



# Hydrodynamics of a non-gradient model for collective dynamics

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# Plan of the talk

## Collective motion & Active matter

- Collective dynamics

- Alignment phase transition

- MIPS

## Model description

- Initial setup

- Description of the dynamics

## Hydrodynamics

- Main result

- Non-gradient hydrodynamics

- Irreducibility

## Continuous angle dynamic

- Extension of the model



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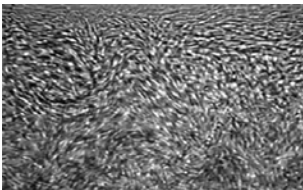
## Continuous angle dynamic

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## Collective motion & Active matter

Collective behavior can be observed among numerous animal species

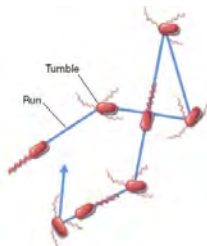




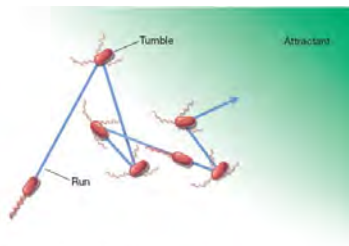
→ Classical representation : *Individual Based Models* (IBM) built around **active matter**.

## Active Matter

System composed of many individuals, maintained out of equilibrium by an *energy influx at the individual level*.



(a) No attractant present: Random movement



(b) Attractant present: Directed movement



Two types of phenomena can arise in active matter models :

→ *Alignment phase transition*

→ *Motility Induced Phase Separation (MIPS).*



# Alignment phase transition

## Alignment

Alignment dynamics will create *groups of particles moving together* and adapting their speed

Original model by Vicsek&al. (1995, *Novel type of phase transition in a system of self driven particles*)

$\mapsto N = \rho L^2$  particles move in the periodic domain  $[0, L]^2$ , with speed  $v_i(t) = v \vec{e}_{\theta_i(t)}$

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t)\Delta t \\ \theta_i(t+1) = \langle \theta(t) \rangle_r + \xi_\eta \end{cases}$$

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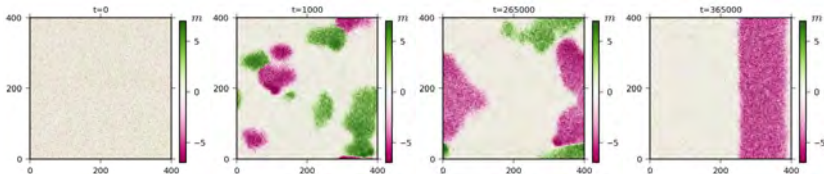




## Numerical results on related models

 Solon & Tailleur, 2013

 Solon & al., 2015



**Figure:** Emergence of global order for an alignment dynamics. From *Flocking with discrete symmetry, the 2d active Ising model*, Solon & Tailleur 2015.



## Exact works

Several analysis and PDE based papers, based on *mean-field interactions* :

→ Each particle interacts with a large number of neighbors

 [Degond, Motsch, 2007](#)

 [Bolley, Carrillo, Canizo, 2011](#)





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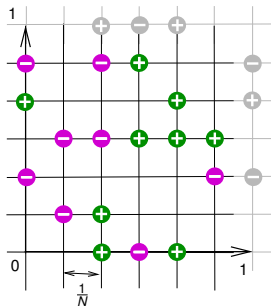
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## Description of the particle system

Two types of particles (+ and -) evolve on the **periodic square lattice**  $\mathbb{T}_N^2 = \{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}^2$

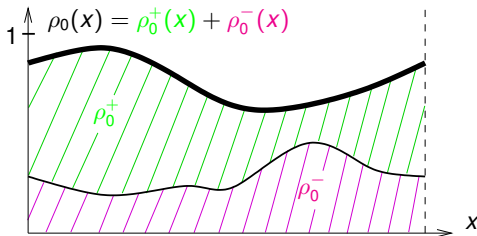
- For each site  $x \in \mathbb{T}_N^2$ , we define  $\eta_x \in \{-1, 0, 1\}$ .  
 $\mapsto \eta_x = 0$ , empty site  
 $\mapsto \eta_x = \pm 1$  site occupied by a particle  $\pm$
- We let  $\eta_x = \eta_x^+ - \eta_x^-$ , where  $\eta_x^\pm = 1$  iff  $x$  is occupied by a  $\pm$  particle.





## Initial configuration

- **Initial macroscopic smooth profiles**  $\rho_0^+$  and  $\rho_0^-$ ,  $[0, 1]^2 \rightarrow [0, 1]$ .
- $\rho_0 = \rho_0^+ + \rho_0^-$  is the initial particle density



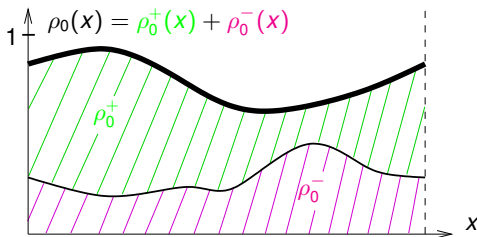
- Initial *local equilibrium* : independently for any  $x \in \mathbb{T}_N^2$

$$\eta_x(0) = \begin{cases} \pm 1 & \text{w.p. } \rho_0^\pm(x) \\ 0 & \text{w.p. } 1 - \rho_0^+(x) - \rho_0^-(x) \end{cases}$$



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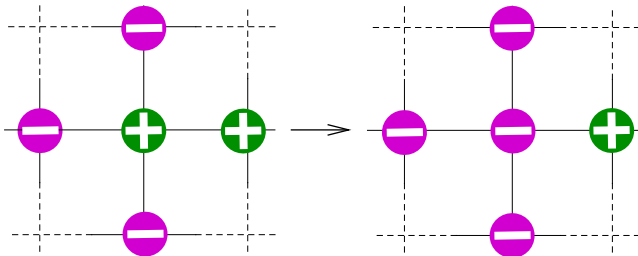
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## Glauber dynamics



⇒ *Ising type* alignment dynamics with inverse temperature  $\beta$

- $\beta = 0$ , no alignment
- $\beta \rightarrow \infty$ , strong alignment



## Generator of the dynamic

The Markov generator of the process is given by

$$L_N = N^2 \mathcal{L}^S + \hat{\lambda} N \mathcal{L}^A + \mathcal{L}^G,$$

- $\mathcal{L}^S$  : generator of the *Symmetric Simple Exclusion Process* (SSEP)

$$\mathcal{L}^S f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{|z|=1} |\eta_x| \underbrace{(1 - |\eta_{x+z}|)}_{\text{Exclusion Rule}} (f(\eta^{x,x+z}) - f(\eta)).$$

- Diffusive scaling :  $\times N^2$ .



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$$L_N = N^2 \mathcal{L}^S + \hat{\lambda} N \mathcal{L}^A + \mathcal{L}^G,$$

- $\mathcal{L}^A$  : generator of the *Asymmetric Simple Exclusion Process* (WASEP)

$$\mathcal{L}^A f(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{\delta = \pm 1} \delta \eta_x \underbrace{(1 - |\eta_{x+\delta \mathbf{e}_1}|)}_{\text{Exclusion Rule}} \left( f(\eta^{x, x+\delta \mathbf{e}_1}) - f(\eta) \right),$$

- Ballistic scaling :  $\times N$ .



## Generator of the dynamic

$$L_N = N^2 \mathcal{L}^S + \hat{\lambda} N \mathcal{L}^A + \mathcal{L}^G,$$

- $\mathcal{L}^G$  : generator of the *Glauber alignment dynamics*

$$\mathcal{L}^G f(\eta) = \sum_{x \in \mathbb{T}_N} c_\beta(x, \eta) |\eta_x| (f(\eta^x) - f(\eta)).$$

- No need for rescaling.

The complete generator and the initial state define a Markov process  $(\eta(t))_{t \in [0, T]}$ .



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## Heuristic formulation of the macroscopic limit

### Theorem

**Assumption :**  $\forall u \in [0, 1]^2$ ,  $\rho_0(u) = \rho_0^+(u) + \rho_0^-(u) < 1$ .

The *macroscopic density* of particles  $+$ , denoted  $\rho^+(t, u)$ , is solution *in a weak sense* of the reaction-diffusion equation

$$\partial_t \rho^+ = \nabla \cdot [d_s(\rho) \nabla \rho^+ + D(\rho^+, \rho) \nabla \rho] + 2\hat{\lambda} \partial_{x_1} \sigma(\rho^+, \rho) + \Gamma_\beta(\rho^+, \rho),$$

and

$$\rho^+(0, u) = \rho_0^+(u).$$

An analogous equation is verified by the  $-$  particle density  $\rho^-$ , and  $\rho = \rho^+ + \rho^-$  is the total particle density.



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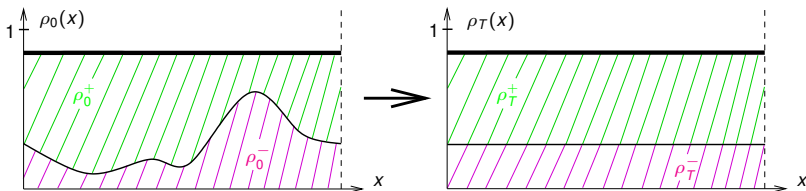
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## Dynamical interpretation

$$\partial_t \rho^+ = \nabla \cdot \left[ d_s(\rho) \nabla \rho^+ + D(\rho^+, \rho) \nabla \rho \right] + 2\hat{\lambda} \partial_{x_1} \sigma(\rho^+, \rho) + \Gamma_\beta(\rho^+, \rho).$$

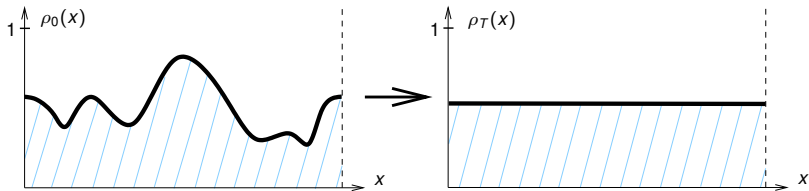


$d_s$  is the **self-diffusion coefficient** of a tracer particle in an homogeneous environment.



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The coefficient  $D$  quantifies the **diffusion** due to heterogeneities of the **total particle density**

$$D(\rho^+, \rho) = \frac{\rho^+}{\rho} (1 - d_s(\rho)) \quad (\text{Quastel, 1992}).$$

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↳ The drift  $\sigma$  can be expressed as

$$\sigma(\rho^+, \rho) = \frac{\rho^+}{\rho} (1 - \rho - d_s(\rho)) (\rho^+ - \rho^-) + \rho^+ d_s(\rho),$$

and is linked to  $D$  and  $d_s$  by the Stokes-Einstein relation.



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- $\Gamma_\beta$  is the creation rate of "+" particles.
- It depends on the alignment jump rates  $c_\beta$

# Hydrodynamic limit

## Empirical measures

$$\pi_t^{+,N} = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x^+(t) \delta_x, \quad \text{and} \quad \pi_t^{-,N} = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \eta_x^-(t) \delta_x.$$

We want to prove  $\pi_t^{+,N} \xrightarrow{\mathbb{P}} \pi_t^+ = \rho^+(t, x) dx$ , i.e. that for any smooth  $H$ ,

$$\langle \pi_t^{+,N}, H \rangle \rightarrow \int_{[0,1]^2} \rho^+(t, x) H(x) dx$$

**Core principle :**

$$\langle \pi_T^{+,N}, H \rangle = \langle \pi_0^{+,N}, H \rangle + \int_0^T L_N \langle \pi_t^{+,N}, H \rangle dt + \overbrace{M_T^N}^{o_N(1)}$$

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## Hydrodynamic limit

Assume we are in **one dimension** :

$$\begin{aligned}
 L_N \langle \pi_t^{+,N}, H \rangle &= \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x) L_N \eta_x^+ \\
 &= \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x) \left( W_{x-\frac{1}{N}, x} - W_{x, x+\frac{1}{N}} + \gamma_{\beta, x} \right) \\
 &= \frac{1}{N} \sum_{x \in \mathbb{T}_N} \left[ \left( H(x) - H\left(x + \frac{1}{N}\right) \right) W_{x, x+\frac{1}{N}} + H(x) \gamma_{\beta, x} \right]
 \end{aligned}$$

and the particle current can be written

$$W_{x, x+\frac{1}{N}} = N^2 \underbrace{w_{x, x+\frac{1}{N}}^S}_{\text{symmetric current}} + \hat{\lambda} N \underbrace{w_{x, x+\frac{1}{N}}^A}_{\text{antisymmetric current}} .$$





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## Non-gradient hydrodynamics

- The partial derivative on  $H$  balances out a factor  $N$  in the current.
- In ***non-gradient systems***, the symmetric current  $w_{x,x+\frac{1}{N}}^S$  is not a discrete gradient.  
 $\mapsto$  the second integration by parts is not immediate
- One must prove

$$w_{x,x+\frac{1}{N}}^S \simeq D(\eta_x - \eta_{x+1}) + d_s(\eta_x^+ - \eta_{x+1}^+) + \mathcal{L}^S f$$



# Out of equilibrium dynamics

## Key issue

Comparison with a product measure on the discrete lattice :

- Distortion of the measure by the Glauber part and the initial configuration are easily controlled
- Distortion by the weak drift, harder to control

⇒ Challenge : prove that the exponential estimates needed in the *non-gradient* method still hold.



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# Irreducibility

Exclusion rule : mixing is compromised in full clusters

- One must prove that the dynamics spreads enough empty sites to ensure mixing.
- This was a major issue with the model.
- The needed estimate is

$$\mathbb{E} \left( \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \mathbb{1}_{\text{Empty site in } \tau_x B_p} \right) \simeq \int_{[0,1]^2} dx \rho_x^{|B_p|} \xrightarrow{p \rightarrow \infty} 0. \quad (1)$$



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### Theorem, continuous angles

The macroscopic density  $\rho^\theta$  of particles with angle  $\theta$  is solution in a weak sense of

$$\partial_t \rho^\theta = \frac{1}{2} \nabla \left[ d_s(\rho) \nabla \rho^\theta + D(\rho^\theta, \rho) \nabla \rho \right] + \hat{\lambda} \nabla \sigma(\rho^\theta, \rho) + \Gamma^\theta$$

- $\rho = \int_\theta \rho^\theta d\theta$  is the total particle density
- $\Gamma^\theta$  alignment term, continuous diffusion, jump process



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Thanks for your attention !

