Scaling limits for symmetric exclusion with open boundary

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Goal: analyse the evolution of a fluid or a gas (the number of components is huge, no precise description of the microscopic state of the system).

Then, we should:

♣ find the equilibrium states;
♣ characterize them by macroscopic quantities: pressure, temperature, density, etc;
♣ do the analysis out of equilibrium!

Statistical mechanics explains and predicts how the properties of atoms determine the physical properties of matter.
Physical motivation

- Find the invariant states of a system.
- Characterize them by a quantity $\rho(\cdot)$.
- Fix $u \in \Lambda$ and a neighborhood $\Lambda_u$ (microscopically big). Due to interaction, the system reaches an equilibrium $\rho(u)$.
- Let time evolve. Now the equilibrium close to $u$ is given by $\rho(t, u)$. How does $\rho(t, u)$ evolve?

- We discretize a volume $\Lambda$ according to a parameter $N$ and get $\Lambda_N$.
- Two scales for space/time.
- In each cell we put a random number of particles.
- The dynamics conserves some quantity of interest.
- Waiting times are given by independent Poisson processes $\rightarrow$ particle system is a Markov chain - loss memory.

 QUESTION: What is the macroscopic law describing the evolution of the conserved quantity of the system?
Example:

Let us fix our (microscopic) space $\Lambda_N$ as the set of points \{0, 1, 2, 3, 4, 5\}.

Now, we fix the initial state we can do the following. Toss a coin, if we get head we put a particle at the site 0 and if we get a tail we leave it empty. Repeat this for each site of the discrete set. Suppose we got at the end to:
The dynamics

Time

$\eta_0$

$\eta_1$

$\eta_2$

$\eta_3$

$\eta_4$

$\eta_5$

$\eta_6$

$\eta_7$

$\eta_8$

$\eta_9$

$\eta_{10}$

$\eta_{11}$

$\eta_{12}$

$\eta_{13}$

$\eta_{14}$

$\eta_{15}$
Particle system

- $N$ is a scaling parameter;
- Microscopic space / macroscopic space;
- Microscopic time $t \theta(N)$ / macroscopic time $t$;
- Each bond has an exponentially distributed clock / clocks at different bonds are independent;
- Fix a transition probability $p(x, y) = p(y - x)$.
- $\eta_t(x)$ denotes the quantity of particles at the site $x$.
- **Markov process** for which the quantity of particles $\sum_x \eta(x)$ is conserved;
Exclusion processes

The dynamics:

After the ring of a clock the particle jumps from $x$ to $y$ at rate $p(y - x)$ if $y$ is empty, otherwise the particle waits a new random time.
A ring of a clock

$p(2)$
After the ring of the clock
The forbidden jumps

not allowed
This cannot happen
Zero-Range processes

The dynamics:

After the ring of a clock a particle jumps from $x$ to $y$ at rate $g(\eta(x))p(y - x)$. 
A ring of a clock

\[ g(3)p(-4) \]
After the ring of the clock
Another ring of a clock

g(1)p(2)
After the ring of the clock
Simulation of Zero-Range:
Simulation of Zero-Range:
The hydrodynamic equation:
The hydrodynamic equation:

*Thanks to F. Hernandez for the simulations.*
Outline of the mini-course:

- We will analyse the hydrodynamic limit for an exclusion process in contact with stochastic reservoirs when jumps are long range given by a symmetric probability transition rate:
  - with finite variance (Lecture 1);
  - with infinite variance (Lecture 2).
- Fluctuations, Large Deviations, open problems!? (Lecture 3).

Let us start with the simplest case: jumps to nearest-neighbors.

Now $\Lambda = [0, 1]$ and $\Lambda_N = \{1, \ldots, N - 1\}$. The state space of the Markov process is $\Omega_N = \{0, 1\}^{\Lambda_N}$. 