(q)dq,$$

as N goes to ∞.
The other terms can be replaced by
$\kappa\langle \alpha - \pi_t^N, Hr^- \rangle + \kappa\langle \beta - \pi_t^N, Hr^+ \rangle$ which converges to

$$\kappa \int_0^1 H(q)r^-(q)(\alpha - \rho_t(q))dq + \kappa \int_0^1 H(q)r^+(q)(\beta - \rho_t(q))dq$$
$$= \kappa \int_0^1 H(q)V_0(q)dq - \kappa \int_0^1 H(q)V_1(q)\rho_t(q)dq,$$

as N goes to ∞.

Uniqueness of weak solution:

To prove it we do the following. Let $\bar{\rho} = \rho^1 - \rho^2$, where ρ^1 and ρ^2 are two weak solutions starting from g. We have $\bar{\rho}_t(0) = \bar{\rho}_t(1) = 0$. Then,

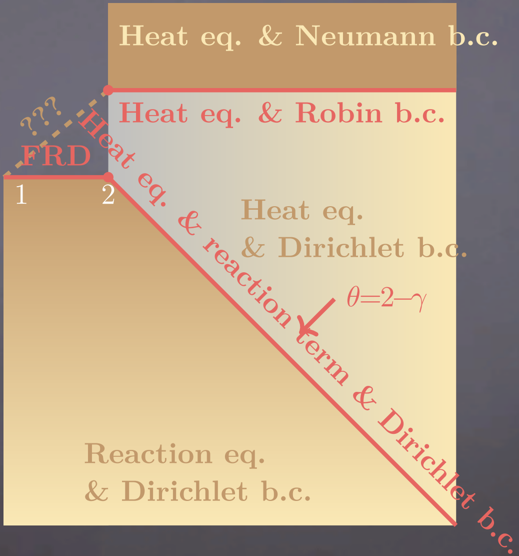
$$\langle \bar{\rho}_t, H_t \rangle - \int_0^t \langle \bar{\rho}_s, \Big(\partial_s + \mathbb{L} \Big) H_s \rangle ds + \kappa \int_0^t \langle V_1 H_s, \bar{\rho}_s \rangle ds = 0.$$

Take now $H_N(s,q) = \int_s^t G_N(r,q)\, dr$ where $(G_N)_{N \geq 0}$ is a sequence of functions in $C_c^{1,\infty}([0,T] \times (0,1))$ converging to $\bar{\rho}$. Plug H_N in the equation and take $N \to \infty$ to get

$$\int_0^t \int_0^1 \bar{\rho}_s^2(q)\, dq ds + \frac{1}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{\gamma/2}^2 + \frac{\kappa}{2} \Big\| \int_0^t \bar{\rho}_s ds \Big\|_{V_1}^2 = 0.$$

From this we conclude the uniqueness.

Open problems:



Conjecture:

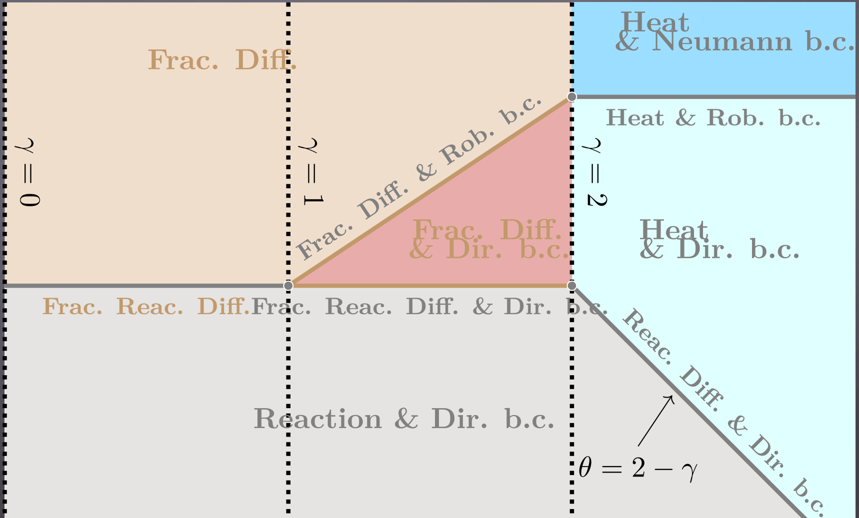
For $\theta > 0$ small and $\gamma \in (1,2)$ the solution should correspond to the solution when $\kappa = 0$. Supported by the result:

Let $g : [0,1] \to [0,1]$ be a measurable function and let ρ^κ be the weak solution of

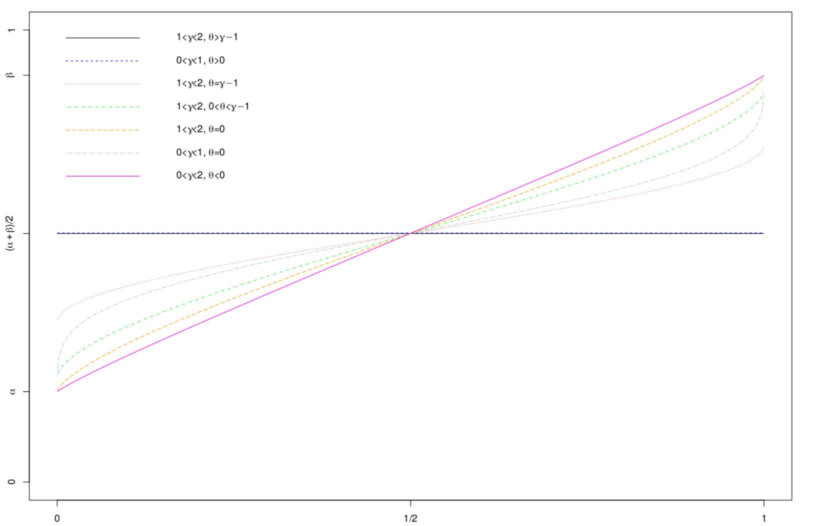
$$\partial_t \rho_t(q) = \mathbb{L}_\kappa \rho_t(q) + \kappa V_0(q),$$

with Dirichlet boundary conditions and initial condition $g(\cdot)$. Then ρ^κ converges strongly to ρ^0 in $L^2(0,T;\mathcal{H}^{\gamma/2})$ as κ goes to 0, where ρ^0 is the weak solution of the equation with $\kappa = 0$ and initial condition $g(\cdot)$.

Solved problem:



Stationary solutions:



Lecture 3: Fluctuations

The space of test functions: Let $\mathcal{S}_\theta$ denote the set of functions $H \in C^\infty([0,1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

(1) for $\theta < 1$: $\partial_u^{2k} H(0) = \partial_u^{2k} H(1) = 0$;
(2) for $\theta = 1$: $\partial_u^{2k+1} H(0) = \partial_u^{2k} H(0)$ and $\partial_u^{2k+1} H(1) = -\partial_u^{2k} H(1)$;
(3) for $\theta > 1$: $\partial_u^{2k+1} H(0) = \partial_u^{2k+1} H(1) = 0$.

The density fluctuation field $\mathcal{Y}_\cdot^N$ is the time-trajectory of linear functionals acting on functions $H \in \mathcal{S}_\theta$ as

$$\mathcal{Y}_t^N(H) = \frac{1}{\sqrt{N-1}} \sum_{x=1}^{N-1} H(\tfrac{x}{N})\Big(\eta_{tN^2}(x) - \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\Big).$$

Operators:

For $\theta \geq 0$, let $-\Delta_\theta$ be the positive self-adjoint operator on $L^2[0,1]$, defined on $H \in \mathcal{S}_\theta$ by

$$\Delta_\theta H(u) \; = \left\{ \begin{array}{ll} \partial_u^2 H(u)\,, & \text{if } \; u \in (0,1), \\ \partial_u^2 H(0^+)\,, & \text{if } \; u = 0, \\ \partial_u^2 H(1^-)\,, & \text{if } \; u = 1. \end{array} \right.$$

Let $\nabla_\theta : \mathcal{S}_\theta \to C^\infty([0,1])$ be the operator given by

$$\nabla_\theta H(u) \; = \left\{ \begin{array}{ll} \partial_u H(u)\,, & \text{if } \; u \in (0,1), \\ \partial_u H(0^+)\,, & \text{if } \; u = 0, \\ \partial_u H(1^-)\,, & \text{if } \; u = 1. \end{array} \right.$$

Let $T_t^\theta : \mathcal{S}_\theta \to \mathcal{S}_\theta$ be the semigroup associated to the PDE with the corresponding boundary conditions with $\alpha = \beta = 0$.

Fluctuations: $\theta = 1$

the initial state?

- For each $N \in \mathbb{N}$, the measure μ_N is associated to a measurable profile $\rho_0 : [0,1] \to [0,1]$ (This is the same condition for hydrodynamics!).
- For $\rho_0^N(x) = \mathbb{E}_{\mu_N}[\eta_0(x)]$

$$\max_{x \in \Lambda_N} |\,\rho_0^N(x) - \rho_0(\tfrac{x}{N})| \lesssim \frac{1}{N}\,.$$

- For

$$\varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta(x)\eta(y)] - \rho_0^N(x)\rho_0^N(y)$$

it holds that

$$\max_{1 \leq x < y \leq N-1} |\varphi_0^N(x,y)| \lesssim \frac{1}{N}\,.$$

Examples - initial measures:

- If for a given measurable profile $\rho_0 : [0,1] \to [0,1]$, we take μ_N as the Bernoulli product measure given by

$$\mu_N\{\eta : \eta(x) = 1\} = \rho_0(\tfrac{x}{N})$$

  then all the conditions above are true.

- If μ_{ss} is the stationary measure, then all the conditions above are true, by choosing the profile ρ_0 as the stationary profile $\bar{\rho}$ given above.

$\theta = 1$:

For each $N \geq 1$, let $\mathbb{Q}_N$ be the probability measure on $\mathcal{D}([0,T],\mathcal{S}'_\theta)$ induced by $\mathcal{Y}_\cdot^N$ and μ_N.

The sequence of measures $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ is tight on $\mathcal{D}([0,T],\mathcal{S}'_\theta)$ and all limit points $\mathbb{Q}$ are p.m. concentrated on paths $\mathcal{Y}_\cdot$ satisfying

$$\mathcal{Y}_t(H) \; = \; \mathcal{Y}_0(T_t^1 H) + \mathcal{W}_t(H)\,,$$

for any $H \in \mathcal{S}_\theta$. Above $\mathcal{W}_t(H)$ is a mean zero Gaussian variable of variance $\int_0^t \|\nabla_1 T_{t-r}^1 H\|^2_{L^{2,1}(\rho_r)} dr$, where $\rho(t,u)$ is the solution of the hydrodynamic equation. Moreover, $\mathbb{E}_{\mathbb{Q}}\Big[\mathcal{Y}_0(H)\,\mathcal{W}_t(G)\Big] = 0$ for all $H, G \in \mathcal{S}_\theta$.

If $\{\mathcal{Y}_0^N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a mean-zero Gaussian field $\mathcal{Y}_0$ with covariance given on $H, G \in \mathcal{S}_\theta$ by

$$\mathbb{E}\left[\mathcal{Y}_0(H)\mathcal{Y}_0(G)\right] := \sigma(H,G),$$

then, the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a generalized Ornstein-Uhlenbeck process, which is the formal solution of: $\partial_t \mathcal{Y}_t = \Delta_1 \mathcal{Y}_t dt + \sqrt{2\chi(\rho_t)}\nabla_1 \mathcal{W}_t$, where $\mathcal{W}_t$ is a space-time white noise of unit variance. As a consequence, the covariance of the limit field $\mathcal{Y}_t$ is given on $H, G \in \mathcal{S}_\theta$ by

$$\begin{aligned}\mathbb{E}\left[\mathcal{Y}_t(H)\mathcal{Y}_s(G)\right] = \sigma(T_t^1 H, T_s^1 G)&\\ + \int_0^s \langle \nabla_1 T_{t-r}^1 H, \nabla_1 T_{s-r}^1 G\rangle_{L^{2,1}(\rho_r)} dr.&\end{aligned}$$

Stationary ($\theta = 1$):

Suppose to start the process from μ_{ss} with $\alpha \neq \beta$. Then, $\mathcal{Y}^N$ converges to the centered Gaussian field $\mathcal{Y}$ with covariance given on $H, G \in \mathcal{S}_\theta$ by:

$$\mathbb{E}_{\mu_{ss}}[\mathcal{Y}(H)\mathcal{Y}(G)] = \int_0^1 \chi(\overline{\rho}(u))H(u)G(u)\,du - \left(\tfrac{\beta-\alpha}{3}\right)^2 \int_0^1 [(-\Delta_1)^{-1}H(u)]G(u)\,du$$

where $\overline{\rho}(\cdot)$ is the stationary solution of the PDE.

Associated martingales:

Let $H : [0,1] \to \mathbb{R}$ be a test function and note that

$$\mathcal{M}_t^N(H) := \mathcal{Y}_t^N(H) - \mathcal{Y}_0^N(H) - \int_0^t N^2 \mathcal{L}_N \mathcal{Y}_s^N(H)\, ds$$

is a martingale where

$$\begin{aligned} N^2 \mathcal{L}_N \mathcal{Y}_s^n(H) &= \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H(\tfrac{x}{N}) \bar{\eta}_{sN^2}(x) \\ &\quad + \sqrt{N} \Big[\nabla_N^+ H(0) - H(\tfrac{1}{N}) \Big] \bar{\eta}_{sN^2}(1) \\ &\quad + \sqrt{N} \Big[H(\tfrac{N-1}{N}) + \nabla_N^- H(1) \Big] \bar{\eta}_{sN^2}(N-1). \end{aligned}$$

Note that the second term at the right hand side of the previous expression is $\mathcal{Y}_s^N(\Delta_N H)$. Above, we have used the notation

$$\nabla_N^+ H(x) = N \Big[H(\tfrac{x+1}{N}) - H(\tfrac{x}{N}) \Big], \quad \nabla_N^- H(x) = N \Big[H(\tfrac{x}{N}) - H(\tfrac{x-1}{N}) \Big].$$

The correlation estimate:

For each $x, y \in V_N \;=\; \{(x,y)\,;\, x,y \in \mathbb{N},\, 0 < x < y < N\}$ and $t \in [0,T]$, let

$$\varphi_t^N(x,y) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)\eta_{tN^2}(y)] - \rho_t^N(x)\rho_t^N(y)\,,$$

and set $\varphi_t^N(x,y) = 0$, for $x = 0$ or $y = N$, we set

Proposition:

If

$$\max_{x,y \in V_N} \left|\varphi_0^N(x,y)\right| \;\lesssim\; \frac{1}{N},$$

then

$$\sup_{t \geq 0} \max_{(x,y) \in V_N} \left|\varphi_t^N(x,y)\right| \;\lesssim\; \frac{1}{N}.$$

$\mathscr{F}luctuations$: $\theta \neq 1$

$\theta \neq 1$:

$$N^2 \mathscr{L}_N \mathscr{Y}_s^N(H) = \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H\left(\tfrac{x}{N}\right)\left(\eta_{sN^2}(x) - \rho_s^N(x)\right)$$
$$+\sqrt{N}\nabla_N^+ H(0)\bar{\eta}_{sN^2}(1) - \sqrt{N}\nabla_N^- H(1)\bar{\eta}_{sN^2}(N-1)$$
$$-\frac{N^{3/2}}{N^\theta} H\left(\tfrac{1}{N}\right)\bar{\eta}_{sN^2}(1) - \frac{N^{3/2}}{N^\theta} H\left(\tfrac{N-1}{N}\right)\bar{\eta}_{sN^2}(N-1).$$

For $x \in \{1, N-1\}$ and $t \in [0,T]$ it holds

$$\mathbb{E}_{\mu_N}\left[\left(\int_0^t C_N^\theta (\eta_{sN^2}(x) - \rho_s^N(x))\, ds\right)^2\right] \lesssim (C_N^\theta)^2 \tfrac{N^\theta}{N^2}.$$

Apply last result with $C_N^\theta = \sqrt{N}\mathbf{1}_{\{\theta<1\}} + N^{3/2-\theta}\mathbf{1}_{\{\theta>1\}}$.

The initial measures:

We fix an initial profile $\rho_0 : [0,1] \to [0,1]$ which is measurable and of class C^6, and we assume that

$$\max_{x \in \Lambda_N} |\rho_0^N(x) - \rho_0(\tfrac{x}{N})| \lesssim \frac{1}{N}.$$

Moreover, we also assume that

$$\max_{y \in \Lambda_N} |\varphi_0^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, \ \theta \le 1, \\ \frac{1}{N}, \ \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1,$$

and that

$$\max_{(x,y) \in V_N} |\varphi_0^N(x,y)| \lesssim \frac{1}{N}.$$

The correlation estimate:

Proposition:

If

$$\max_{y\in\Lambda_N} |\varphi_0^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, & \theta \le 1, \\ \frac{1}{N}, & \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1,$$

$$\max_{(x,y)\in V_N} |\varphi_0^N(x,y)| \lesssim \frac{1}{N},$$

then,

$$\sup_{t\ge 0} \max_{y\in\Lambda_N} |\varphi_t^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, & \theta \le 1, \\ \frac{1}{N}, & \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1\,,$$

$$\sup_{t\ge 0} \max_{(x,y)\in V_N} |\varphi_t^N(x,y)| \lesssim \frac{1}{N}.$$

Ingredients for correlations

Show that $\varphi_t^N(x,y)$ is solution of

$$\begin{cases} \partial_t \varphi_t^N(x,y) = N^2 \mathcal{A}_N^\theta \varphi_t^N(x,y) - (\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1}, (x,y) \in V_N, \\ \varphi_t^N(x,y) = 0, (x,y) \in \partial V_N, \\ \varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta_0(x)\eta_0(y)] - \rho_0^N(x)\rho_0^N(y), (x,y) \in V_N \cup \partial V_N, \end{cases}$$

where $\mathcal{A}_N^\theta$ acts on $f : V_N \cup \partial V_N \to \mathbb{R}$ as

$$(\mathcal{A}_N^\theta f)(u) = \sum_{v \in V_N} c_N^\theta(u,v)\big[f(v) - f(u)\big],$$

and it is the infinitesimal generator of the RW in $V_N \cup \partial V_N$ which is absorbed at ∂V_N. Above,

$$c_N^\theta(u,v) = \begin{cases} 1, & \text{if } \|u-v\| = 1 \text{ and } u, v \in V_N, \\ N^{-\theta}, & \text{if } \|u-v\| = 1 \text{ and } u \in V_N,\ v \in \partial V_N, \\ 0, & \text{otherwise.} \end{cases}$$

Ingredients for correlations

Show that $\rho_t^N(\cdot)$ is a solution of

$$\begin{cases} \partial_t \rho_t^N(x) \; = \; (N^2 \mathfrak{B}_N^\theta \rho_t^N)(x)\,, \;\; x \in \Lambda_N\,, \;\; t \geq 0\,, \\ \rho_t^N(0) \; = \alpha\,, \rho_t^N(N) \; = \; \beta\,, \;\; t \geq 0\,, \end{cases}$$

where $\mathfrak{B}_N^\theta$ acts on $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$(\mathfrak{B}_N^\theta f)(x) \; = \; \sum_{y=0}^{N} \xi_{x,y}^{N,\theta} (f(y) - f(x)), \quad \text{for } x \in \Lambda_N$$

and it is the infinitesimal generator of the RW in $\overline{\Lambda}_N$ which is absorbed at the points $\{0, N\}$. Above

$$\xi_{x,y}^{N,\theta} \; = \; \begin{cases} 1\,, & \text{if } \; |y - x| = 1 \text{ and } x, y \in \Lambda_N, \\ N^{-\theta}\,, & \text{if } \; x = 1, y = 0 \text{ and } x = N-1\,, y = N, \\ 0\,, & \text{otherwise.} \end{cases}$$

Ingredients for correlations:

The stationary solutions of the equations above are given by

$$\varphi_{ss}^N(x,y) = -\frac{(\alpha-\beta)^2(x+N^\theta-1)(N-y+N^\theta-1)}{(2N^\theta+N-2)^2(2N^\theta+N-3)}$$

and $\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$, where

$$a_N = \frac{\beta-\alpha}{2N^\theta+(N-2)} \quad \text{and} \quad b_N = a_N(N^\theta-1)+\alpha.$$

The time spent by the 1-d RW at the points $x=1$ and $x=N-1$ is of order $O(\frac{N^\theta}{N^2})$ (good bound when $\theta<1$ but not when $\theta>1$). When $\theta>1$ we compare with the reflected RW and we prove that the time now is of order $O(\frac{1}{N})$. We need the same estimates in the 2-d setting for the time spent by the RW on the diagonal.

For the future:

- What about hydrostatics, for the long jumps case?
- Fluctuations?
- Other boundary conditions?

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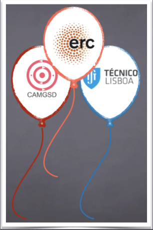
Thank you and...



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Thank you and...