How to initialise a second class particle?

Joint with
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Márton Balázs

University of Bristol

Eindhoven, YEP XIII (LD for IPS and PDE)
8 March, 2016.

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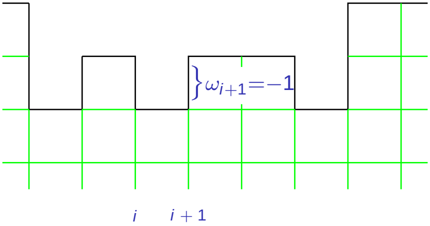
  - Examples:
    - $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$: classical zero range; $\omega_i \sim$ Geom(θ).
    - $r(\omega_i) = \omega_i$: independent walkers; $\omega_i \sim$ Poi(θ).

Totally asymmetric bricklayers process



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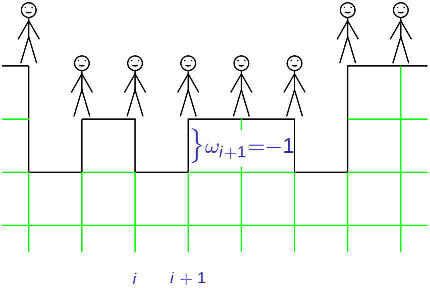

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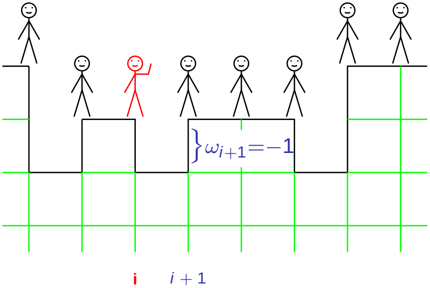

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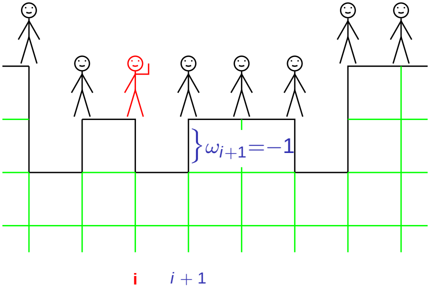

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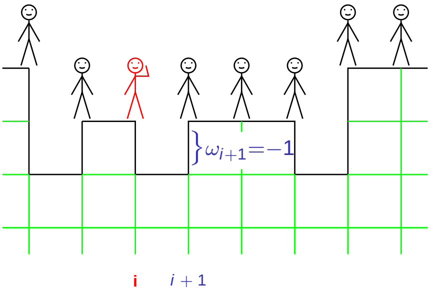

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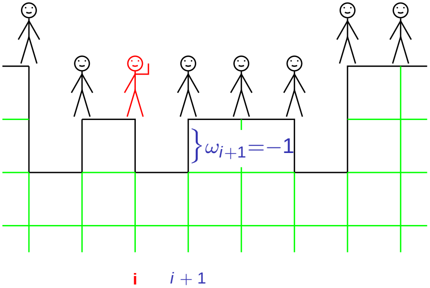

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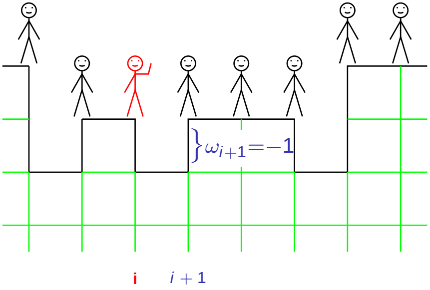

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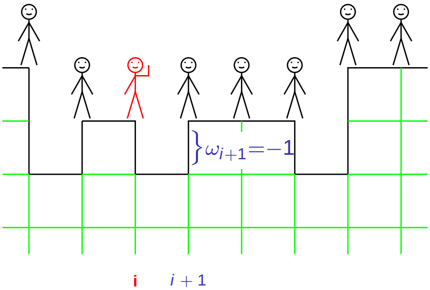

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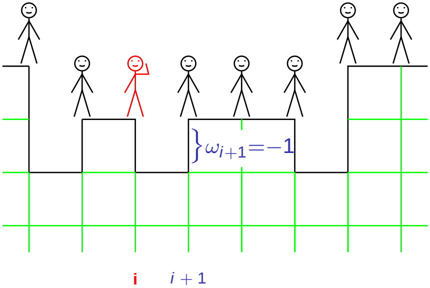

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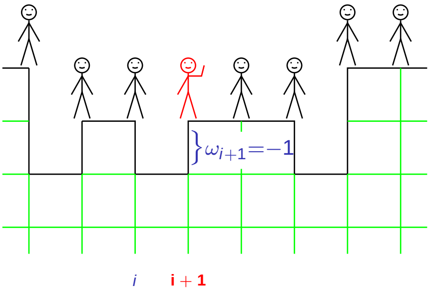

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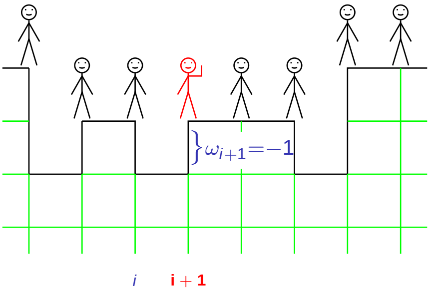

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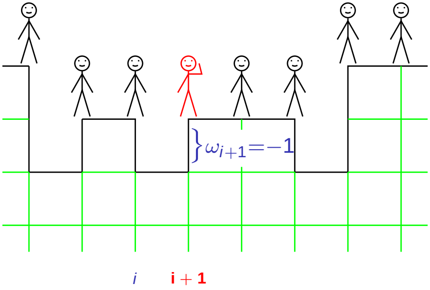

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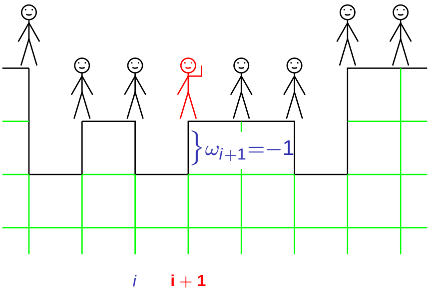

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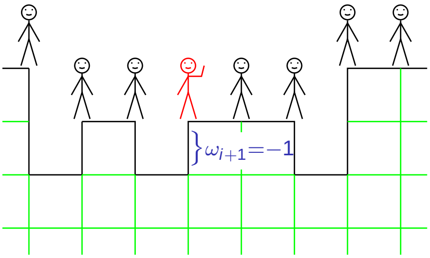

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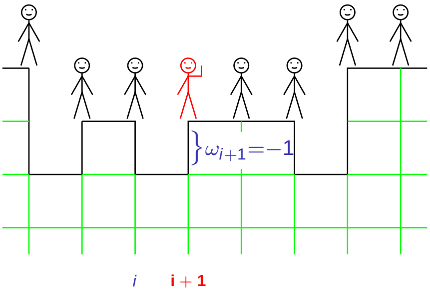

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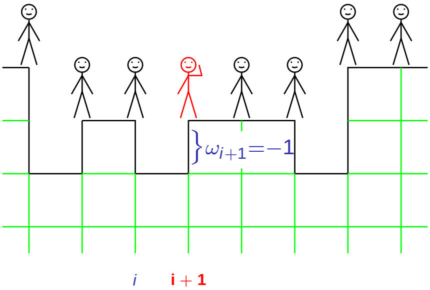

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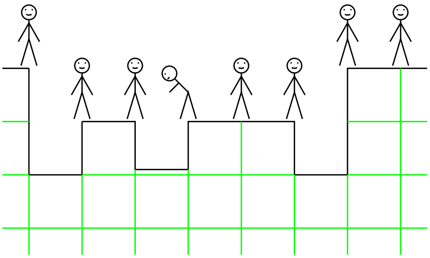

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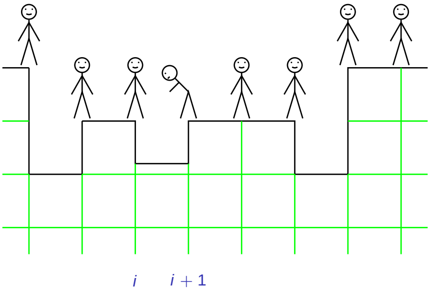

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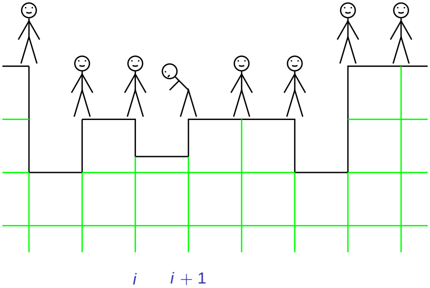

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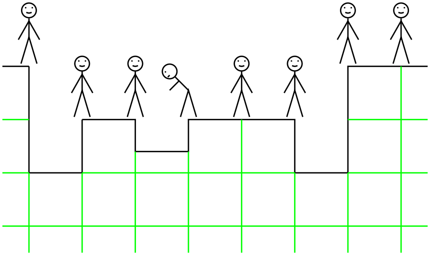

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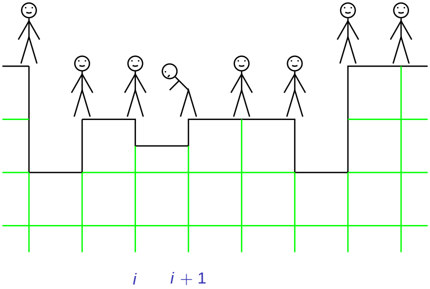

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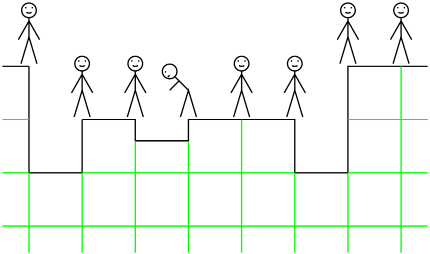

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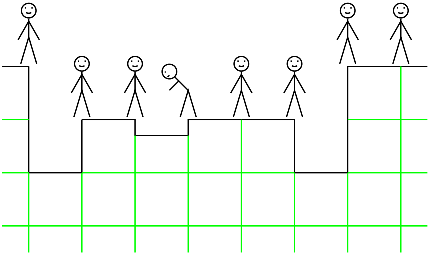

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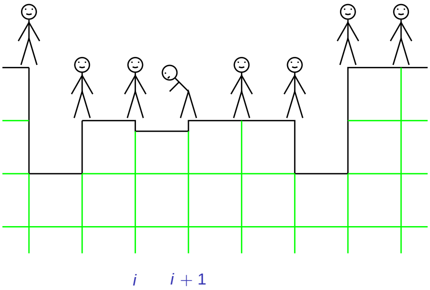

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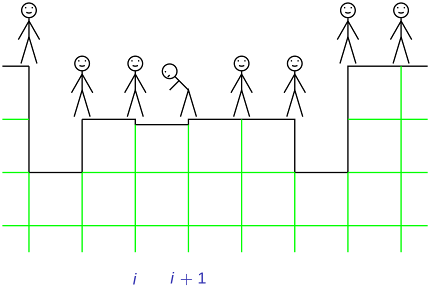

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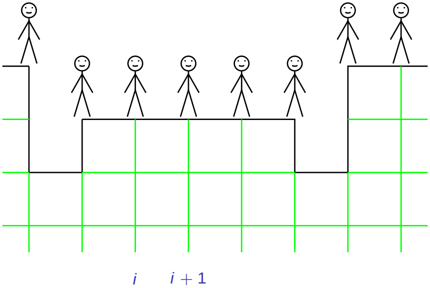

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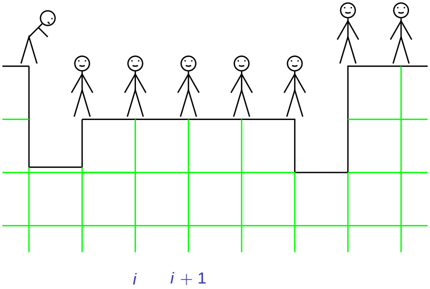

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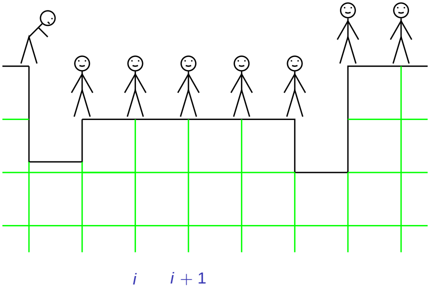

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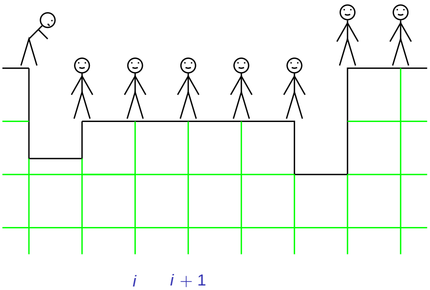

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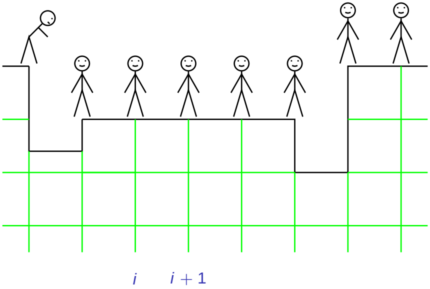

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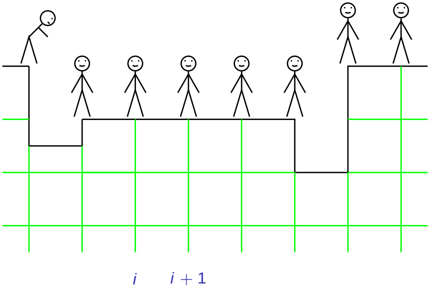

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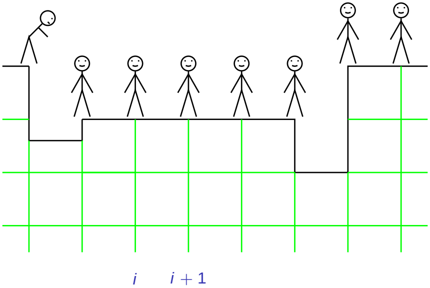

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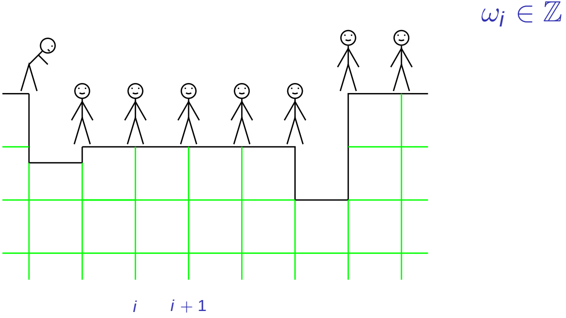

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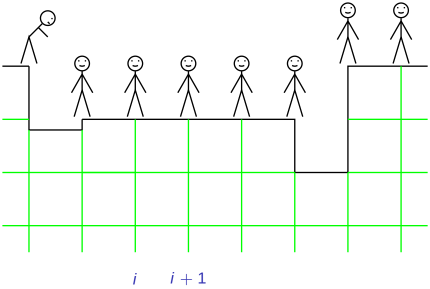

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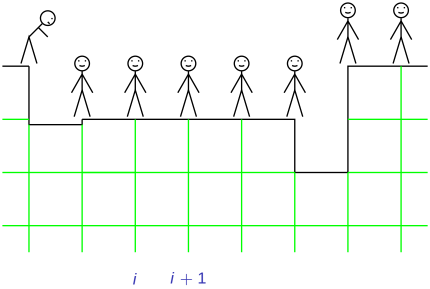
Totally asymmetric bricklayers process



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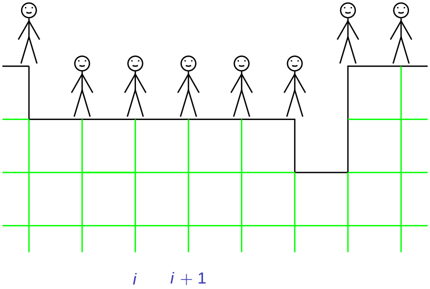

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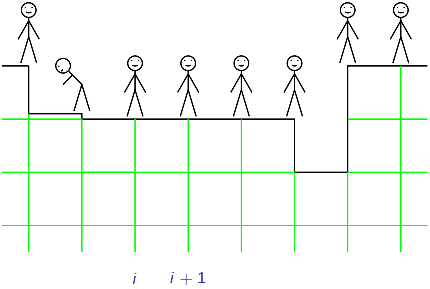

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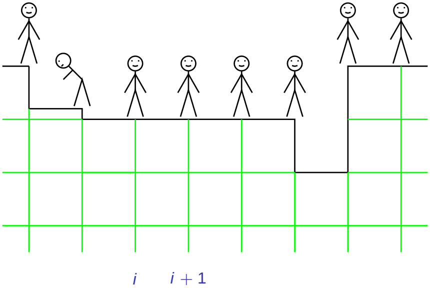

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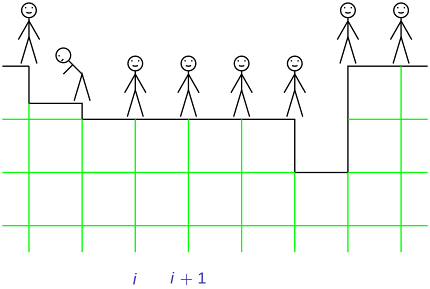

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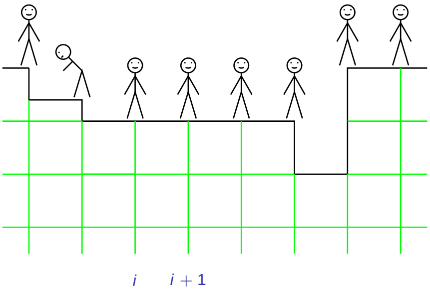

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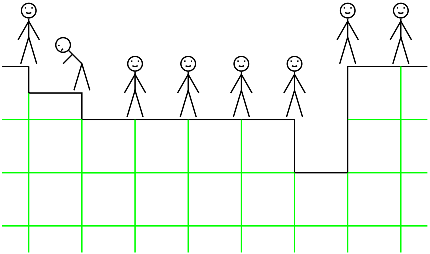

i $\quad i+1$

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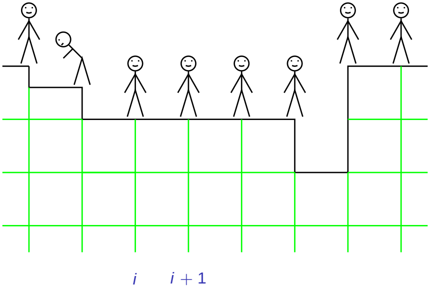

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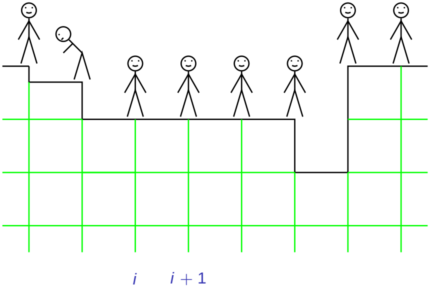

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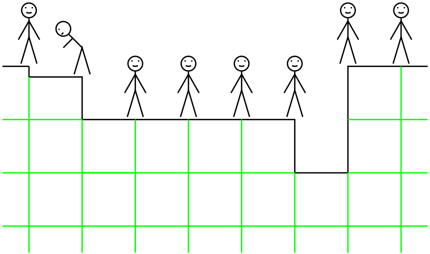

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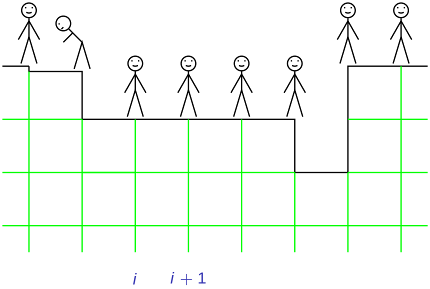

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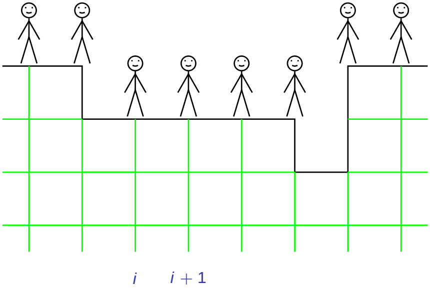

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Totally asymmetric bricklayers process

Extremal translation-invariant distributions are still product, and rather explicit in terms of $r(\cdot)$.

A special case: $r(\omega_i) = \mathrm{e}^{\beta\omega_i}$: $\omega_i \sim$ discrete Gaussian($\frac{\theta}{\beta}$, $\frac{1}{\sqrt{\beta}}$).

Hydrodynamics (very briefly)

Define the *density* $\varrho := \mathbf{E}(\omega)$
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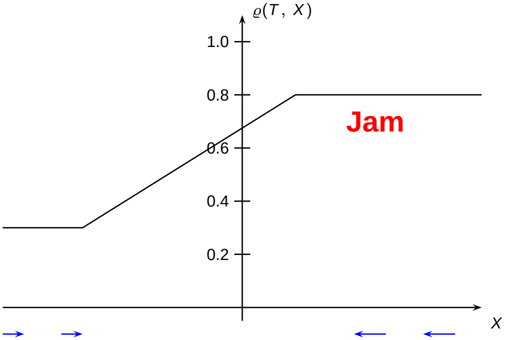
- The *characteristic velocity* is $H'(\varrho)$.

Shock



$H'(\varrho) \searrow$ (H concave)

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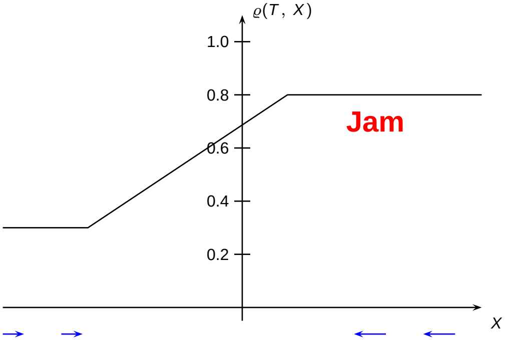

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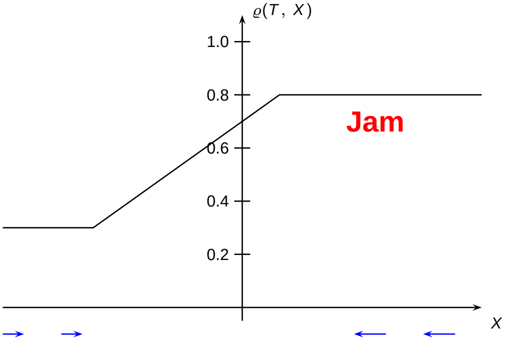

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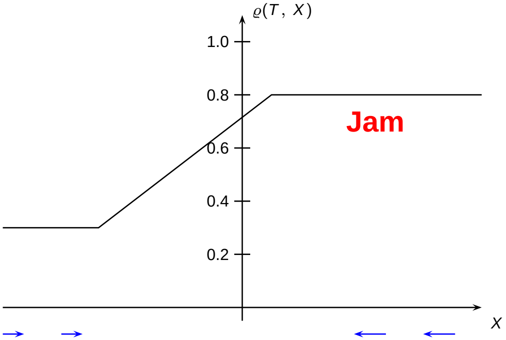

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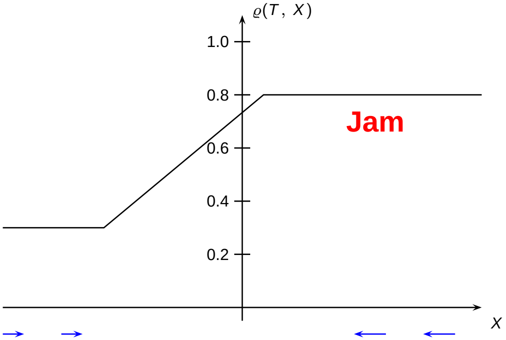

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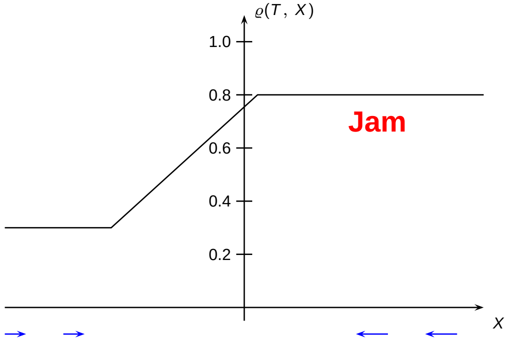

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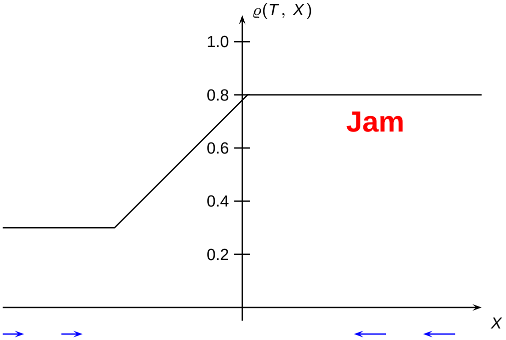

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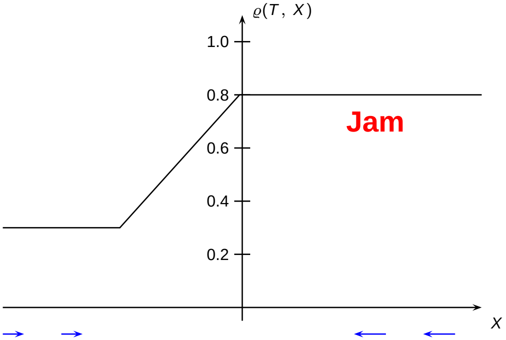

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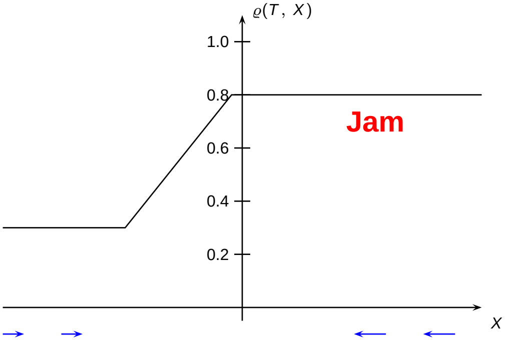

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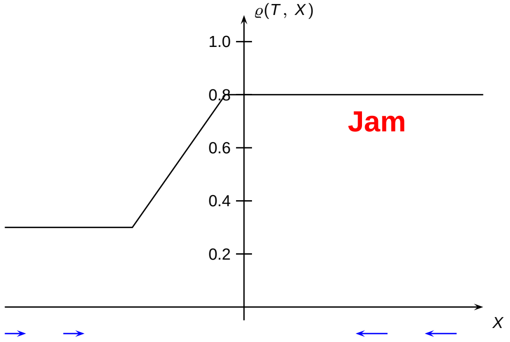

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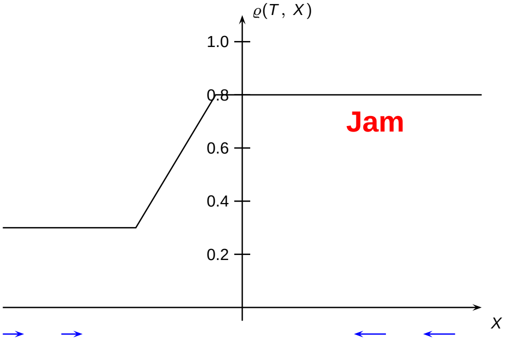

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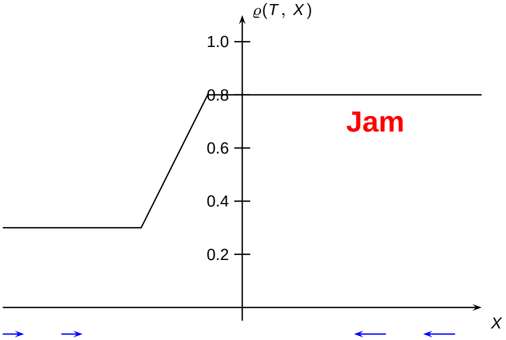

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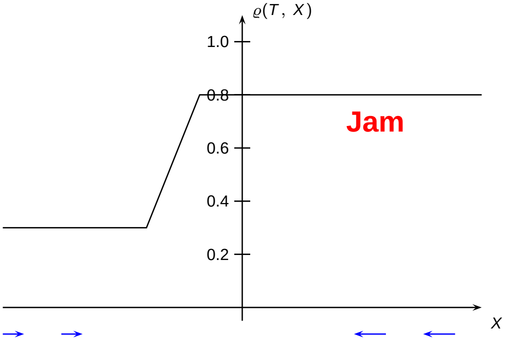

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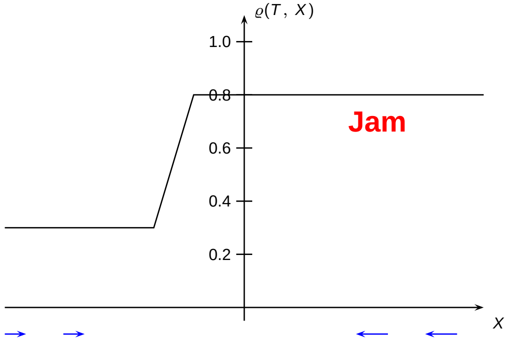

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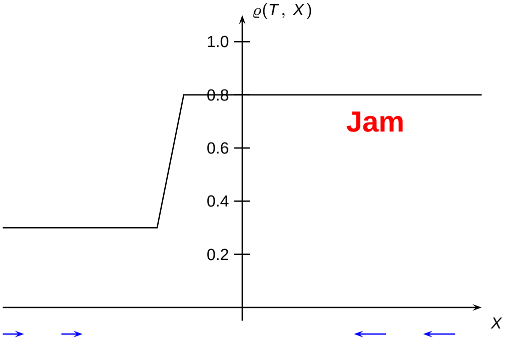

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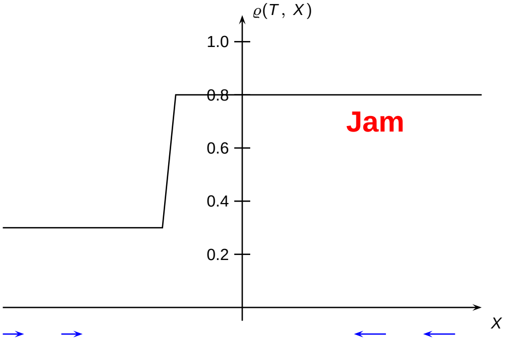

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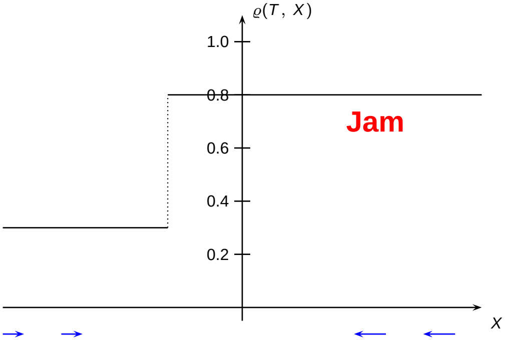

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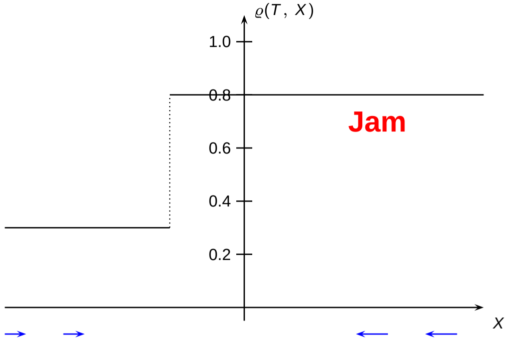

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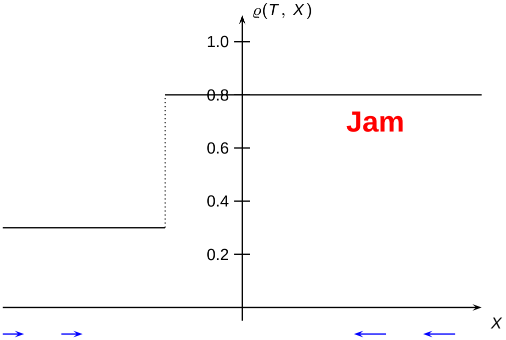

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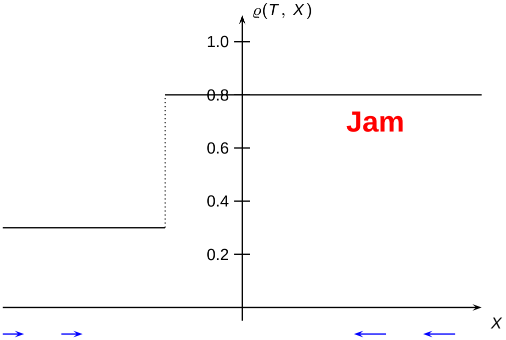

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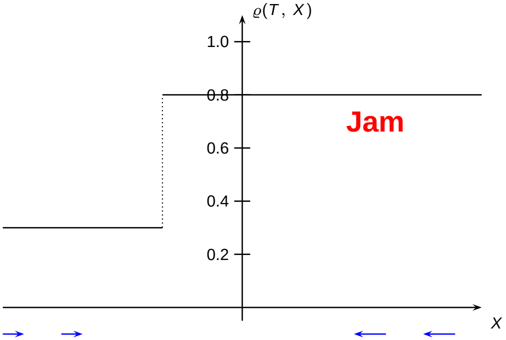

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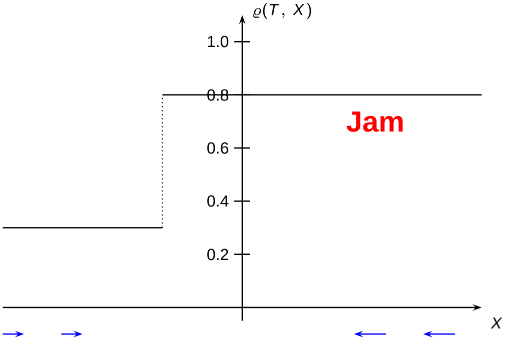

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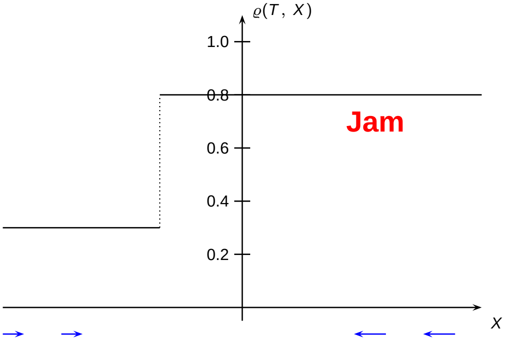

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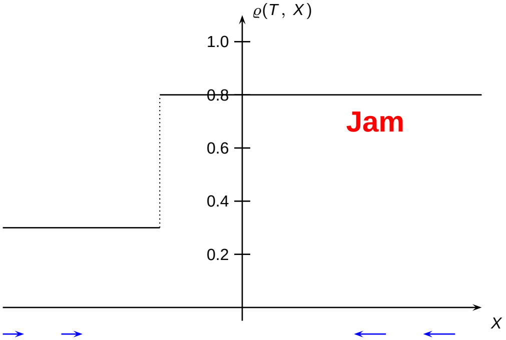

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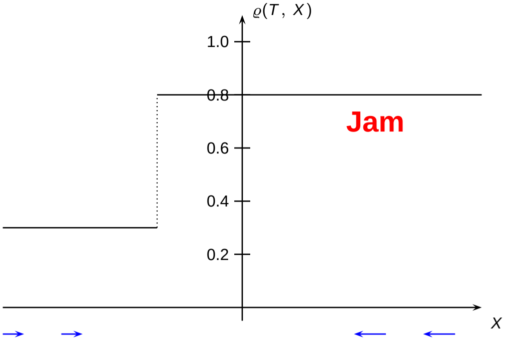

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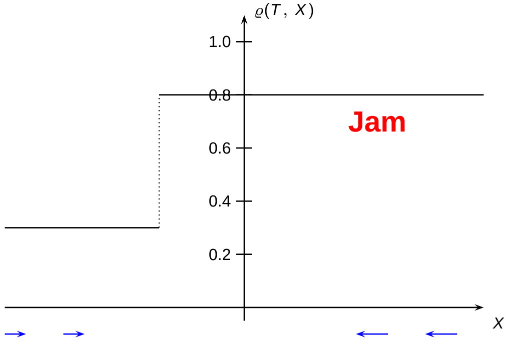

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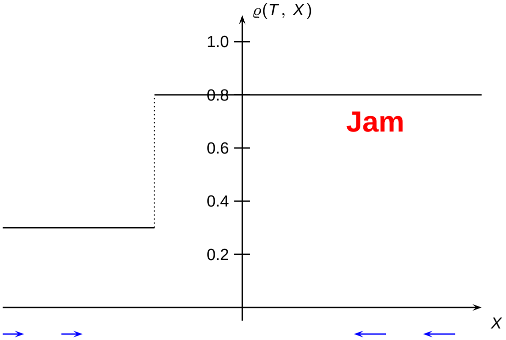

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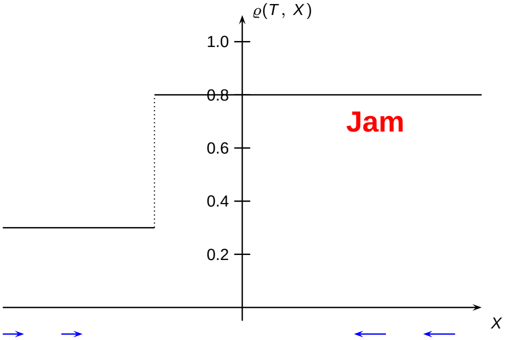

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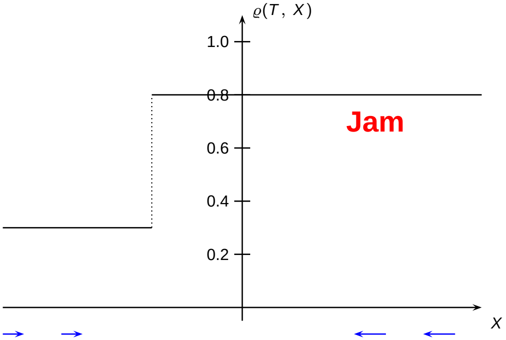

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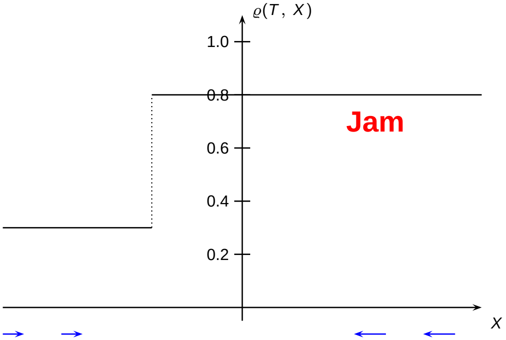

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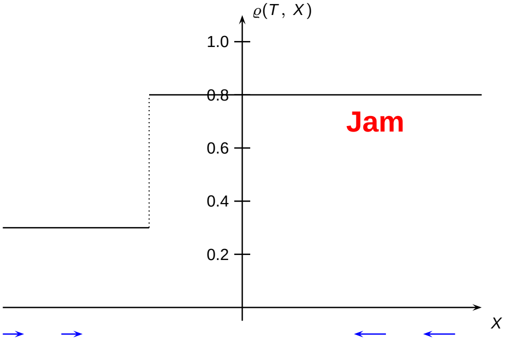

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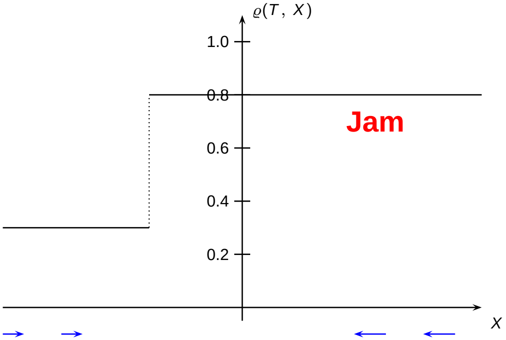

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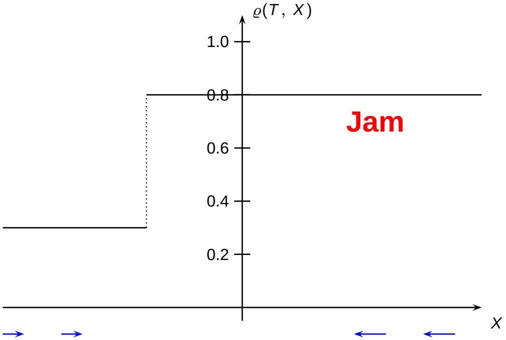

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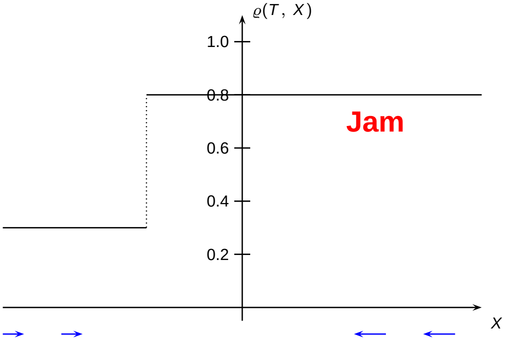

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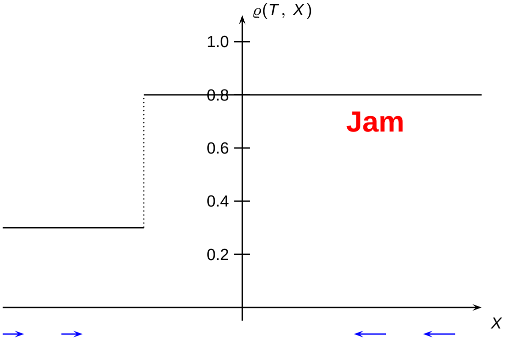

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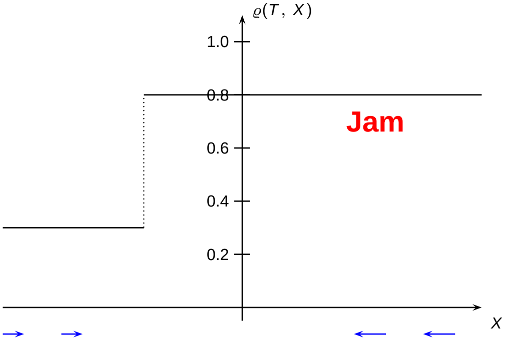

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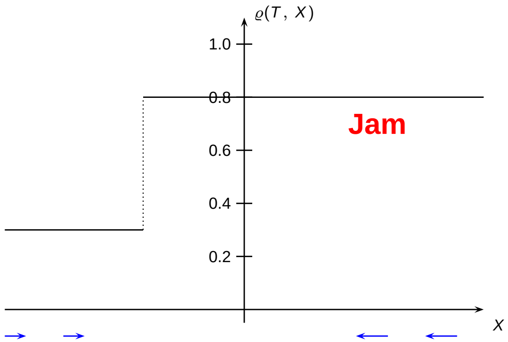

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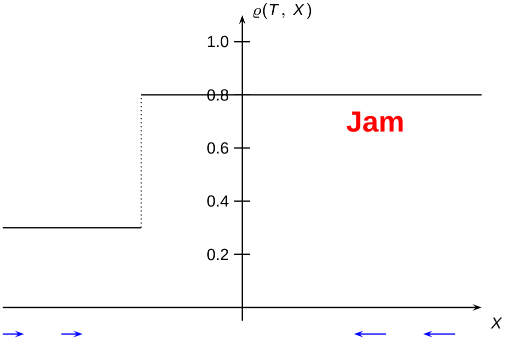

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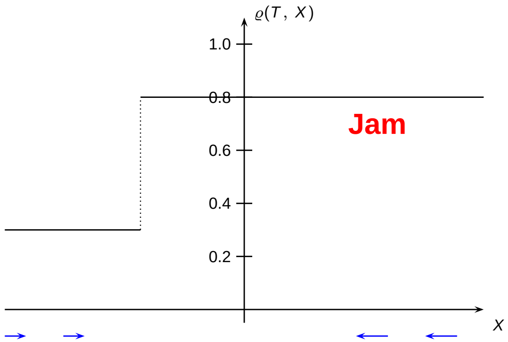

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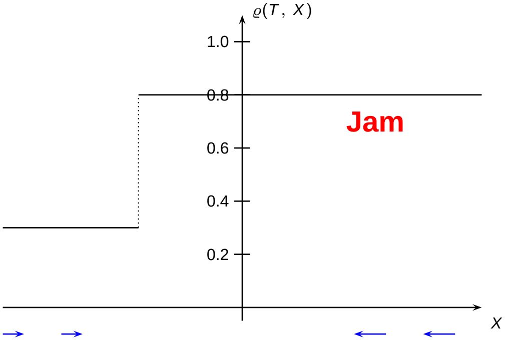

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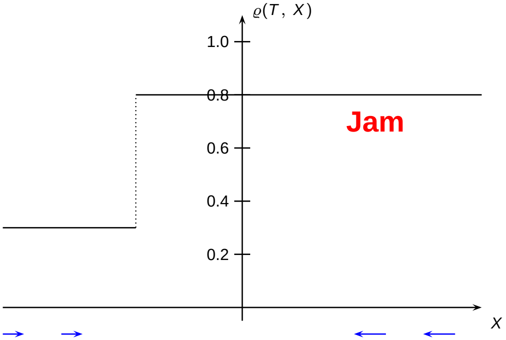

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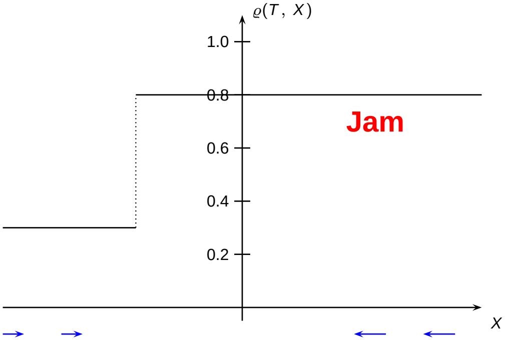

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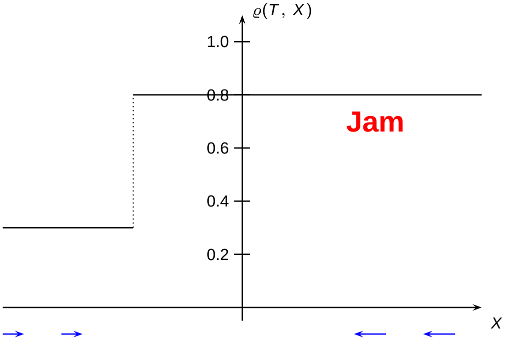

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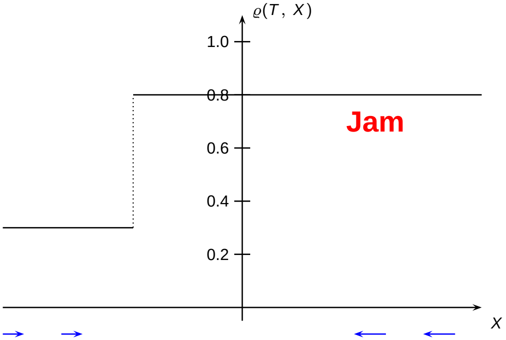

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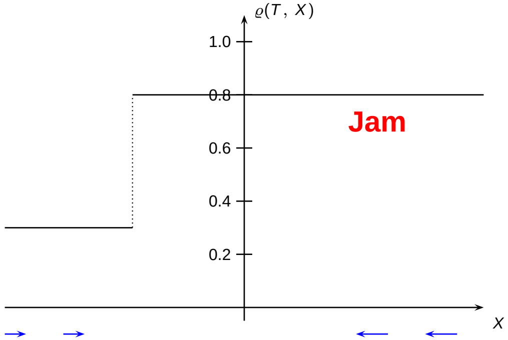
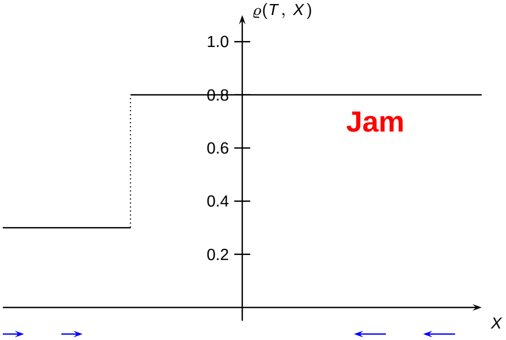


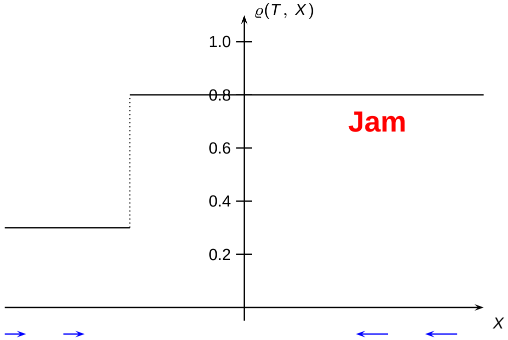
$H'(\varrho)$ ↘ (H concave)

Shock



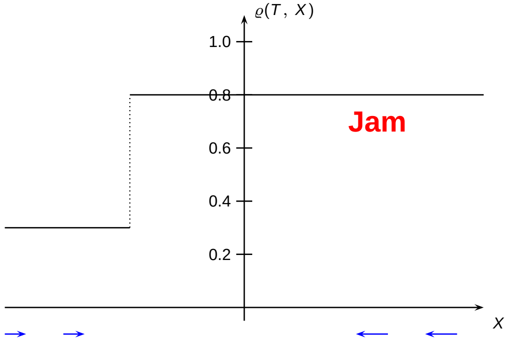
$H'(\varrho) \searrow$ (H concave)

Shock



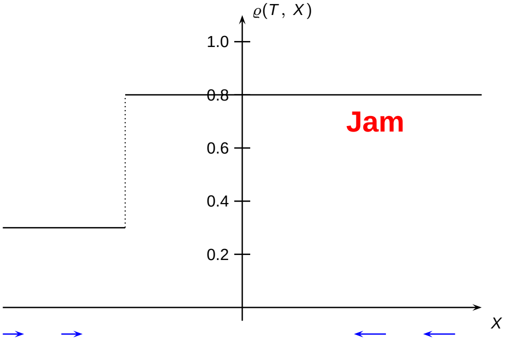
$H'(\varrho) \searrow$ (H concave)

Shock



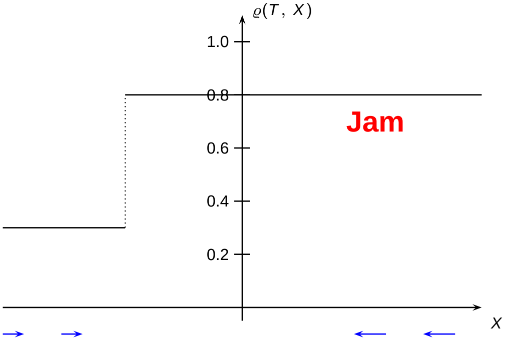
$H'(\varrho) \searrow$ (H concave)

Shock



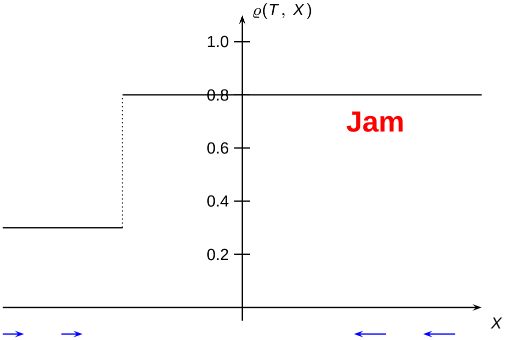
$H'(\varrho)$ ↘ (H concave)

Shock



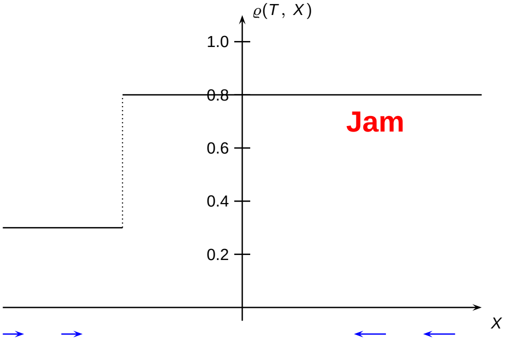
$H'(\varrho) \searrow$ (H concave)

Shock



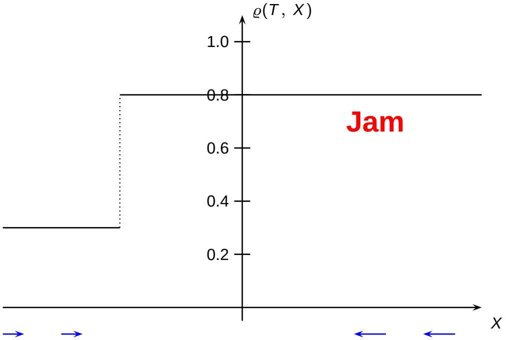
$H'(\varrho) \searrow$ (H concave)

Shock



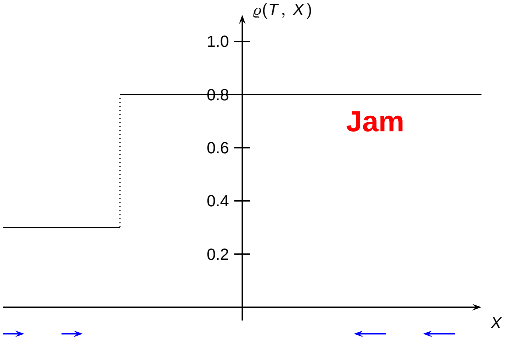
$H'(\varrho) \searrow$ (H concave)

Shock



$H'(\varrho) \searrow$ (H concave)

Shock



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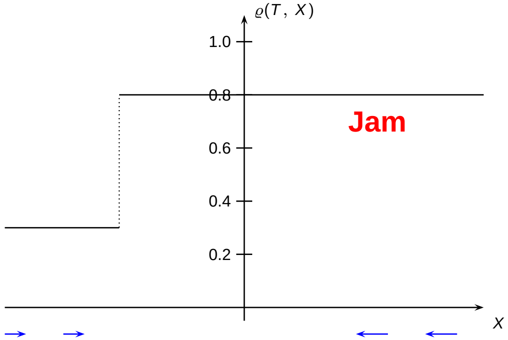
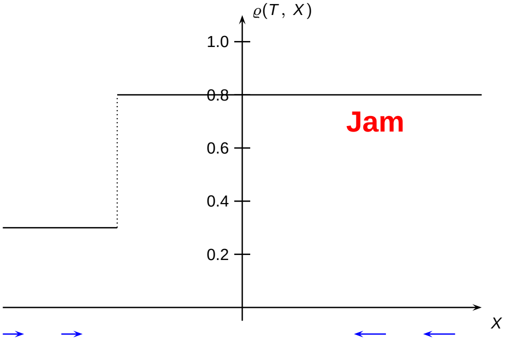
Shock



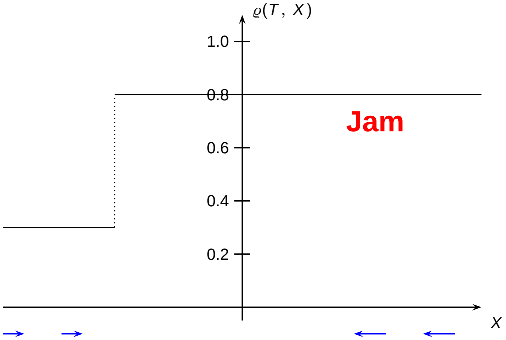
$H'(\varrho) \searrow$ (H concave)

Shock



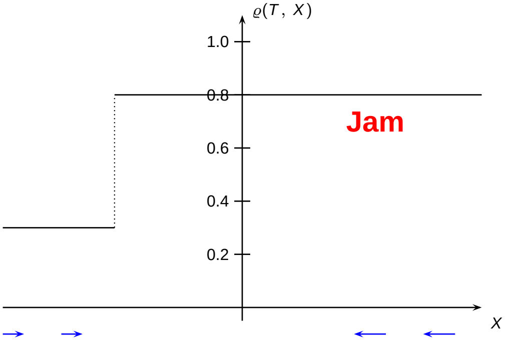
$H'(\varrho) \searrow$ (H concave)

Shock



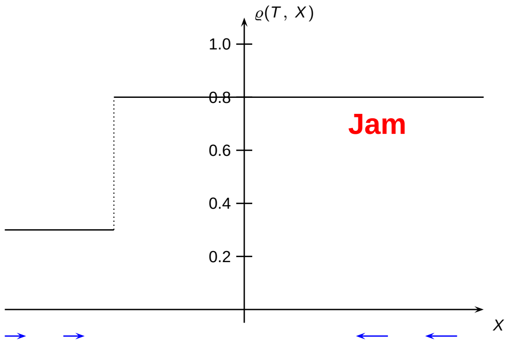
$H'(\varrho) \searrow$ (H concave)

Shock



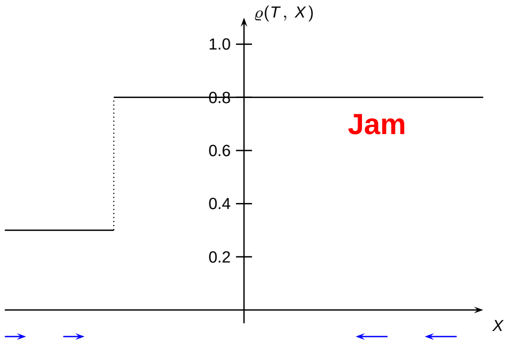
$H'(\varrho) \searrow$ (H concave)

Shock



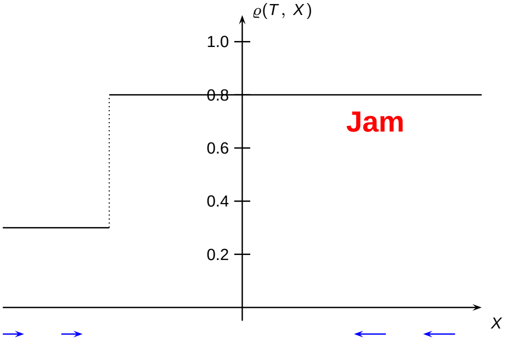
$H'(\varrho) \searrow$ (H concave)

Shock



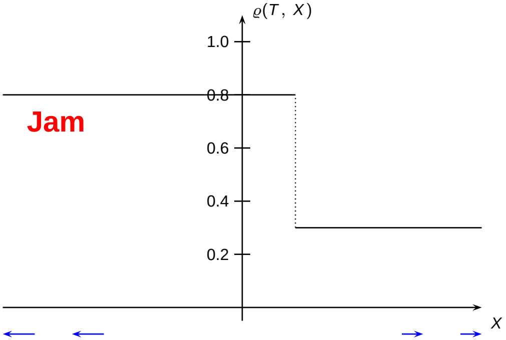
$H'(\varrho) \searrow$ (H concave)

Shock



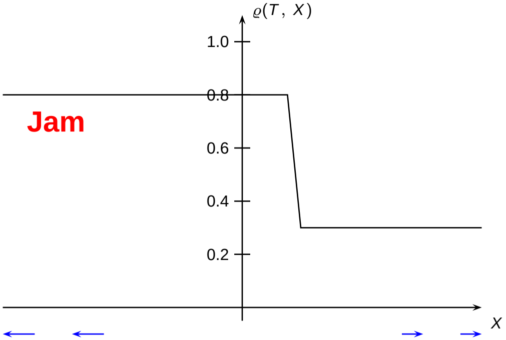
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



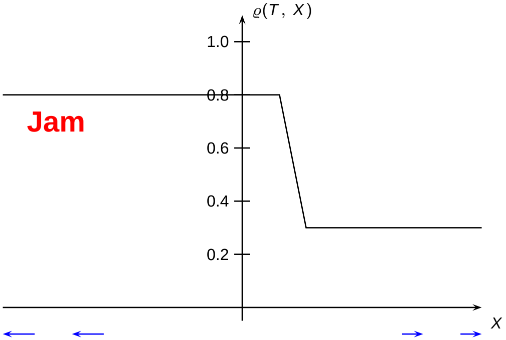
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



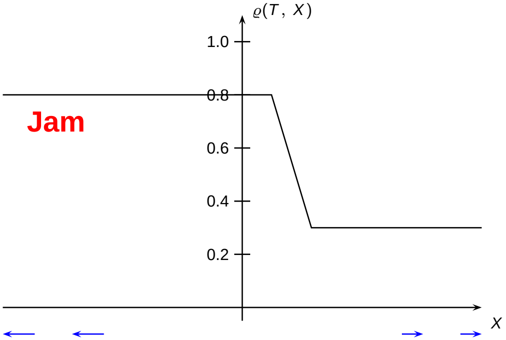
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



$H'(\varrho) \searrow$ (H concave)

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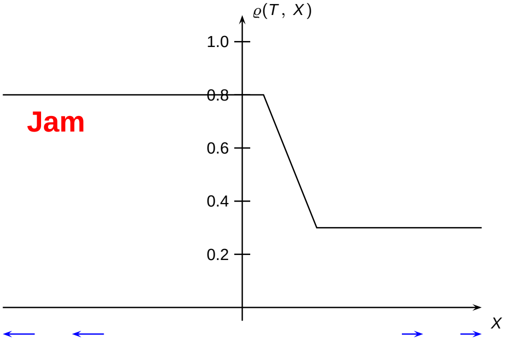

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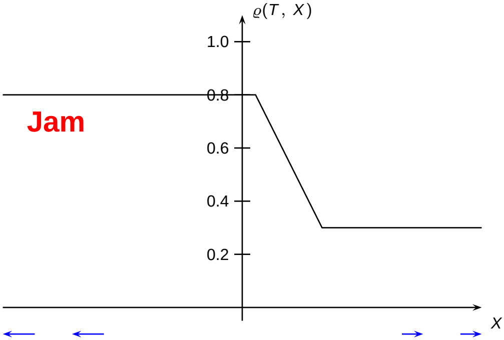
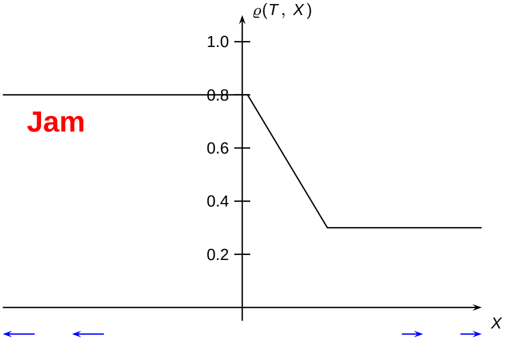
Rarefaction fan



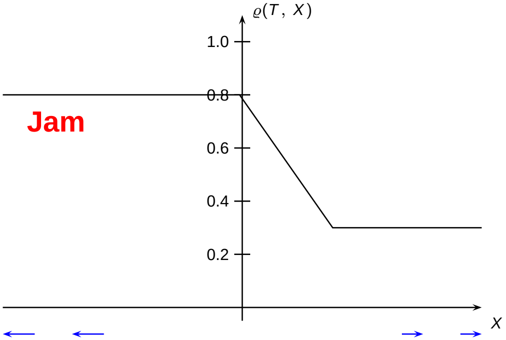
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



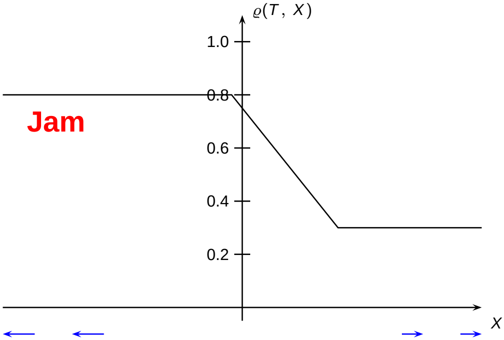
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



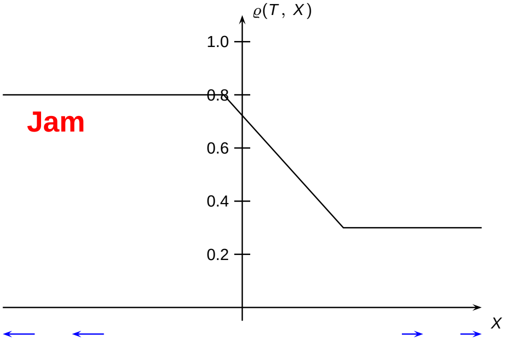
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



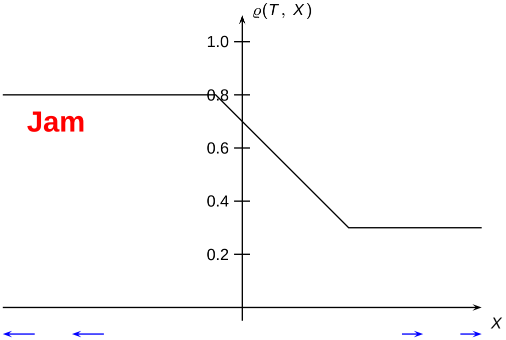
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



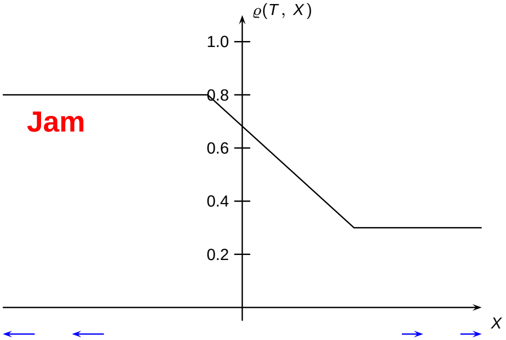
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



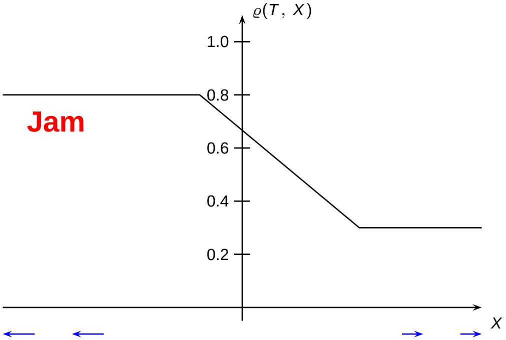
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



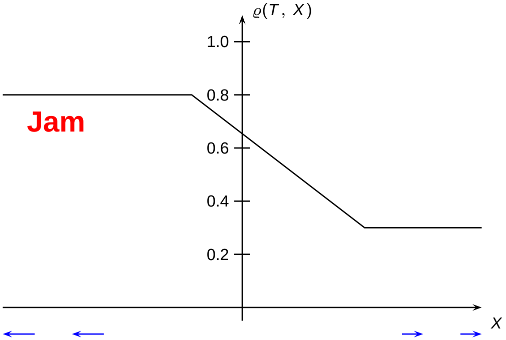
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



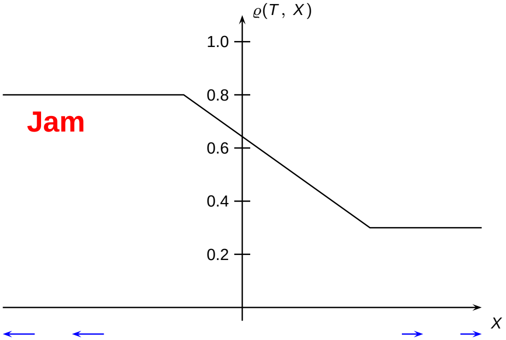
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



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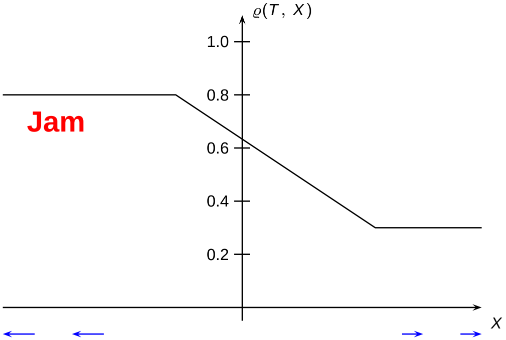
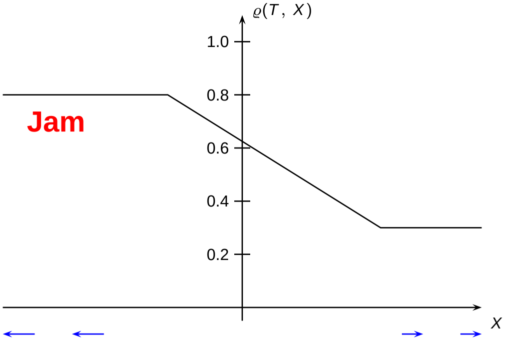


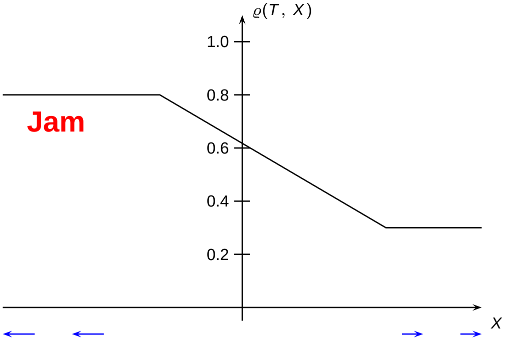
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



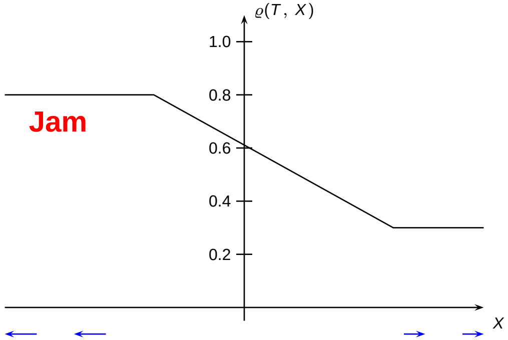
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



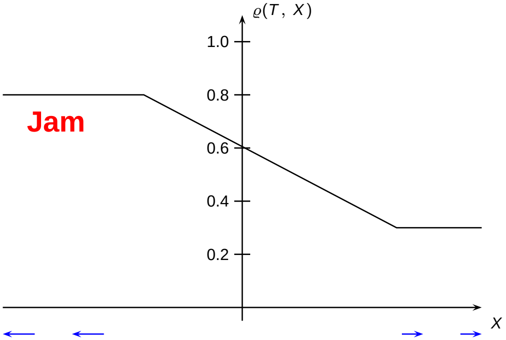
$H'(\varrho)$ ↘ (H concave)

Rarefaction fan



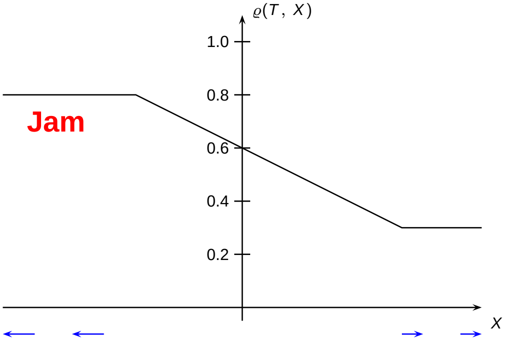
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



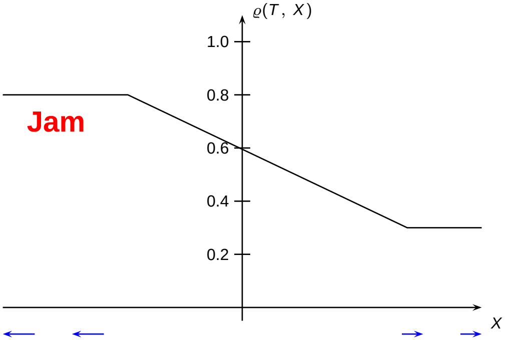
$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



$H'(\varrho) \searrow$ (H concave)

Rarefaction fan



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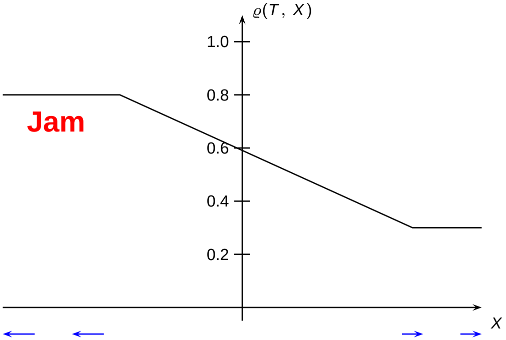
Rarefaction fan



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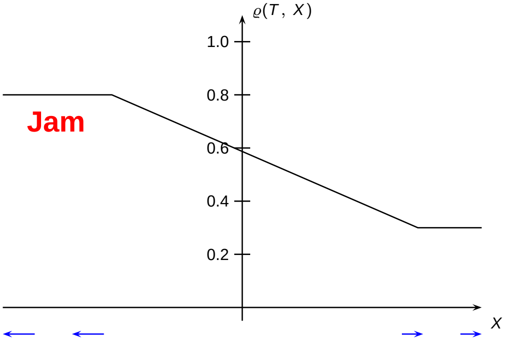

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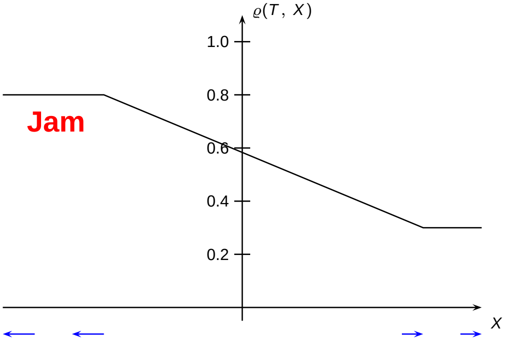
Rarefaction fan



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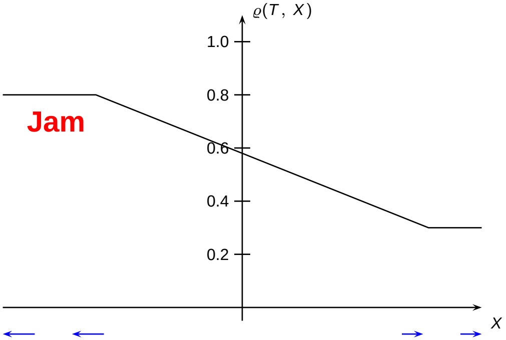
Rarefaction fan



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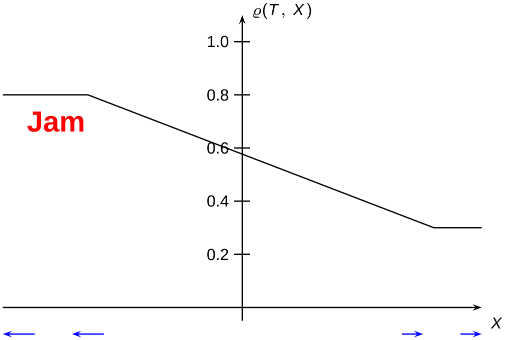

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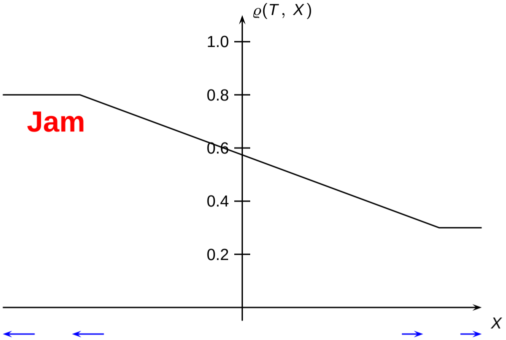

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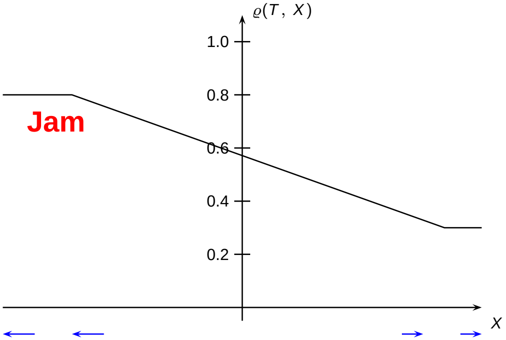

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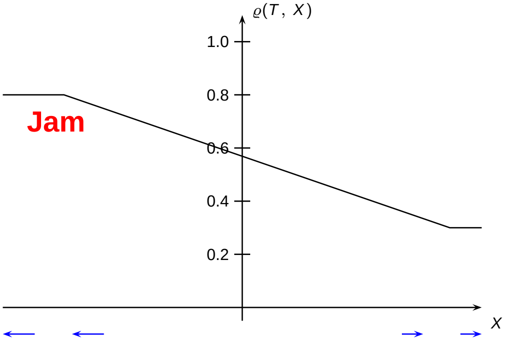

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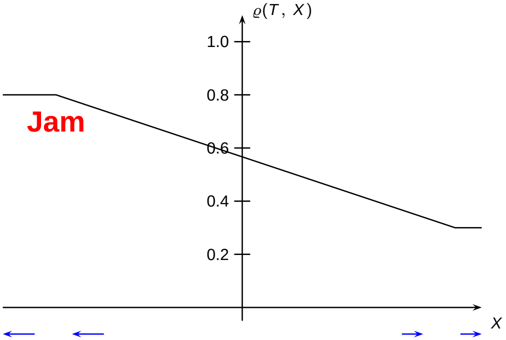
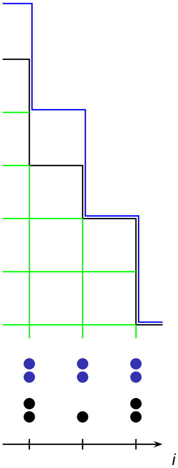


$H'(\varrho) \searrow$ (H concave)

The second class particle

States ω and η only differ at one site.



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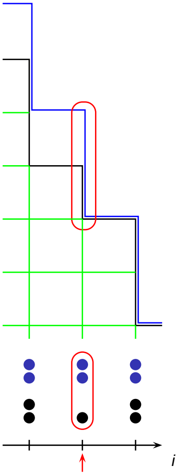

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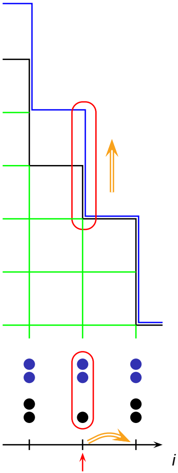
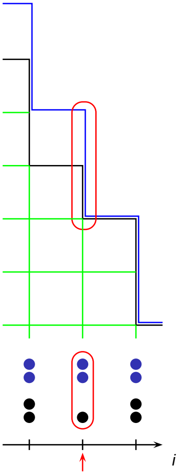


Growth on the right:

rate$\leq$rate

The second class particle

States ω and η only differ at one site.



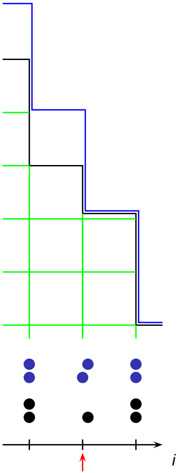
Growth on the right:

rate$\leq$rate

with rate:

The second class particle

States ω and η only differ at one site.



Growth on the right:
rate$\leq$rate
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The second class particle

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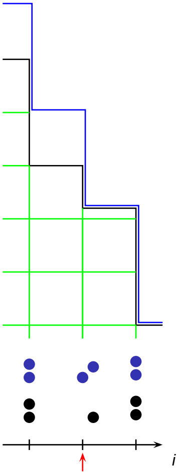

Growth on the right:
rate≤rate
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The second class particle

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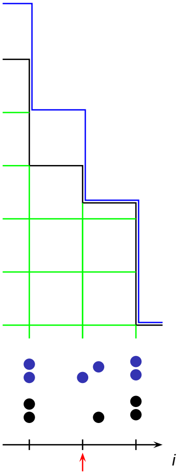

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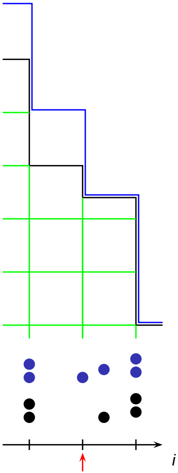

Growth on the right:

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with rate:

The second class particle

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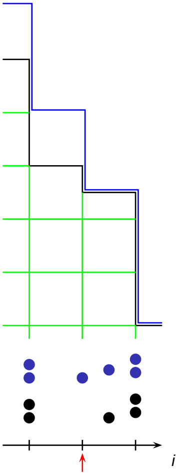


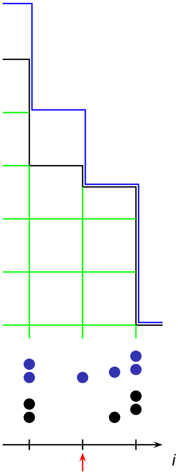
Growth on the right:

rate$\leq$rate

with rate:

The second class particle

States ω and η only differ at one site.



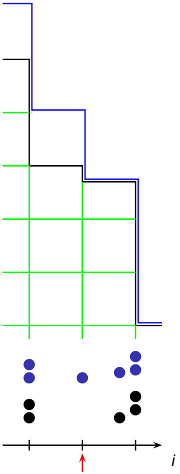
Growth on the right:

rate$\leq$rate

with rate:

The second class particle

States ω and η only differ at one site.



Growth on the right:

rate$\leq$rate

with rate:

The second class particle

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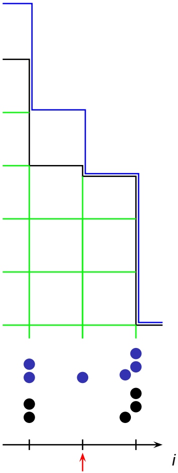

Growth on the right:

rate$\leq$rate

with rate:

The second class particle

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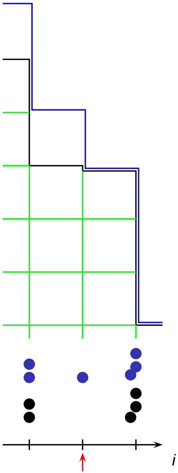
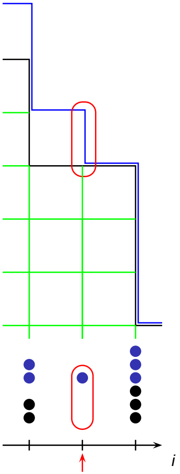


Growth on the right:
rate$\leq$rate
with rate:

The second class particle

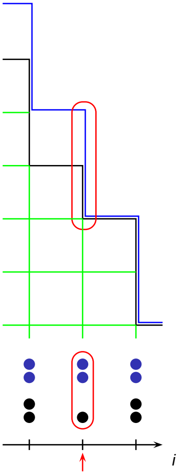
States ω and η only differ at one site.



Growth on the right:
rate≤rate
with rate:

The second class particle

States ω and η only differ at one site.



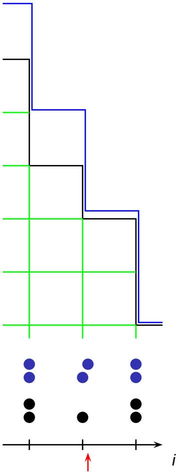
Growth on the right:

rate$\leq$rate

with rate-rate:

The second class particle

States ω and η only differ at one site.



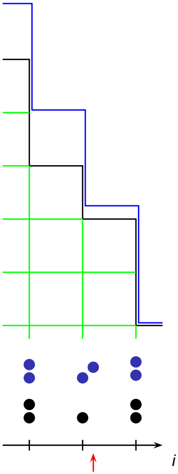
Growth on the right:

rate$\leq$rate

with rate-rate:

The second class particle

States ω and η only differ at one site.



Growth on the right:

rate$\leq$rate

with rate-rate:

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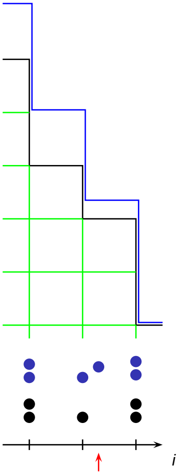

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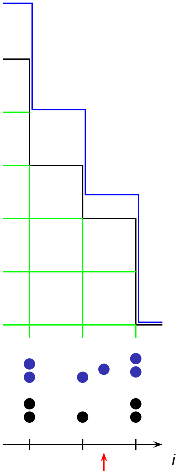

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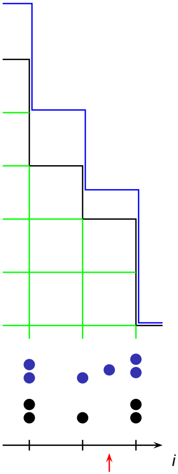

Growth on the right:

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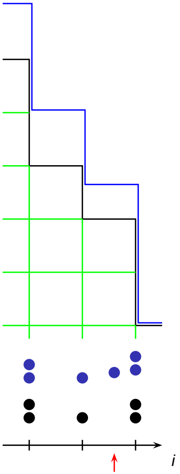

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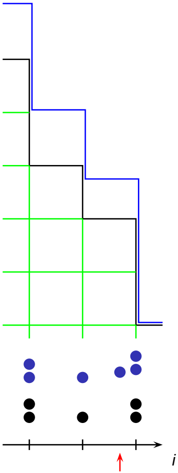

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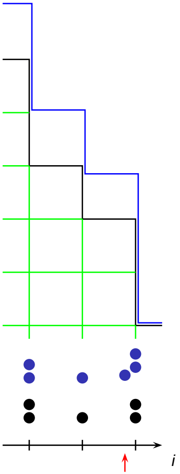

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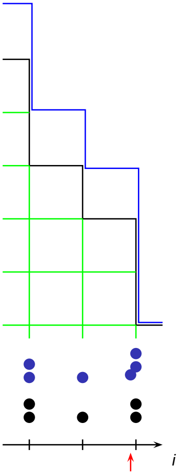

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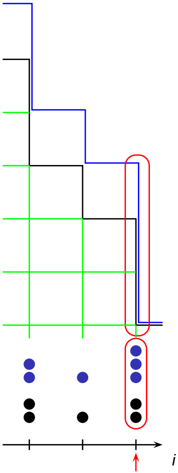

Growth on the right:
rate$\leq$rate
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The second class particle

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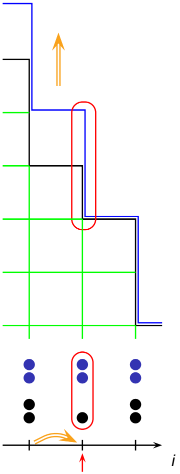
Growth on the left:
rate$\geq$rate



The second class particle

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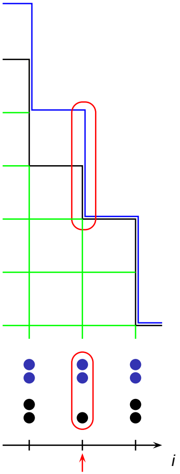

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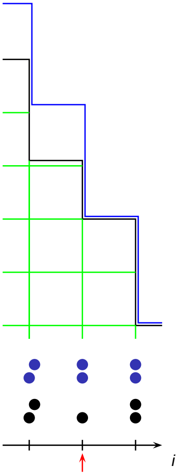

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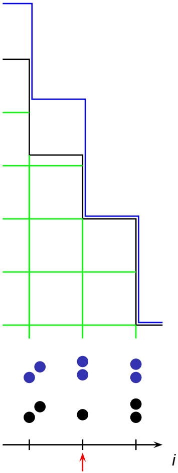

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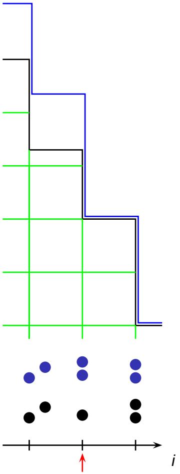

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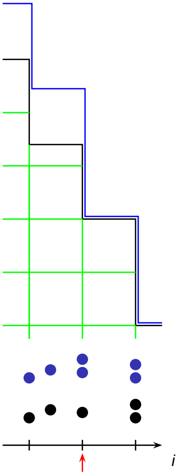

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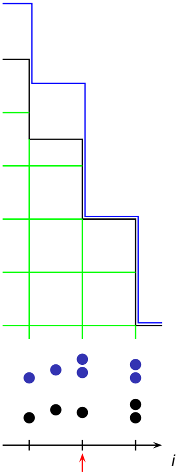

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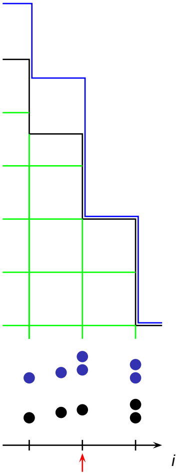

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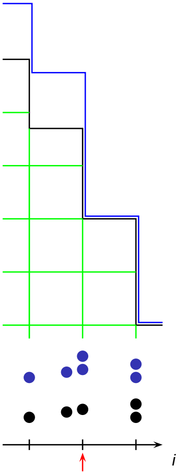

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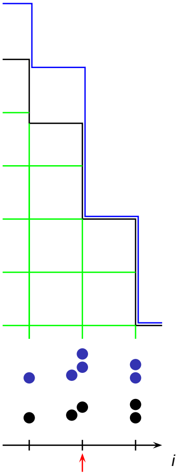

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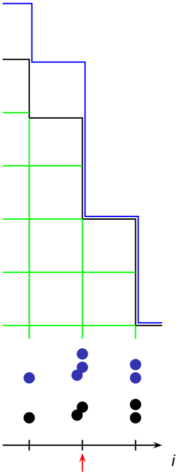

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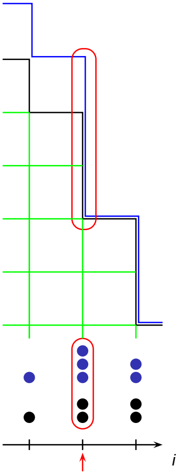

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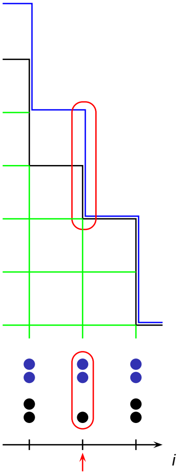

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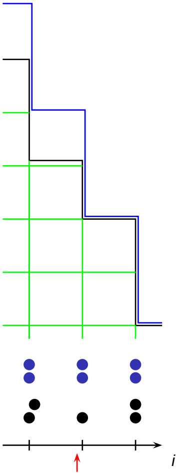

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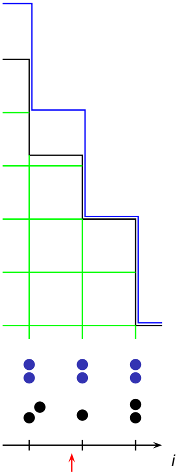

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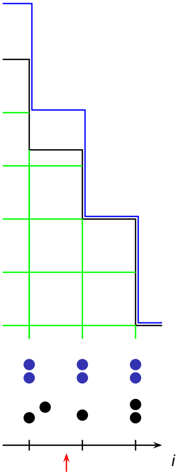

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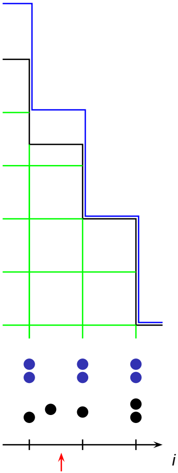

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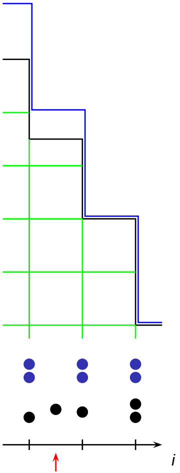

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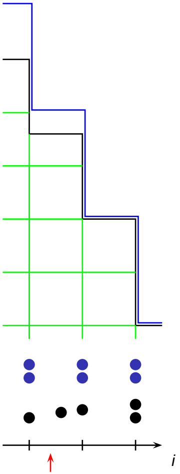

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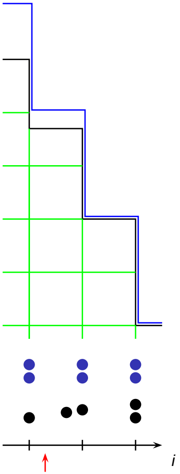

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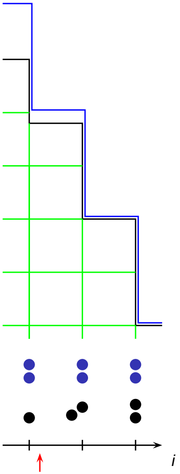

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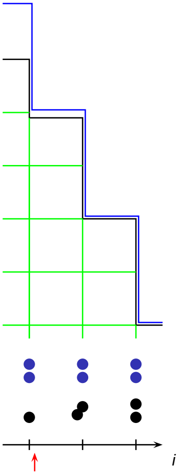

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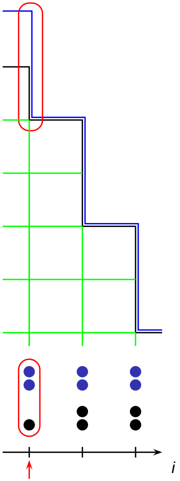

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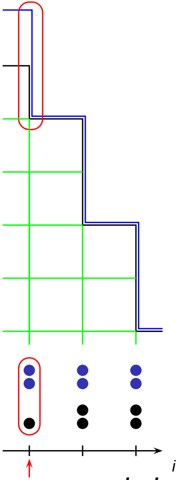
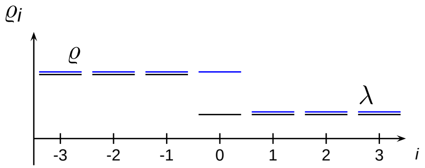


A single discrepancy↑, the *second class particle*, is conserved.
Its position at time t is $Q(t)$.

Ferrari-Kipnis '95 for TASEP

Blue TASEP ω:
Bernoulli(ϱ) for sites $\{\dots, -2, -1, 0\}$,
Bernoulli(λ) for sites $\{1, 2, 3, \dots\}$.

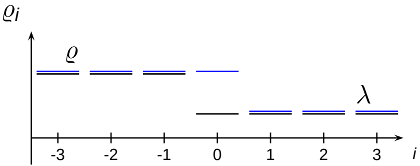
Black TASEP η:
Bernoulli(ϱ) for sites $\{\dots, -3, -2, -1\}$,
Bernoulli(λ) for sites $\{0, 1, 2, \dots\}$.



$h_i(t)$, $g_i(t)$ are the respective numbers of particles jumping over the edge $(i, i+1)$ by time t ($i > 0$).

Ferrari-Kipnis ’95 for TASEP, Part 1

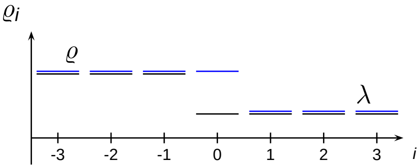
First realization:



Ferrari-Kipnis ’95 for TASEP, Part 1

First realization:

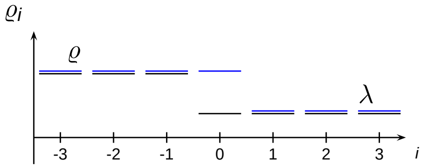
- $\omega_i(0) = \eta_i(0) \sim$ Bernoulli(ϱ) for $i < 0$



Ferrari-Kipnis ’95 for TASEP, Part 1

First realization:

- $\omega_i(0) = \eta_i(0) \sim \text{Bernoulli}(\varrho)$ for $i < 0$
- $(\omega_0(0), \eta_0(0)) = (0, 0)$ w. prob. $1 - \varrho$
  $(\omega_0(0), \eta_0(0)) = (1, 0)$ w. prob. $\varrho - \lambda$
  $(\omega_0(0), \eta_0(0)) = (1, 1)$ w. prob. λ



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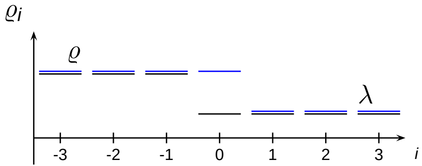

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  $(\omega_0(0), \eta_0(0)) = (1, 0)$ w. prob. $\varrho - \lambda$ 2^{nd} class particle
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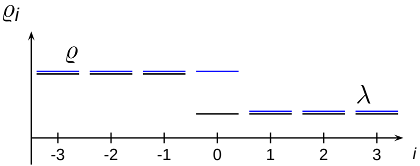

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  $(\omega_0(0), \eta_0(0)) = (1, 0)$ w. prob. $\varrho - \lambda$ — 2^{nd} class particle
  $(\omega_0(0), \eta_0(0)) = (1, 1)$ w. prob. λ
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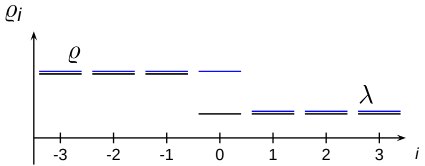
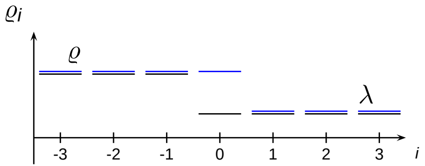


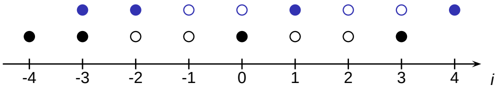
$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}\{Q(t) > i\}.$$

Ferrari-Kipnis ’95 for TASEP, Part 2

Second realization:

$$\omega_i(t) \equiv \eta_{i-1}(t) \qquad \forall i,\ \forall t.$$





Ferrari-Kipnis ’95 for TASEP, Part 2

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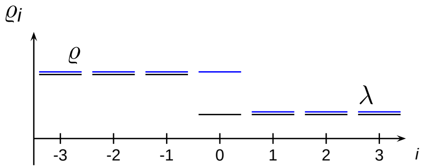

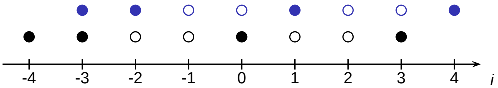

$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = \mathbf{E}(\eta_i(t) - \eta_i(0)) = \mathbf{E}\eta_i(t) - \mathbf{E}\eta_i(0).$$

Ferrari-Kipnis '95 for TASEP

Thus,

$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = (\varrho - \lambda) \cdot \mathbf{P}\{Q(t) > i\},$$
$$\mathbf{E}h_i(t) - \mathbf{E}g_i(t) = \mathbf{E}(h_i(t) - g_i(t)) = \mathbf{E}\eta_i(t) - \mathbf{E}\eta_i(0),$$

Ferrari-Kipnis '95 for TASEP

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Ferrari-Kipnis '95 for TASEP

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$$\mathbf{P}\{Q(t) > i\} = \frac{\mathbf{E}\eta_i(t) - \mathbf{E}\eta_i(0)}{\varrho - \lambda}.$$

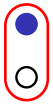
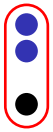
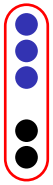
Combine with hydrodynamics to conclude

$$\frac{Q(t)}{t} \Rightarrow \begin{cases} \text{shock velocity} & \text{in a shock,} \\ \mathrm{U}\big(H'(\varrho), H'(\lambda)\big) & \text{in a rarefaction wave.} \end{cases}$$

Let's generalise

Other models have more than 0 or 1 particles per site. How do we start the second class particle?

- Shall we do

 ? Or  ? Or  ? Or . . .

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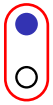
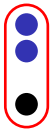
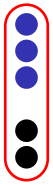
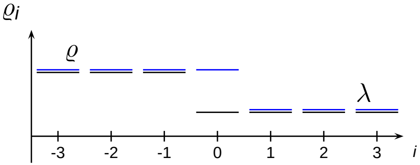
Other models have more than 0 or 1 particles per site. How do we start the second class particle?

- Shall we do

 ? Or  ? Or  ? Or ...

- Recall for TASEP we increased λ to ϱ by adding or not adding a 2^{nd} class particle.

$(\omega_0(0), \eta_0(0)) = (0, 0)$ w. prob. $1 - \varrho$
$(\omega_0(0), \eta_0(0)) = (1, 0)$ w. prob. $\varrho - \lambda$
$(\omega_0(0), \eta_0(0)) = (1, 1)$ w. prob. λ



Let’s generalise: problems with coupling

Fix $\lambda < \varrho \leq \lambda + 1$. Is there a joint distribution of (ω_0, η_0) such that

- the first marginal is $\omega_0 \sim$ stati. μ^ϱ;
- the second marginal is $\eta_0 \sim$ stati. μ^λ;
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- *Yes for discrete Gaussian (bricklayers with $r(\omega_i) = \mathrm{e}^{\beta\omega_i}$).*

Let's generalise

Keep calm and couple anyway.

Find a coupling measure ν with

- first marginal $\omega_0 \sim$ stati. μ^ϱ;
- second marginal $\eta_0 \sim$ stati. μ^λ;
- zero weight whenever $\omega_0 \notin \{\eta_0, \eta_0 + 1\}$.

Not many choices:

$$\nu(x, x) = \mu^\varrho\{-\infty \dots x\} - \mu^\lambda\{-\infty \dots x - 1\},$$
$$\nu(x + 1, x) = \mu^\lambda\{-\infty \dots x\} - \mu^\varrho\{-\infty \dots x\},$$
$$\nu = \text{zero elsewhere.}$$

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We can still use the *signed measure* ν formally, as we only care about $\nu(x+1, x)$. Scale this up to get the initial distribution at the site of the second class particle:

$$\mu(\omega_0, \eta_0) = \mu(\eta_0 + 1, \eta_0) = \frac{\nu(\eta_0+1, \eta_0)}{\sum_x \nu(x+1, x)} = \frac{\nu(\eta_0+1, \eta_0)}{\varrho - \lambda}.$$

Let's generalise

$$\mu(\omega_0, \eta_0) = \frac{\nu(\eta_0 + 1, \eta_0)}{\varrho - \lambda}$$

- is a proper probability distribution;
- actually agrees with the coupling measure ν conditioned on a 2^{nd} class particle when ν behaves nicely (Bernoulli, discr.Gaussian);
- allows the extension of Ferrari-Kipnis:

Let's generalise

Theorem

Starting in

$$\bigotimes_{i<0} \mu_i^{\varrho} \otimes \mu_0 \otimes \bigotimes_{i>0} \mu_i^{\lambda},$$

$$\lim_{N\to\infty} \mathbf{P}\left\{ \frac{Q(NT)}{N} > X \right\} = \frac{\varrho(X, T) - \lambda}{\varrho - \lambda}$$

where $\varrho(X, T)$ is the entropy solution of the hydrodynamic equation with initial data

ϱ *on the left*

λ *on the right.*

What do we have?

$$\lim_{N\to\infty} \mathbf{P}\left\{\frac{Q(NT)}{N} > X\right\} = \frac{\varrho(X, T) - \lambda}{\varrho - \lambda}$$

$\rightsquigarrow$ The solution $\varrho(X, T)$ is the distribution of the velocity for Q.

- Shock: distribution is step function, velocity is deterministic (LLN).
- Rarefaction wave: distribution is continuous, velocity is random (e.g., Uniform for TASEP).

A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

$\omega_i = -1, 0, 1;$

$$
\begin{aligned}
(0, -1) &\to (-1, 0) && \text{with rate } \frac{1}{2}, \\
(1, 0) &\to (0, 1) && \text{with rate } \frac{1}{2}, \\
(1, -1) &\to (0, 0) && \text{with rate } 1, \\
(0, 0) &\to (-1, 1) && \text{with rate } c.
\end{aligned}
$$

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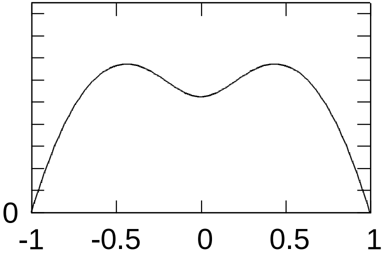
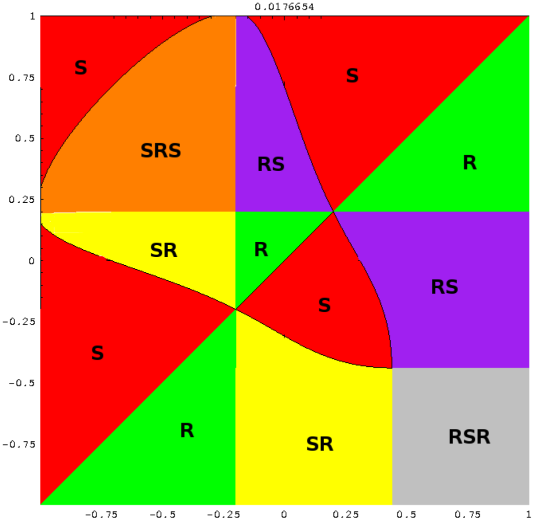
Hydrodynamic flux $H(\varrho)$, for certain c:



A fun model (B., A.L. Nagy, I. Tóth, B. Tóth)

Here is what can happen (**R**: rarefaction wave, **S**: Shock):



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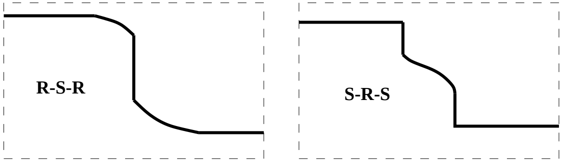
Examples for $\varrho(T, X)$:



$$\lim_{N\to\infty} \mathbf{P}\left\{\frac{Q(NT)}{N} > X\right\} = \frac{\varrho(X, T) - \lambda}{\varrho - \lambda}$$

$\rightsquigarrow$ The solution $\varrho(X, T)$ is the distribution of the velocity for Q.

I haven't seen a walk with a random velocity of *mixed distribution* before.

A few more remarks

- This work sheds light on a measure $\hat{\mu}$ we came up with in the 1/3-fluctuations papers (B., J. Komjáthy, T. Seppäläinen). *At that time we had no idea why $\hat{\mu}$.* It just worked nice with our formulas.

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