On the asymptotic behavior of slowed exclusion processes

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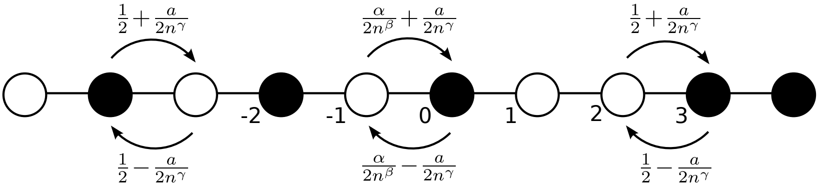
Slowed exclusion processes: the dynamics

- η_t is an exclusion process with space state $\Omega = \{0,1\}^{\mathbb{Z}}$, so that for $x \in \mathbb{Z}$, $\eta(x) = 1$ if the site is occupied, otherwise $\eta(x) = 0$.

The rates are:



We assume $\gamma > \beta$ or $\beta = \gamma$ and $\alpha \geq a$ (in last case if $a = \alpha$ then $\{-1, 0\}$ is totally asymmetric).

- For $a = 0$, we obtain the SSEP with a slow bond.
- For $\alpha = 1$ and $\beta = 0$ we obtain the WASEP - weak asymmetry.
- ν_ρ the Bernoulli product measure of parameter ρ is invariant.

Hydrodynamic limit: the case $a = 0$

• For η let $\pi_t^n(\eta; du) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \eta_{tn^2}(x)\delta_{x/n}(du)$.

• Fix $\rho_0 : \mathbb{R} \to [0,1]$ and μ_n such that for every $\delta > 0$ and every continuous function $H : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{n}\sum_{x\in\mathbb{Z}} H\left(\tfrac{x}{n}\right)\eta(x) \to_{n\to\infty} \int_{\mathbb{R}} H(u)\,\rho_0(u)du,$$

wrt μ_n. Then for any $t > 0$, $\pi_t^n \to \rho(t,u)du$, as $n \to \infty$, where $\rho(t,u)$ evolves according to:

• $\beta < 1$: Heat equation $\partial_t\rho(t,u) = \Delta\rho(t,u)$

• $\beta = 1$: Heat equation $\partial_t\rho(t,u) = \Delta\rho(t,u)$ with a type of **Robin's** boundary conditions $\partial_u\rho(t,0^-) = \partial_u\rho(t,0^+) = \alpha(\rho(t,0^+) - \rho(t,0^-))$.

• $\beta > 1$: Heat equation $\partial_t\rho(t,u) = \Delta\rho(t,u)$ with **Neumann's** boundary conditions $\partial_u\rho(t,0^-) = \partial_u\rho(t,0^+) = 0$.

Equilibrium density fluctuations: $a = 0$

- Fix a density $\rho \in (0,1)$ and consider the process starting from ν_ρ.

- The *density fluctuation field* $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}$ is given on $H \in \mathcal{S}_\beta(\mathbb{R})$ by

$$\mathcal{Y}_t^{\beta,\gamma,n}(H) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{n}\Big)(\eta_{tn^2}(x) - \rho).$$

Definition

Let $\mathcal{S}(\mathbb{R}\backslash\{0\})$ be the space of functions $H : \mathbb{R} \to \mathbb{R}$ such that: 1) H is smooth on $\mathbb{R}\backslash\{0\}$, 2) H is continuous from the right at 0, 3) for all non-negative integers k, ℓ, the function H satisfies

$$\|H\|_{k,\ell} := \sup_{u \neq 0} \Big|(1 + |u|^\ell)\frac{d^k H}{du^k}(u)\Big| < \infty.$$

Space of test functions

Definition

1. For $\beta < 1$, $\mathcal{S}_\beta(\mathbb{R}) := \mathcal{S}(\mathbb{R})$, the usual Schwartz space $\mathcal{S}(\mathbb{R})$.
2. For $\beta = 1$, $\mathcal{S}_\beta(\mathbb{R})$ is the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions H such that
$$\frac{d^{2k+1}H}{du^{2k+1}}(0^+) = \frac{d^{2k+1}H}{du^{2k+1}}(0^-) = \alpha\Big(\frac{d^{2k}H}{du^{2k}}(0^+) - \frac{d^{2k}H}{du^{2k}}(0^-)\Big)$$
for any integer $k \geq 0$.
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Density fluctuation field for $a = 0$

Theorem (Franco, G., Neumann - 2013)

If $a = 0$, the sequence of processes $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}_{n\in\mathbb{N}}$ converges to the Ornstein-Uhlenbeck process given by

$$d\mathcal{Y}_t^{\beta} = \frac{1}{2}\Delta_{\beta}\mathcal{Y}_t^{\beta}dt + \sqrt{\chi(\rho)}\nabla_{\beta}d\mathcal{W}_t^{\beta},$$

where $\{\mathcal{W}_t^{\beta}\ ;\ t \in [0,T]\}$ is an $\mathcal{S}'_{\beta}(\mathbb{R})$-valued Brownian motion and $\chi(\rho) = \rho(1-\rho)$.

Density fluctuation field for $a \neq 0$: removing the drift

We redefine for any $H \in \mathcal{S}_\beta(\mathbb{R})$

$$\mathcal{Y}_t^{\beta,\gamma,n}(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\left(\frac{x - n^{2-\gamma} a(1-2\rho)t}{n}\right)(\eta_{tn^2}(x) - \rho).$$

Theorem (Ornstein-Uhlenbeck process)

If one of these two conditions are satisfied:

- $\beta \leq 1/2$ *and* $\gamma > 1/2$,
- $\beta > 1/2$ *and* $\gamma \geq \beta$

then $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}_{n \in \mathbb{N}}$ *converges to OU as in the case* $a = 0$.

- The influence of the asymmetry is NOT SEEN in the limit.

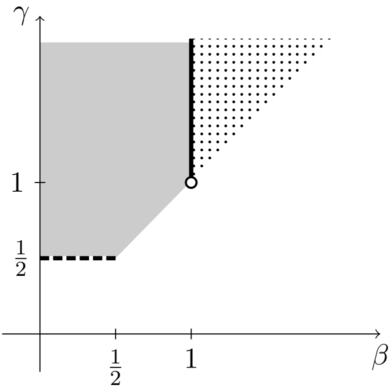
Effect of a stronger asymmetry $a \neq 0$: the KPZ scaling

Theorem (Stochastic Burgers equation)

Fix $\rho = 1/2$. For $\beta \leq 1/2$ and $\gamma = 1/2$, $\{\mathcal{Y}_t^{\beta,\gamma,n} \, ; \, t \in [0,T]\}_{n\in\mathbb{N}}$ is tight and any limit point is a stationary energy solution of the stochastic Burgers equation

$$d\mathcal{Y}_t = \frac{1}{2}\Delta\mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

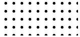
where $\{\mathcal{W}_t \, ; \, t \in [0,T]\}$ is an $\mathcal{S}'(\mathbb{R})$-valued Brownian motion.



 Stochastic Burgers equation (KPZ regime)

 OU process with no boundary conditions

 OU process with Robin's boundary conditions

 OU process with Neumann's boundary conditions

The KPZ scaling: stationary energy solution

To show that $\mathcal{Y}_t$ is a stationary energy solution of

$$d\mathcal{Y}_t = \frac{1}{2}\Delta\mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

we need to prove that $\{\mathcal{M}_t : t \in [0,T]\}$ given by

$$\mathcal{M}_t(H) \ := \ \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{1}{2}\int_0^t \mathcal{Y}_s(\Delta H)ds + a\mathcal{A}_t(H)$$

is a continuous martingale with quadratic variation

$$\langle \mathcal{M}(H)\rangle_t = \rho(1-\rho)\|\nabla H\|_2^2,$$

where

$$\mathcal{A}_t(H) = \lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}} \nabla H(x)\Big[\mathcal{Y}_u(\iota_\varepsilon(x))\Big]^2 dxdu$$

in $\mathbb{L}^2$, where $\iota_\varepsilon(x,y) = \frac{1}{\varepsilon}\mathbf{1}_{x\le y<x+\varepsilon}$, for $y \in \mathbb{R}$.

The instantaneous current

Note that

$$j^{n}_{x,x+1}(\eta) = j^{n,S}_{x,x+1}(\eta) + j^{n,A}_{x,x+1}(\eta)$$

with

$$\begin{aligned} j^{n,A}_{x,x+1}(\eta) &= \frac{an^2}{2n^\gamma}(\eta(x+1)-\eta(x))^2, \qquad && x \in \mathbb{Z}, \\ j^{n,S}_{x,x+1}(\eta) &= \frac{n^2}{2}(\eta(x)-\eta(x+1)), \qquad && x \neq -1, \\ j^{n,S}_{-1,0}(\eta) &= \frac{\alpha n^2}{2n^\beta}(\eta(-1)-\eta(0)). \end{aligned}$$

The martingale problem

Simple computations show that

$$\mathcal{M}_t^n(H) := \mathcal{Y}_t^n(H) - \mathcal{Y}_0^n(H) - \mathcal{I}_t^n(H) - \mathcal{B}_t^n(H),$$

plus some negligible term, where

$$\mathcal{I}_t^n(H) := \frac{1}{2}\int_0^t \mathcal{Y}_s^n(\Delta H)\, ds = \frac{1}{2}\int_0^t \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} (\eta_{sn^2}(x) - \rho)\Delta H\Big(\frac{x}{n}\Big)\, ds,$$

and

$$\mathcal{B}_t^n(H) = -a\frac{\sqrt{n}}{n^\gamma}\int_0^t \sum_{x\in\mathbb{Z}} \bar{\eta}_{sn^2}(x+1)\bar{\eta}_{sn^2}(x)\nabla H\Big(\frac{x}{n}\Big)\, ds.$$

Last term is the hard one!

The second-order Boltzmann-Gibbs Principle

Theorem

Let $v : \mathbb{Z} \to \mathbb{R}$ *be a function such that* $\|v\|_{2,n}^2 := \frac{1}{n}\sum_{x\in\mathbb{Z}} v^2(x) < \infty$. *Then, there exists* $C > 0$ *such that for any* $t > 0$ *and* $\ell = \varepsilon n$*:*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} v(x)\Big\{\bar\eta_{sn^2}(x)\bar\eta_{sn^2}(x+1) - \Big(\big(\bar\eta^\ell_{sn^2}(x)\big)^2 - \frac{\chi(\rho)}{\ell}\Big)\Big\}ds\Big)^2\Big]$$

$$\le Ct\Big\{\frac{\ell}{n} + \frac{n^\beta}{\alpha n} + \frac{tn}{\ell^2}\Big\}\|v\|_{2,n}^2 + Ct\Big\{\frac{n^\beta(\log_2(\ell))^2}{\alpha n}\Big\}\frac{1}{n}\sum_{x\neq -1} v^2(x),$$

where

$$\bar\eta^\ell(x) = \frac{1}{\ell}\sum_{y=x+1}^{x+\ell}\bar\eta(y).$$

On the universality of KPZ: exclusion processes

- Let $r : \Omega \to \mathbb{R}$ be a local function that satisfies:

[i] There exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta \in \Omega$.

[ii] For any η, ξ such that $\eta(x) = \xi(x)$ for $x \neq 0, 1$, then $r(\eta) = r(\xi)$.

[iii] *Gradient condition.* There exists $\omega : \Omega \to \mathbb{R}$ such that

$$r(\eta)(\eta(1) - \eta(0)) = \tau_1 \omega(\eta) - \omega(\eta), \text{for any}\, \eta \in \Omega.$$

On the universality of KPZ: zero-range processes

• η_t a Markov process with space state $\Omega := \mathbb{N}^{\mathbb{Z}}$.

• the jump rate from x only depends on the number of particles at x and is given by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and g is Lipschitz: $\sup_{k\geq 0} |g(k+1) - g(k)| < \infty$.

As examples:

• If g is Lipschitz and there exists x_0 and $\varepsilon_0 > 0$ such that $g(x + x_0) - g(x) \geq \varepsilon_0$ for all $x \geq 0$.
• If g is sublinear, that is $C^{-1}x^\gamma \leq g(x+1) - g(x) \leq Cx^\gamma$ for $0 < \gamma < 1$ and $C > 0$.
• If $g(x) = \mathbf{1}_{x\geq 1}$.

On the universality of KPZ: kinetically constrained exclusion processes

• η_t is a Markov process with space state $\Omega = \{0,1\}^{\mathbb{Z}}$.

• here particles more likely hop to unoccupied nearest-neighbor sites when at least $m-1 \geq 1$ other neighboring sites are full.

• for $m=2$, the jump rate to the right is given by:

$$\eta(x)(1-\eta(x+1))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big]$$

and the jump rate to the left is given by

$$\eta(x+1)(1-\eta(x))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big].$$

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THANK YOU!