Scaling limits for symmetric exclusion with open boundary

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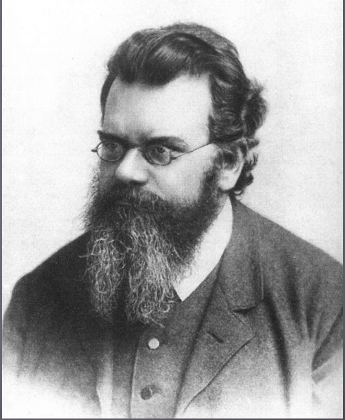
Historical context

Goal: analyse the evolution of a fluid or a gas (the number of components is huge, no precise description of the microscopic state of the system).

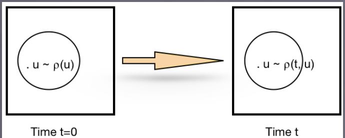
Then, we should:

- find the equilibrium states;
- characterize them by macroscopic quantities: pressure, temperature, density, etc;
- do the analysis out of equilibrium!



Statistical mechanics explains and predicts how the properties of atoms determine the physical properties of matter.

Physical motivation



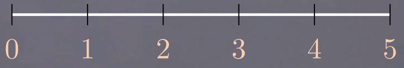
- Find the invariant states of a system.
- Characterize them by a quantity $\rho(\cdot)$.
- Fix $u \in \Lambda$ and a neighborhood Λ_u (microscopically big). Due to interaction, the system reaches an equilibrium $\rho(u)$.
- Let time evolve. Now the equilibrium close to u is given by $\rho(t, u)$. How does $\rho(t, u)$ evolve?

- We discretize a volume Λ according to a parameter N and get Λ_N.
- Two scales for space/time.
- In each cell we put a random number of particles.
- The dynamics conserves some quantity of interest.
- Waiting times are given by independent Poisson processes -> particle system is a Markov chain - *loss memory*.

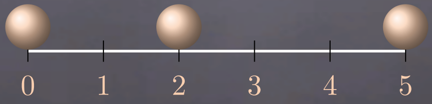
QUESTION: What is the macroscopic law describing the evolution of the conserved quantity of the system?

Example:

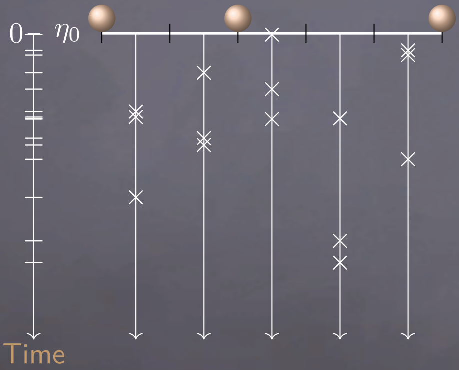
Let us fix our (microscopic) space Λ_N as the set of points $\{0, 1, 2, 3, 4, 5\}$.



Now, we fix the initial state we can do the following. Toss a coin, if we get head we put a particle at the site 0 and if we get a tail we leave it empty. Repeat this for each site of the discrete set. Suppose we got at the end to:



The dynamics



Particle system

- N is a scaling parameter;
- Microscopic space /macroscopic space;

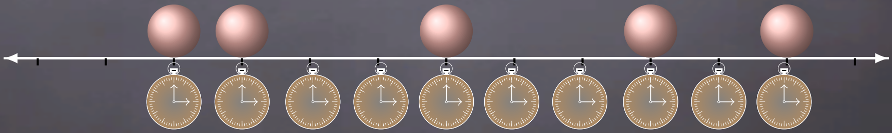
- Microscopic time $t\theta(N)$/macroscopic time t;
- Each bond has an exponentially distributed clock/ clocks at different bonds are independent;

- Fix a transition probability $p(x, y) = p(y - x)$.
- $\eta_t(x)$ denotes the quantity of particles at the site x. 
- **Markov process** for which the quantity of particles $\sum_x \eta(x)$ is conserved;

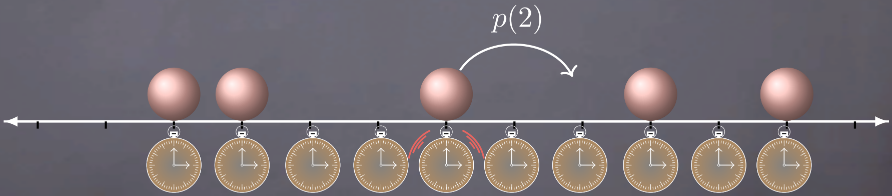
Exclusion processes

The dynamics:

After the ring of a clock the particle jumps from x to y at rate $p(y-x)$ if y is empty, otherwise the particle waits a new random time.

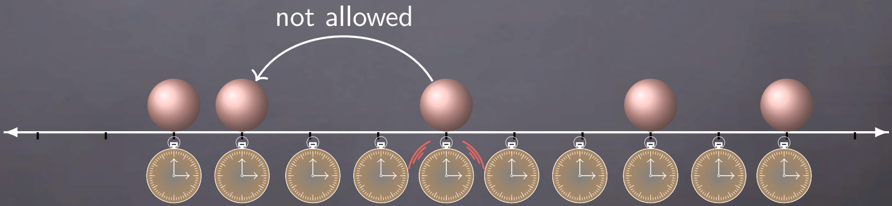


A ring of a clock

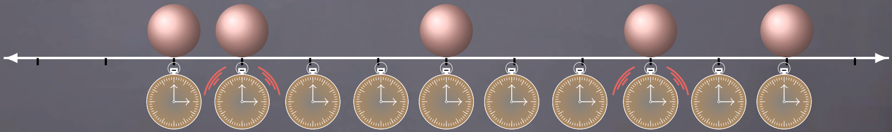


After the ring of the clock

The forbidden jumps



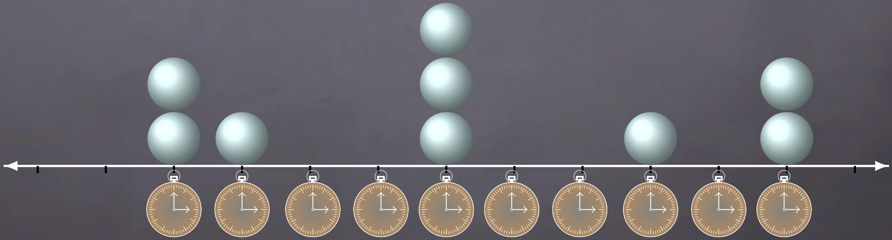
This cannot happen



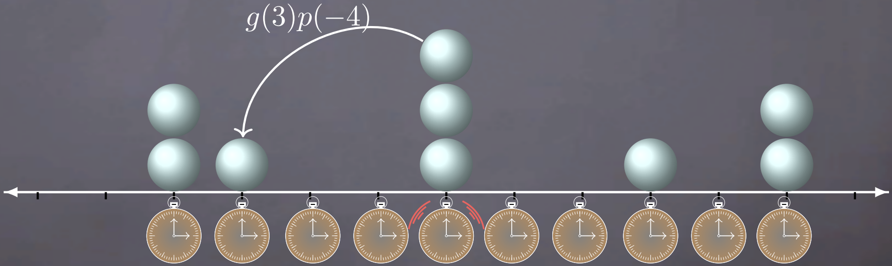
Zero-Range processes

The dynamics:

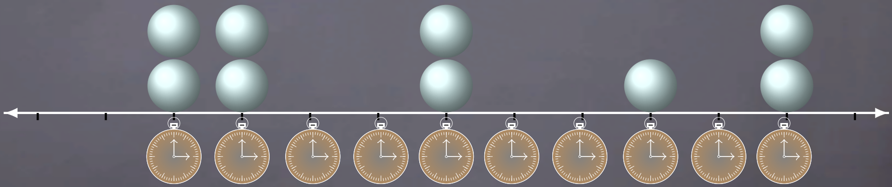
After the ring of a clock a particle jumps from x to y at rate $g(\eta(x))p(y-x)$.



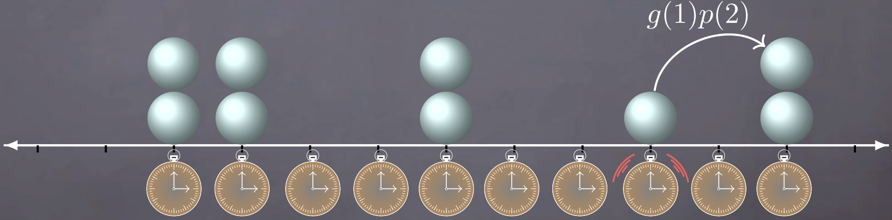
A ring of a clock



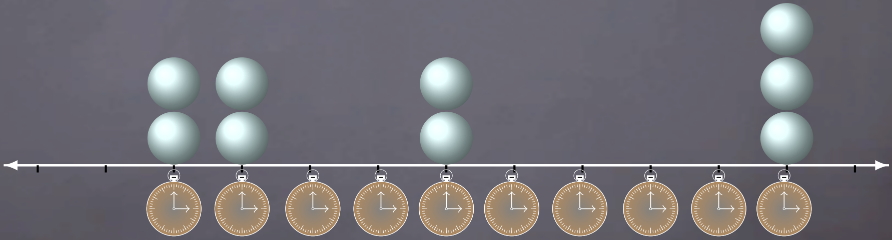
After the ring of the clock



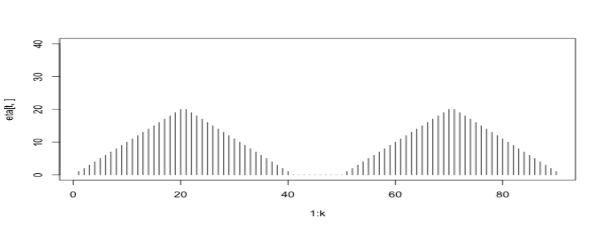
Another ring of a clock



After the ring of the clock



Simulation of Zero-Range:



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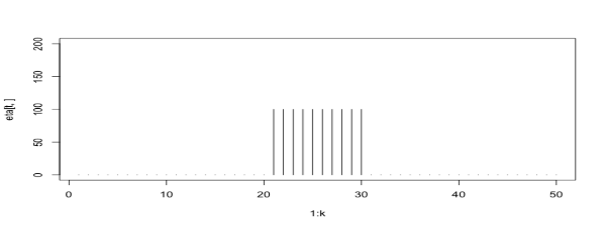
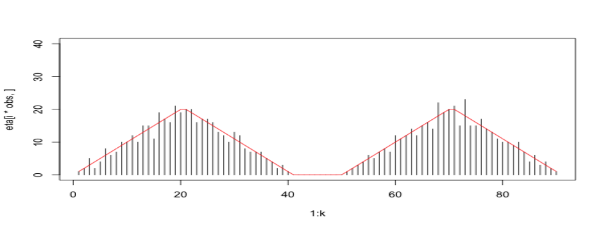


The hydrodynamic equation:



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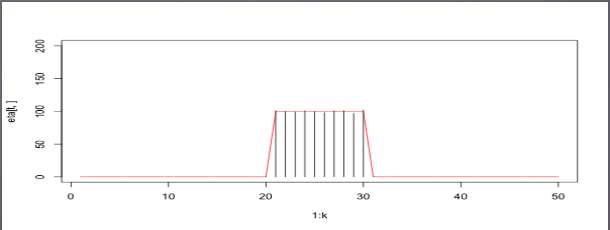

*Thanks to F. Hernandez for the simulations.

Outline of the mini-course:

- ♣ We will analyse the hydrodynamic limit for an exclusion process in contact with stochastic reservoirs when jumps are long range given by a symmetric probability transition rate:
  - ♣ with finite variance (Lecture 1);
  - ♣ with infinite variance (Lecture 2).
- ♣ Fluctuations, Large Deviations, open problems!? (Lecture 3).

Let us start with the simplest case: jumps to nearest-neighbors.

Now $\Lambda = [0, 1]$ and $\Lambda_N = \{1, ..., N - 1\}$. The state space of the Markov process is $\Omega_N = \{0, 1\}^{\Lambda_N}$.