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Complex Networks: Structure and Functionality

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Complex Networks: Structure and Functionality

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Abstract.

The goal of the present paper is to describe three topics in the area of complex networks that constituted a mini-course delivered at the 12th Mathematical Society of Japan – Seasonal Institute, held at Kyushu University in Fukuoka, Japan, 31/07–09/08, 2019. The topics are: (I) Spectra of adjacency matrices, (II) Equivalence of ensembles, (III) Exploration and mixing times. Topics (I) and (II) are part of the ongoing attempt to understand *structure* of networks, topic (III) to elucidate *functionality* of networks.

§1. Outline

In Section 2, which is based on joint work with Arijit Chakrabarty, Rajat Hazra and Matteo Sfragara [7], we consider inhomogeneous *Erdős-Rényi random graphs* on n vertices. We study the *empirical spectral distribution* of the *adjacency matrix* A_n in the limit as $n \to \infty$ in a regime that interpolates between sparse and dense. In particular, we show that the empirical spectral distribution of A_n when properly scaled converges to a deterministic limit weakly in probability. For the special case where the connectivity probability between two vertices has the *product property*, we give an explicit characterisation of the limit distribution. The result is applied to statistical inference of *sociability patterns in social networks*.

In Section 3, which is based on joint work with Diego Garlaschelli, Michel Mandjes, Joey de Mol, Andrea Roccaverde, Tiziano Squartini and Nicos Starreveld [9], [10], [11], [12], [13], we consider random graphs subject to topological constraints. We compare two probability distributions on the set of simple graphs on n vertices induced by a given constraint: (1) The *microcanonical ensemble*, where the constraint is *hard*, i.e., has to be satisfied for every realisation of the graph; (2) The

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canonical ensemble, where the constraint is *soft*, i.e., has to be satisfied on average. We say that *breaking of ensemble equivalence* occurs in the limit as $n \to \infty$ when the *relative entropy* of the two ensembles per vertex (in the sparse regime), respectively, per edge (in the dense regime) is strictly positive. We present two examples of constraints where breaking of ensemble equivalence occurs, namely, when the constraint is on the degree sequence and when the constraint is on the total number of edges and triangles. The result is applied to model selection for real-world networks.

In Section 4, which is based on joint work with Luca Avena, Hakan Guldas and Remco van der Hofstad [1], [2], we consider the mixing time of random walks on random graphs. Many real-world networks, such as WWW, are dynamic in nature. It is therefore natural to study random walks on *dynamic random graphs*. We consider random walk on a random graph with prescribed degrees. We investigate what happens when at each unit of time a fraction α_n of the edges is randomly rewired, where n is the number of vertices. We identify *three regimes* for the mixing time in the limit as $n \to \infty$, depending on the choice of α_n. These regimes exhibit surprising behaviour. The results are relevant for Google PageRank.

§2. Spectra of adjacency matrices

Spectra of random matrices have been analysed for almost a century. In recent years, many interesting results have been derived for spectra of random matrices associated with networks. The question that will be addressed in this section is: *What can be said about the spectrum of the adjacency matrix of a large inhomogeneous Erdős-Rényi random graph?*

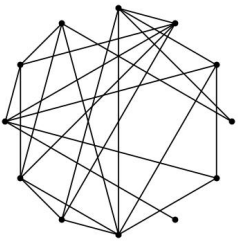

Fig. 1. Erdős-Rényi random graph.

2.1. SETTING

Let $(\varepsilon_N)_{N\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\lim_{N\to\infty} \varepsilon_N = 0, \qquad \lim_{N\to\infty} N\varepsilon_N = \infty.$$

Let $f\colon [0,1]\times[0,1] \to [0,\infty)$ be a continuous function such that $f(x,y) = f(y,x)$ for all $x, y \in [0,1]$. Fix $N \in \mathbb{N}$, and consider the inhomogeneous Erdös-Rényi random graph ER_N on N vertices where an edge is placed between the pair of vertices $\{i,j\}$ with probability

$$\varepsilon_N f\left(\tfrac{i}{N}, \tfrac{j}{N}\right), \quad 1 \le i,j \le N,$$

independently for different edges. Write $\mathbb{P}$ for the law of ER_N.

Let A_N be the adjacency matrix of ER_N. Write

$$\lambda_i(A_N), \qquad 1 \le i \le N,$$

for the real eigenvalues of A_N. The *empirical spectral distribution* of A_N is defined as

$$\mathrm{ESD}(A_N) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(A_N)},$$

which is a random probability distribution on $\mathbb{R}$.

2.2. Scaling

Theorem 1. *There exists a compactly supported symmetric probability measure μ on $\mathbb{R}$ such that, weakly in $\mathbb{P}$-probability,*

$$\lim_{N\to\infty} \mathrm{ESD}\left(A_N \Big/ \sqrt{N\varepsilon_N}\right) = \mu.$$

Furthermore, if

$$\min_{x,y\in[0,1]} f(x,y) > 0,$$

then μ is absolutely continuous with respect to Lebesgue measure. The density of μ can be characterised implicitly via an integral equation for its Stieltjes transform.

(A weaker version of the above theorem also appeared in [18].)

It is possible to identify μ when

$$f(x,y) = r(x)r(y), \qquad x,y \in [0,1],$$

for some continuous function $r\colon [0,1] \to [0,\infty)$.

Theorem 2. *If f is of product form, then*

$$\mu = \mu_r \boxtimes \mu_s,$$

where

$$\mu_r = \mathrm{LAW}[r(U)], \quad U = \mathrm{UNIF}[0,1],$$
$$\mu_s = \textit{standard Wigner semicircle law},$$

and $\boxtimes$ denotes free multiplicative convolution.

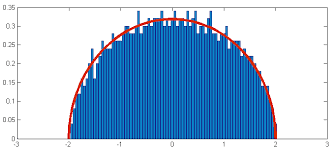

Fig. 2. Wigner semicircle law.

In free probability, the Wigner semicircle law takes over the role of the normal law in classical probability. The so-called free cumulants replace the classical cumulants, in the sense that partitions are replaced by non-crossing partitions. Just as the cumulants of degree ≥ 2 are all zero if and only if the distribution is normal, the free cumulants of degree ≥ 2 are all zero if and only if the distribution is the Wigner semicircle law.

Theorem 3. *Theorems* 1–2 *can be generalized to the situation where the function f is random, depends on N and converges to a deterministic limit as $N \to \infty$.*

Key ingredients of the proof are: centering, Gaussianisation, perturbation, decoupling, and combinatorics from free probability.

2.3. Application 1: social networks

Consider a community of N individuals, represented by the vertices in ER_N. Data is available about which individuals are acquainted. Based on this data, the sociability pattern of the community has to be inferred statistically.

Let ρ denote a probability measure on $[0,\infty)$ with bounded support. Let $(R_i)_{1\leq i\leq N}$ be i.i.d. random variables drawn from ρ. Think of R_i as the sociability index of individual i. Pick N so large that

$$0 \leq \varepsilon_N R_i R_j \leq 1 \qquad \forall\, 1 \leq i,j \leq N.$$

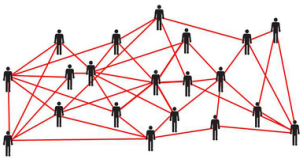

Fig. 3. A social network.

Suppose that i, j are acquainted with probability $\varepsilon_N R_i R_j$, which is represented by an edge in ER_N between vertices i, j. The data that is available is the adjacency matrix A_N. The statistical inference problem is to estimate ρ from A_N. To standardise ρ, we assume that

$$\int_0^\infty x\rho(dx) = 1.$$

Since, weakly $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{R_i} = \rho,$$

Theorem 3 gives that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\left(A_N \Big/ \sqrt{N\varepsilon_N}\right) = \rho \boxtimes \mu_s.$$

In practice, ε_N is unknown, which can be worked around by arguing that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\left(\sqrt{\frac{N}{\mathrm{Tr}(A_N^2)}}\, A_N\right) = \rho \boxtimes \mu_s.$$

The procedure is that $\rho \boxtimes \mu_s$ can be *statistically estimated* from A_N. Subsequently, ρ can be estimated because the moments of $\rho \boxtimes \mu_s$ are functions of the moments of ρ and μ_s. Indeed, since the moments of μ_s are known, the moments of ρ can be recursively computed from the moments of $\rho \boxtimes \mu_s$. Since ρ is compactly supported, it can in turn be computed via its moments.

2.4. Application 2: configuration model

Let $\mathcal{S}_N$ be the set of simple graphs on N vertices. We fix the degrees of all the vertices, namely, vertex i has degree d_i^*, where

$$\vec{d}_N^* = \{d_i^*\}_{1\le i\le N}$$

is a sequence of positive integers of which we only require that it is graphical, i.e., there is at least one simple graph matching these degrees. The Gibbs canonical ensemble P_N is the unique probability distribution on $\mathcal{S}_N$ with the following two properties:

(I) The average degree of vertex i, defined by

$$\sum_{G\in\mathcal{S}_N} d_i(G)P_N(G),$$

equals d_i^* for all $i \leq i \leq N$.

(II) The entropy of P_N, defined by

$$-\sum_{G\in\mathcal{S}_N} P_N(G)\log P_N(G),$$

is maximal.

P_N models a random graph of which we have *no prior information* other than the average degrees.

Property (II) forces P_N to take the form [14]

$$P_N(G) = \frac{1}{Z_N(\vec{\theta}^*)}\exp\left[-\sum_{i=1}^N \theta_i^* d_i(G)\right], \qquad G\in\mathcal{S}_N,$$

where $\vec{\theta}_N^* = \{\theta_i^*\}_{1\leq i\leq N}$ is the unique sequence of Lagrange multipliers such that property (I) is satisfied. Reparametrisation yields

$$P_N(G) = \prod_{1\leq i<j\leq N} (p_{ij}^*)^{A_N[G](i,j)}\,(1-p_{ij}^*)^{1-A_N[G](i,j)},\ G\in\mathcal{S}_N,$$

where $A_N[G]$ is the adjacency matrix of G, and

$$p_{ij}^* = \frac{x_i^* x_j^*}{1+x_i^* x_j^*}, \quad x_i^* = e^{-\theta_i^*}, \qquad 1\leq i\neq j\leq N.$$

Property (I) requires that

$$d_i^* = \sum_{\substack{1\leq j\leq N\\ j\neq i}} p_{ij}^*, \qquad 1\leq i\leq N,$$

which constitutes a set of N equations for N unknowns.

Abbreviate

$$m_N = \max_{1\leq i\leq N} d_i^*.$$

We focus on the regime

$$\lim_{N\to\infty} m_N = \infty, \quad \lim_{N\to\infty} m_N/\sqrt{N} = 0.$$

It turns out that in this regime

$$p^*_{ij} = [1 + o(1)]\,\frac{d^*_i d^*_j}{\sigma_N}, \qquad N \to \infty,$$

with

$$\sigma_N = \sum_{1\le i\le N} d^*_i.$$

Pick

$$\varepsilon_N = m_N^2/\sigma_N.$$

Then

$$\lim_{N\to\infty} \varepsilon_N = 0, \qquad \lim_{N\to\infty} N\varepsilon_N = \infty,$$

and

$$p^*_{ij} = [1 + o(1)]\,\varepsilon_N\,(d^*_i/m_N)(d^*_j/m_N).$$

Under the assumption that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{d^*_i/m_N} = \rho$$

for some probability measure ρ, Theorem 3 gives that, weakly in $\mathbb{P}$-probability,

$$\lim_{N\to\infty} \mathrm{ESD}\big(A_N \Big/ \sqrt{N\varepsilon_N}\big) = \rho \boxtimes \mu_s.$$

This identifies the scaling of the ESD for the network that is modeled by the soft configuration model as a function of the imposed average degrees.

Challenges for the future are:

- ▷ What can we say in the sparse regime and in the dense regime?
- ▷ How can we deal with more general classes of random graphs?

§3. Equivalence of ensembles

3.1. Statistical physics

Systems consisting of a very large number of interacting particles can be described by *statistical ensembles*, i.e., probability distributions on spaces of configurations. Two important examples are:

I. *micro-canonical* ensemble.
II. *canonical* ensemble.

The former fixes the energy of the system, the latter fixes the average energy of the system, with temperature as the control parameter. The two ensembles capture physically different microscopic situations. For both the entropy is maximal subject to the constraint. The canonical ensemble is easier to compute with than the micro-canonical ensemble, because the constraint is soft rather than hard.

In textbooks of statistical physics the two ensembles are assumed (!) to be *thermodynamically equivalent*, i.e., to have the same macroscopic behaviour. Here the idea is that for large systems the energy is typically close to its average value.This assumption is certainly reasonable for systems with interactions that are short-ranged. But, counterexamples have been found for systems with interactions that are long-ranged.

3.2. Complex networks

We will be interested in large random graphs, i.e., the two ensembles live on the set $\mathcal{S}_N$ of all simple graphs with N vertices where $N \to \infty$.

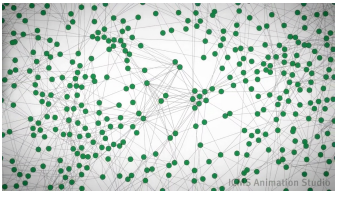

Fig. 4. A realisation of a large random graph.

Given are a vector-valued function $\vec{C}$ on $\mathcal{S}_N$, and a specific vector $\vec{C}^*$ called the constraint.

I. The *micro-canonical ensemble* is defined by

$$P_N^{\mathrm{mic}}(G) = \begin{cases} 1/\Omega_{\vec{C}^*} & \text{if } \vec{C}(G) = \vec{C}^*, \\ 0 & \text{else,} \end{cases}$$

where $\Omega_{\vec{C}^*} = |\{G \in \mathcal{S}_N \colon\ \vec{C}(G) = \vec{C}^*\}|$.

II. The *canonical ensemble* is defined by

$$P_N^{\mathrm{can}}(G) = \frac{1}{\mathcal{N}(\vec{\theta}^*)}\, \mathrm{e}^{-\vec{\theta}^* \cdot \vec{C}(G)},$$

where $\mathcal{N}(\vec{\theta}^*)$ is the normalising constant and $\vec{\theta}^*$ is to be chosen such that $\sum_{G\in\mathcal{S}_N} \vec{C}(G)P_N^{\mathrm{can}}(G) = \vec{C}^*$.

Interpretation:

- P_N^{mic} models a random graph of which no information is available other than the *constraint*.
- P_N^{can} models a random graph of which no information is available other than the *average constraint*.

Which of the two ensembles should be used to model a real-world network depends on the *a priori knowledge* that is available about the network.

3.3. Ensemble equivalence

P_N^{mic} and P_N^{can} are said to be equivalent when their *relative entropy per vertex* defined by

$$s_N\left(P_N^{\mathrm{mic}} \mid P_N^{\mathrm{can}}\right) = \frac{1}{N}\sum_{G\in\mathcal{S}_N} P_N^{\mathrm{mic}}(G)\log\left(\frac{P_N^{\mathrm{mic}}(G)}{P_N^{\mathrm{can}}(G)}\right).$$

tends to zero as $N\to\infty$. Because in both ensembles all $G\in\mathcal{S}_N$ such that $\vec{C}(G) = \vec{C}^*$ have the same probability, we get the simpler formula

$$s_N\left(P_N^{\mathrm{mic}} \mid P_N^{\mathrm{can}}\right) = \frac{1}{N}\log\left(\frac{P_N^{\mathrm{mic}}(G^*)}{P_N^{\mathrm{can}}(G^*)}\right)$$

for any G^* such that $\vec{C}(G^*) = \vec{C}^*$. This greatly *simplifies* the computation, since we need not carry out the sum over $\mathcal{S}_N$ and only need to compute with a single graph G^*.

As shown in [17], relative entropy is the sharpest tool to detect breaking of ensemble equivalence. In the remainder we illustrate breaking of ensemble equivalence via a number of examples.

3.4. Constraint of the degree sequence

In the configuration model, each vertex gets a prescribed number of half-edges, which are paired off randomly to form edges.

Consider a graph $G = (V, E)$ with vertex set $V = \{1,\dots,N\}$ and edge set E such that all the vertices have prescribed degrees. In other words, consider the constraint

$$\vec{C}^* = \vec{d}_N^* = (d_1^*,\dots,d_N^*) \in \mathbb{N}_0^N.$$

Suppose that the degrees are moderate, corresponding to what is called the sparse regime:

$$\max_{1\le i\le N} d_i^* = o(\sqrt{N}), \qquad N\to\infty.$$

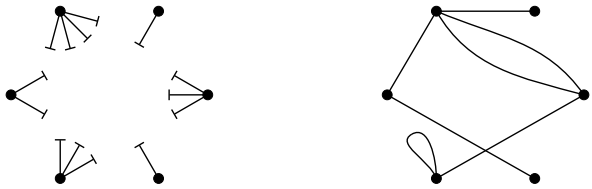

Fig. 5. Example with $N = 6$ and $\vec{d}_N = (1, 3, 1, 3, 2, 4)$.

Let

$$f_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{d_i^*} = \text{empirical degree distribution.}$$

Define

$$g(k) = \log\left(\frac{k!}{k^k e^{-k}}\right), \qquad k \in \mathbb{N}_0.$$

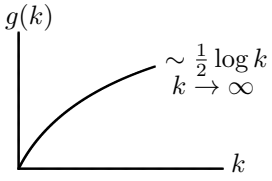

Fig. 6. Picture of $k \mapsto g(k)$.

Theorem 4. *Suppose that*

$$\lim_{N\to\infty} \|f_N - f\|_{\ell^1(g)} = 0$$

for some limiting degree distribution f. Then

$$s_\infty = \lim_{N\to\infty} s_N\left(P_N^{\text{mic}} \mid P_N^{\text{can}}\right) = \|f\|_{\ell^1(g)}.$$

The interpretation is that each vertex with degree k contributes an amount $g(k)$ to the relative entropy. Note that there is breaking of ensemble equivalence for all $f \neq \delta_0$, i.e., breaking is the rule rather than the exception. The proof is based on graph counting (micro-canonical) and percolation theory (canonical).

It turns out that $g(k)$ is the relative entropy of Dirac(k) with respect to Poisson(k). What this says is that, in the limit as $N \to \infty$,

- Micro-canonical ensemble: vertices have a *fixed* degree.

- Canonical ensemble: vertices have a *random* degree.

We illustrate the above with two examples.

Example 1: $f_N = \delta_k$ with $k = o(\sqrt{N})$. For k-regular graphs:

$$s_\infty = g(k) > 0.$$

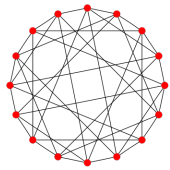

Fig. 7. The 5-regular graph.

Example 2: $f_N(k) = C_N\, k^{-\tau}$, $1 \le k \le k_{\text{cutoff}}(N)$, with $k_{\text{cutoff}}(N) = o(\sqrt{N})$ and $\tau \in (1, \infty)$ a tail exponent. For scale-free graphs:

$$s_\infty \approx \frac{1}{2(\tau - 1)} + \tfrac{1}{2}\log(2\pi) > 0.$$

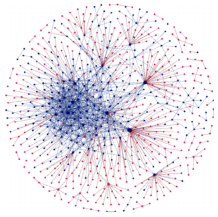

Fig. 8. A graph with hubs.

3.5. Constraint of the total number of edges and triangles

Interesting behaviour shows up when we pick

$$\vec{C}^* = (\text{number of edges}, \text{number of triangles})$$

$$= \left(T_1^* \begin{pmatrix} N \\ 2 \end{pmatrix}, T_2^* \begin{pmatrix} N \\ 3 \end{pmatrix}\right), \qquad T_1^*, T_2^* \in [0, 1].$$

This corresponds to the so-called dense regime, in which the number of edges per vertex is of order N. The quantity of interest is now

$$s_\infty = \lim_{N\to\infty} \frac{1}{N^2} \log\left(\frac{P_N^{\rm mic}(G^*)}{P_N^{\rm can}(G^*)}\right),$$

where we scale by N^2 instead of N.

Theorem 5. *See the figure below.*

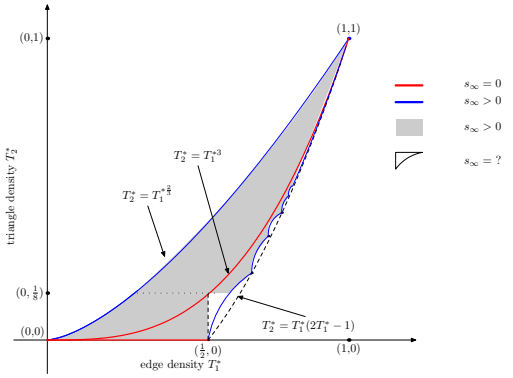

Fig. 9. Between the blue curves the edge-triangle densities are admissible [16]. Breaking of ensemble equivalence occurs everywhere except on the red curves. In the white region between the red curve and the lower blue curve it is not known what happens.

Breaking of ensemble equivalence occurs when (T_1^*, T_2^*) is *frustrated*. The proof is based on the theory of graphons, which are continuum limits of adjacency matrices of graphs [5], [6]. We derive a variational formula for s_∞ with the help of the large deviation principle for graphons associated with the Erdős-Rényi random graph [8].

What happens close the line $T_2^* = T_1^{*3}$? It turns out that anomalous behaviour shows up:

Theorem 6. *For $T_1^* \in (0,1)$,*

$$\lim_{\epsilon\downarrow 0} \epsilon^{-1}\, s_\infty(T_1^*, T_1^{*3} + \epsilon) = C^+ \in (0,\infty),$$

$$\lim_{\epsilon\downarrow 0} \epsilon^{-2/3}\, s_\infty(T_1^*, T_1^{*3} - \epsilon) = C^- \in (0,\infty),$$

where C^+, C^- are computable functions of T_1^* that are, however, not so easy to identify.

In summary, we have obtained a complete classification of breaking of ensemble equivalence in random graphs with constraints on the degree sequence, respectively, the total number of edges and triangles. Breaking occurs when the number of constraints is extensive or when the constraints are frustrated.

Challenges for the future are:

- ▷ Can we estimate the relative entropy away from the Erdős-Rényi curve?
- ▷ What happens when other constraints than edge-triangles are considered?

§4. Exploration and mixing times

4.1. Searching on networks

Search algorithms on networks are important tools for the organisation of large data sets. A key example is Google PageRank, which assigns a weight to each element of a hyperlinked set of documents, such as the World Wide Web, with the purpose of measuring its relative importance within the set.

The weights are assigned via exploration and are obtained recursively. A hyperlink counts as a vote of support: a page that is linked to by many pages with a high rank receives a high rank itself.

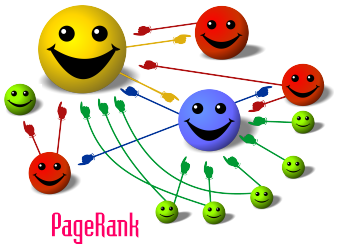

Fig. 10. An example of GooglePageRanks weights in a small network.

4.2. Searching on complex networks

- ▷ Networks are modelled as *graphs*, consisting of a set of vertices and a set of edges connecting pairs of vertices.
- ▷ Complex networks are modelled as *random graphs*, where the vertices and the edges are chosen according to some probability distribution.

▷ Search algorithms are modelled as *random walks*, moving along the network by randomly picking an edge incident to the vertex currently visited and jumping to the vertex at the other end.

The question we ask is: *How long does it take the random walk to explore the random graph properly?* The answer to this question is important because it tells us how long the search algorithm must run.

The *mixing time* of a random walk is the time it needs to approach its stationary distribution. For random walks on *static* random graphs, the mixing time has been the subject of intensive study. However, since many networks are dynamic in nature, it is natural to study random walks on *dynamic* random graphs. This line of research is very recent in the mathematics literature.

As we saw in Section 3, the configuration model is a random graph with a prescribed degree sequence. It is popular because of its mathematical tractability and its flexibility in modelling real-world networks. In what follows we consider a discrete-time dynamic version of the configuration model, where at each unit of time a certain fraction of the edges is *rewired*.

Static version. Let $\mathcal{G}(\vec{d}_N)$ denote the set of all graphs on N vertices with a prescribed degree sequence

$$\vec{d}_N = (d_i)_{i=1}^N, \quad \sum_{i=1}^N d_i = \text{ even}.$$

We draw a random graph uniformly from the set $\mathcal{G}(\vec{d}_N)$. The outcome may have self-loops and multiple edges. The *stationary distribution* of the random walk equals

$$\pi(i) = \frac{d_i}{\sum_{j=1}^N d_j}, \qquad 1 \le i \le N,$$

and does *not* depend on the outcome of the draw. One way to generate the random graph is by randomly pairing half-edges (recall Section 3).

For random walk on the static configuration model, the mixing time is known to be

$$[1 + o(1)]\, c \log N, \quad N \to \infty,$$

with

$$\frac{1}{c} = \lim_{N\to\infty} \frac{\sum_{i=1}^N \log d_i}{\sum_{i=1}^N d_i},$$

subject to certain *regularity assumptions* on the degrees [15], [3], [4].

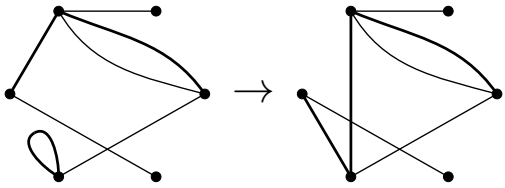

Fig. 11. A transition for the dynamic configuration model. Bold edges on the left are the ones chosen to be rewired. Bold edges on the right are the newly formed edges.

Dynamic version. For fixed N, draw a starting graph η and a starting vertex i, and proceed as follows. At each time $t \in \mathbb{N}$:

(1) Draw edges randomly with probability $\alpha_N \in (0,1)$.
(2) Rewire these edges by breaking them into half-edges and pairing these half-edges again randomly.
(3) After the rewiring, let the random walk make a step to a randomly chosen neighbouring vertex.

We make the following assumptions:

- The degrees must be *moderate*, i.e., not too large.
- The random walk is *non-backtracking*, i.e., immediate jumps back along edges are not allowed.
- $\lim_{N\to\infty} \alpha_N = 0$, i.e., the dynamics is *slow*.

4.3. Mixing time

Let $\mathbb{P}_{\eta,i}$ denote probability with respect to the joint process of random graph and random walk with starting graph η and starting vertex i. Let X_t denote the location of the random walk at time $t \in \mathbb{N}$, and write

$$\mathcal{D}_{\eta,i}(t) = \tfrac{1}{2} \sum_{j=1}^{N} |\mathbb{P}_{\eta,i}(X_t = j) - \pi(j)|$$

to denote total variation distance between the distribution of X_t and the stationary distribution π. It turns out that there is are *three regimes*:

(1) $\lim_{N\to\infty} \alpha_N (\log N)^2 = \infty$: supercritical regime.
(2) $\lim_{N\to\infty} \alpha_N (\log N)^2 = \beta \in (0,\infty)$: critical regime.
(3) $\lim_{N\to\infty} \alpha_N (\log N)^2 = 0$: subcritical regime.

Theorem 7. *With high probability, i.e., for a set of* (η, i) *with probability tending to* 1 *as* $N \to \infty$*, the following hold.*

(1) *Supercritical regime:*

$$\mathcal{D}_{\eta,i}(s/\sqrt{\alpha_N}) = e^{-s^2/2} + o(1), \quad s \in [0,\infty).$$

(2) *Critical regime:*

$$\mathcal{D}_{\eta,i}(s \log N) = \begin{cases} e^{-\beta s^2/2} + o(1), & s \in [0,c), \\ o(1), & s \in [c,\infty). \end{cases}$$

(3) *Subcritical regime:*

$$\mathcal{D}_{\eta,i}(s \log N) = \begin{cases} 1 - o(1), & s \in [0,c), \\ o(1), & s \in [c,\infty). \end{cases}$$

Here, c is the constant in the static version.

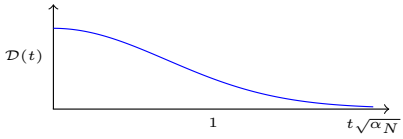

Fig. 12. Scaled mixing profile in the supercritical regime.

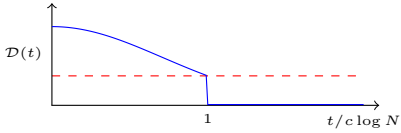

Fig. 13. Scaled mixing profile in the critical regime.

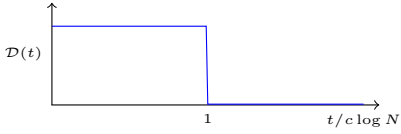

Fig. 14. Scaled mixing profile in the subcritical regime.

In the supercritical regime the mixing time is of order

$$1/\sqrt{\alpha_N} \ll \log N,$$

and does not depend on the degree sequence. In the critical regime and the subcritical regime the mixing time is of order $\log N$ and depends on the degree sequence.

The proof is based on a *stopping time* argument: the first time the random walk moves along an edge that has been relocated is close to a strong uniform time.

Challenges for the future are:

- ▷ What effect do hubs have on the mixing time?
- ▷ What happens when only edges touched by the random walk can be rewired?

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