Compactness and large deviations

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A weak LDP for occupation measures

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- Equivalently: then, exponential decay of probabilities:

$$\mathbb{P}\big(L_t \simeq f^2\mathrm{d}x \ \text{on}\ B\big) \sim \exp\Big\{-tI(f^2)\Big\} \quad \|f\|_2 = 1, f \in H_0^1(B)$$

$I(f^2) = \frac{1}{2}||\nabla f||_2^2$ Donsker-Varadhan rate function.

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- Here is a problem where it does not work.

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- Even more: Can we say something about the measures

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- Continue recursively: $\{\mu_n\}_n$ concentrates on compact pieces of mass $\{p_j\}_j$, which are widely separated, while the rest of the mass $1 - \sum_j p_j$ dissipates. μ_n on these compact pieces, when suitably shifted, converges along subsequences.

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- Conclude: $\mathbf{M}^\star$ is the compactfication of $\widetilde{\mathcal{M}}_1$.

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optimal strategy: move "independently" on distant regions

$$\leq \exp\Big\{ - \sum_j p_j \underbrace{I(\alpha_j)} \Big\}$$

where $I(\alpha)$ is the Donsker-Varadhan rate function.

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Our model is shift-invariant: Does not care about equivalence classes!

Theorem (M-Varadhan 2014)

The family of distributions $\widetilde{L}_t$ satisfies a (strong) LDP in the compact space $\mathbf{M}^\star$ with rate function

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- We have a model on a non-compact space. If model is shift-invariant, we can address questions for exponential growth of integrals/ exponential decay of probabilities!

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- Culmination: Mean-field approximation of the Polaron:

Theorem (Bolthausen-König-M 2015)

$$Q_t \circ L_t^{-1} \Rightarrow \frac{\int_{\mathbb{R}^3} dx\, \psi_0(x) \delta_{\theta_x \psi_0^2}}{\int_{\mathbb{R}^3} dx\, \psi_0(x)}$$