EQUILIBRIUM FLUCTUATIONS FOR ONE-DIMENSIONAL CONSERVATIVE SYSTEMS

Marielle Simon
(INRIA Lille)
in collaboration with O. Blondel and P. Gonçalves

YEP XIII, Eurandom, Eindhoven

March 2016
I. PARTICLE SYSTEMS IN 1D

- Time-dependent system with exclusion rule
I. Particle systems in 1D

- Time-dependent system with exclusion rule

\[ \eta_t(x) = \text{particle number} \in \{0, 1\} \]
I. Particle systems in 1D

- **Time-dependent system with exclusion rule**
  \[ \eta_t(x) = \text{particle number} \in \{0, 1\} \]

- **Invariant measures:**
  - one conserved quantity (**density**):
    \[ \sum_{x \in \mathbb{Z}} \eta_t(x) = \sum_{x \in \mathbb{Z}} \eta_0(x) < \infty \]
I. Particle systems in 1D

- Time-dependent system with exclusion rule
  \[ \eta_t(x) = \text{particle number } \in \{0, 1\} \]

- Invariant measures:
  ▶ one conserved quantity (density):
    \[ \sum_{x \in \mathbb{Z}} \eta_t(x) = \sum_{x \in \mathbb{Z}} \eta_0(x) < \infty \]
  ▶ equilibrium measures: product of Bernoulli’s
    \[ \nu_\rho (d\eta) , \]
    \[ \nu_\rho \{ \eta(x) = 1 \} = \rho \in (0, 1). \]
I. Particle systems in 1D

- Time-dependent system with exclusion rule
  \[ \eta_t(x) = \text{particle number } \in \{0, 1\} \]

- Invariant measures:
  - one conserved quantity (density):
    \[ \sum_{x \in \mathbb{Z}} \eta_t(x) = \sum_{x \in \mathbb{Z}} \eta_0(x) < \infty \]
  - equilibrium measures: product of Bernoulli's
    \[ \nu_\rho (d\eta) , \quad \nu_\rho \left\{ \eta(x) = 1 \right\} = \rho \in (0, 1). \]

- Markov process with jumps:
  \[ \mu(t) = \text{probability law of } \{ \eta_t(x) ; x \in \mathbb{Z} \} \]
Example: porous media

\[ \eta(x) \text{ and } \eta(x+1) \]

\[ r(\eta) = \eta(x) - \eta(x+1) \eta(x-1) + \eta(x+2) \]

\[ \frac{1}{2} + b \]

Time-invariant measures = Bernoulli \[ \nu \rho \]

Generator of the Markov process:

\[ d\mu/dt = \mu L, \text{ and } L = A + S \]

\[ \in L^2(\nu \rho) \]
Example: porous media

\[
\frac{1}{2} \quad \frac{1}{2}
\]

Time-invariant measures = Bernoulli \( \nu \rho \) with \( \rho \in (0, 1) \).

Generator of the Markov process:

\[
d\mu = \mu_L, \quad L = A + S \text{ in } L^2(\nu \rho).
\]
Example: porous media

\[ \frac{1}{2} \]

Generator of the Markov process:
\[ d\mu(t) = \mu L, \quad L = A + S \]
Example: porous media

\[
\frac{1}{2} \quad \frac{1}{2}
\]
Example: porous media

\[ \text{Rate to exchange } \eta(x) \text{ and } \eta(x+1) \]

\[ r(\eta) = \eta(x) \left( 1 - \eta(x+1) \right) + \eta(x+2) \left( \frac{1}{2} + \beta \right) \]

\[ \text{Time-invariant measures } = \text{Bernoulli } \nu \rho \text{ with } \rho \in (0, 1) \]

\[ \text{Generator of the Markov process: } d\mu/dt = \mu L, \text{ and } L = A + S \]
Example: porous media

\[ \frac{1}{2} \quad \frac{1}{2} \]

\[ \uparrow \quad \text{empty} \]

\[ \text{empty} \quad \uparrow \]
Example: porous media

\[ \frac{1}{2} - \frac{b}{2n^\gamma} \quad \frac{1}{2} + \frac{b}{2n^\gamma} \] 

\( (b, \gamma > 0) \)

Weakly asymmetric
Example: porous media

\[ \frac{1}{2} - \frac{b}{2n\gamma} \quad \frac{1}{2} + \frac{b}{2n\gamma} \quad (b, \gamma > 0) \]

- **Rate** to exchange \( \eta(x) \) and \( \eta(x + 1) \)

\[
r(\eta) = \eta(x) \left( 1 - \eta(x + 1) \right) \left( \eta(x - 1) + \eta(x + 2) \right) \left( \frac{1}{2} + \frac{b}{2n\gamma} \right)
\]
Example: porous media

\[ \frac{1}{2} \frac{b}{2^{n\gamma}} \quad \frac{1}{2} \frac{b}{2^{n\gamma}} \quad (b, \gamma > 0) \]

\[ \begin{array}{c}
\text{Rate to exchange } \eta(x) \text{ and } \eta(x + 1) \\
\hline
r(\eta) = \eta(x) \left(1 - \eta(x + 1)\right) \left(\eta(x - 1) + \eta(x + 2)\right) \left(\frac{1}{2} + \frac{b}{2^{n\gamma}}\right) \\
\end{array} \]

- **Rate** to exchange \( \eta(x) \) and \( \eta(x + 1) \)

- **Time-invariant measures** = Bernoulli \( \nu_\rho \) with \( \rho \in (0, 1) \)
Example: porous media

\[
\begin{aligned}
\frac{1}{2} - \frac{b}{2n\gamma} & \quad \frac{1}{2} + \frac{b}{2n\gamma} \\
(b, \gamma > 0)
\end{aligned}
\]

- **Rate** to exchange \(\eta(x)\) and \(\eta(x + 1)\)

\[
r(\eta) = \eta(x) \left(1 - \eta(x + 1)\right) \left(\eta(x - 1) + \eta(x + 2)\right) \left(\frac{1}{2} + \frac{b}{2n\gamma}\right)
\]

- **Time-invariant measures** = Bernoulli \(\nu_\rho\) with \(\rho \in (0, 1)\)
- **Generator** of the Markov process:

\[
\frac{d\mu}{dt} = \mu \mathcal{L}, \quad \text{and} \quad \mathcal{L} = \mathcal{A} + \mathcal{S} \quad \text{in} \ \mathbb{L}^2(\nu_\rho).
\]
Specificities of the dynamics

- Blocked configuration:

No possible jump!
Specificities of the dynamics

• Blocked configuration:

   ![Diagram showing no possible jump]

   *No possible jump!*

• Practical tool: mobile cluster = pair at distance ≤ 2

   ▶ Initial configuration:

   ![Diagram showing initial configuration with x and y]
Specificities of the dynamics

- Blocked configuration:
  
  \[
  \text{No possible jump!}
  \]

- Practical tool: mobile cluster = pair at distance \( \leq 2 \)
  
  ▶ Initial configuration:

  \[
  x \quad \text{mobile cluster} \quad y
  \]

  \[\text{sequence of moves to bring } x \rightarrow y \text{ ??}\]
Specificities of the dynamics

• Blocked configuration:

   No possible jump!

• Practical tool: mobile cluster = pair at distance $\leq 2$

   ▶ Sequence of allowed jumps:

   ▶ Sequence of moves to bring $x \rightarrow y$
Specificities of the dynamics

• Blocked configuration:  
  \[ \text{No possible jump!} \]

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

➤ Sequence of allowed jumps:

\[ x \quad \text{move} \quad y \]
Specificities of the dynamics

• Blocked configuration:

No possible jump!

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

▷ Sequence of allowed jumps:
Specificities of the dynamics

- Blocked configuration:
  
  No possible jump!

- Practical tool: mobile cluster = pair at distance \( \leq 2 \)

  ▶ Sequence of allowed jumps:
Specificities of the dynamics

• Blocked configuration:

  \[\text{No possible jump!}\]

![Diagram of blocked configuration]

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

  \(\Rightarrow\) Sequence of allowed jumps:

![Diagram of sequence of moves]

\(\quad x \rightarrow y \)
Specificities of the dynamics

• Blocked configuration:

  No possible jump!

• Practical tool: mobile cluster = pair at distance $\leq 2$

  ▶ Sequence of allowed jumps:

$x$  $y$
Specificities of the dynamics

• Blocked configuration:

\[ \text{No possible jump!} \]

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

▷ Sequence of allowed jumps:
Specificities of the dynamics

• Blocked configuration:

No possible jump!

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

▶ Sequence of allowed jumps:

\[ x \quad \text{etc.} \quad y \]
Specificities of the dynamics

• Blocked configuration:

\[ \text{No possible jump!} \]

[Diagram showing a blocked configuration with a sequence of moves]

• Practical tool: mobile cluster = pair at distance \( \leq 2 \)

▷ Final configuration:

[Diagram showing the final configuration with labels 'x' and 'y']
Specificities of the dynamics

• Blocked configuration: No possible jump!

• Practical tool: mobile cluster = pair at distance $\leq 2$

For any $x, y$ that do not belong to the cluster, there exists an allowed path that transports the cluster to the vicinity of $x, y$ and uses it to exchange $\eta(x), \eta(y)$. 
Weight of blocked configurations

For $x \in \mathbb{Z}$ and $\ell \in \mathbb{N}$,

$$\Lambda_\ell(x) := \{x + 1, \ldots, x + \ell\} = \text{box of size } \ell.$$
Weight of blocked configurations

For \( x \in \mathbb{Z} \) and \( \ell \in \mathbb{N} \),

\[
\Lambda_\ell(x) := \{ x + 1, \ldots, x + \ell \} = \text{box of size } \ell.
\]

▷ Good configurations

\[
\mathcal{G}_\ell(x) := \{ \eta \ ; \ \eta \text{ contains a mobile cluster in } \Lambda_\ell(x) \}
\]
Weight of blocked configurations

For \( x \in \mathbb{Z} \) and \( \ell \in \mathbb{N} \),

\[
\Lambda_\ell(x) := \{x + 1, \ldots, x + \ell\} = \text{box of size } \ell.
\]

- **Good configurations**

\[
\mathcal{G}_\ell(x) := \{\eta \ ; \ \eta \text{ contains a mobile cluster in } \Lambda_\ell(x)\}
\]

- **Bad configurations**

\[
\mathcal{B}_\ell(x) := \left(\mathcal{G}_\ell(x)\right)^c
\]

Exponentially small weight

\[
\nu_{\rho} B_\ell(x) \leq \left(1 - \rho^2 \right)^{\ell/2}
\]
Weight of blocked configurations

For $x \in \mathbb{Z}$ and $\ell \in \mathbb{N}$,

$$\Lambda_\ell(x) := \{x + 1, \ldots, x + \ell\} = \text{box of size } \ell.$$

- **Good configurations**

$$\mathcal{G}_\ell(x) := \{\eta \; ; \; \eta \text{ contains a mobile cluster in } \Lambda_\ell(x)\}$$

- **Bad configurations**

$$\mathcal{B}_\ell(x) := \left(\mathcal{G}_\ell(x)\right)^c$$

Exponentially small weight

$$\nu_\rho(\mathcal{B}_\ell(x)) \leq (1 - \rho^2)^{\ell/2}$$
• **Initial distribution:** $\nu_\rho$ (equilibrium)

  ▶ **Stationarity:** for any $t \geq 0$, the law of $\eta_t(\cdot)$ is $\nu_\rho$
• Initial distribution: $\nu_\rho$ (equilibrium)
  
  ▶ Stationarity: for any $t \geq 0$, the law of $\eta_t(\cdot)$ is $\nu_\rho$
  
  ▶ Diffusive acceleration of time:

\[
\mathbb{P}_\rho = \text{law of the process } \left\{ \eta_{tn^2}(\cdot) \right\}_{t \in [0,T]} \Rightarrow \mathbb{E}_\rho[\eta_{tn^2}(x)] = \rho
\]
• **Initial distribution:** \( \nu_\rho \) (equilibrium)

  ▶ **Stationarity:** for any \( t \geq 0 \), the law of \( \eta_t(\cdot) \) is \( \nu_\rho \)

  ▶ **Diffusive acceleration of time:**

\[
P_\rho = \text{law of the process } \{ \eta_{tn^2}(\cdot) \}_{t \in [0,T]} \implies \mathbb{E}_\rho[\eta_{tn^2}(x)] = \rho
\]

• **Density fluctuation field:**

\[
\mathcal{Y}^n_t(\varphi) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{n}\right)(\eta_{tn^2}(x) - \rho)
\]

\[\varphi \in S(\mathbb{R})\]
• Initial distribution: $\nu_\rho$ (equilibrium)

▷ **Stationarity**: for any $t \geq 0$, the law of $\eta_t(\cdot)$ is $\nu_\rho$

▷ **Diffusive acceleration of time:**

$$P_\rho = \text{law of the process } \{ \eta_{tn^2}(\cdot) \}_{t \in [0,T]} \quad \Rightarrow \quad E_\rho[\eta_{tn^2}(x)] = \rho$$

• Density fluctuation field:

$$Y_t^n(\varphi) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{n}\right)(\eta_{tn^2}(x) - \rho) \quad \varphi \in S(\mathbb{R})$$

▷ For each fixed time

$$Y_t^n(\cdot) \xrightarrow{\text{distr.}} \chi(\rho)W(\cdot) \quad W = \text{white noise} \quad \text{(CLT)}$$
• **Initial distribution:** $\nu_\rho$ (equilibrium)

  ▶ **Stationarity:** for any $t \geq 0$, the law of $\eta_t(\cdot)$ is $\nu_\rho$

  ▶ **Diffusive acceleration of time:**

  \[
  \mathbb{P}_\rho = \text{law of the process } \{ \eta_{tn^2}(\cdot) \}_{t \in [0,T]} \quad \Rightarrow \quad \mathbb{E}_\rho[\eta_{tn^2}(x)] = \rho
  \]

• **Density fluctuation field:**

  \[
  \gamma^n_t(\varphi) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x}{n} \right) (\eta_{tn^2}(x) - \rho) \quad \varphi \in S(\mathbb{R})
  \]

  ▶ For each **fixed time**

  \[
  \gamma^n_t(\cdot) \xrightarrow{\text{distr.}} \chi(\rho)\mathcal{W}(\cdot) \quad \mathcal{W} = \text{white noise} \quad \text{(CLT)}
  \]

  ▶ **Limiting process for** $t \in [0, T]$ **and** $n \to \infty$ ??
Previous result when $b = 0$

**Ornstein-Uhlenbeck process (OU)**

\[ \{ Y^n_t(\cdot); t \in [0, T]\} \] converges to the stationary solution of

\[
\begin{align*}
dY_t &= D(\rho) \Delta Y_t \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t)
\end{align*}
\]

where

\[ B_t = \text{Brownian motion} \quad \text{and} \quad D(\rho) = \rho \text{ is the diffusion coefficient of the porous media equation} \]

\[ \begin{align*}
\partial_t \rho(t, u) &= \frac{1}{2} \Delta \rho^2(t, u), \quad (t, u) \in \mathbb{R}_+ \times \mathbb{R}.
\end{align*} \]
Previous result when $b = 0$

Ornstein-Uhlenbeck process (OU)

$\{Y^n_t(\cdot); t \in [0, T]\}$ converges to the stationary solution of

$$dY_t = D(\rho) \Delta Y_t \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t)$$

where $B_t = \text{Brownian motion}$ and

$$D(\rho) = \rho$$
Previous result when $b = 0$

### Ornstein-Uhlenbeck process (OU)

\[
\{ \mathcal{Y}_t^n(\cdot) ; t \in [0,T] \} \text{ converges to the stationary solution of}
\]

\[
d \mathcal{Y}_t = D(\rho) \Delta \mathcal{Y}_t \, dt + \sqrt{2 \chi(\rho) D(\rho)} \nabla (dB_t)
\]

where $B_t = \text{Brownian motion and}$

\[
D(\rho) = \rho
\]

is the diffusion coefficient of the porous media equation

\[
\partial_t \rho(t,u) = \frac{1}{2} \Delta (\rho^2)(t,u), \quad (t,u) \in \mathbb{R}_+ \times \mathbb{R}.
\]

[ Gonçalves-Landim-Toninelli 2008]
II. BOLTZMANN-GIBBS PRINCIPLES (BG)

- Remember:

\[ Y^n_t(\varphi) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{n}\right)(\eta_{tn^2}(x) - \rho) \]

\[ \Rightarrow \quad M^n_t(\varphi) = Y^n_t(\varphi) - Y^n_0(\varphi) - \int_0^t n^2 \mathcal{L}(Y^n_s(\varphi)) \, ds \]
II. BOLTZMANN-GIBBS PRINCIPLES (BG)

• Remember:

$$\mathcal{Y}_t^n(\varphi) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi\left(\frac{x}{n}\right) (\eta_{tn^2}(x) - \rho)$$

$$\Rightarrow \mathcal{M}_t^n(\varphi) = \mathcal{Y}_t^n(\varphi) - \mathcal{Y}_0^n(\varphi) - \int_0^t n^2 \mathcal{L}(\mathcal{Y}_s^n(\varphi)) \, ds$$

• Density current:

指引 Conservation law:

$$\mathcal{L}(\eta(x)) = -\nabla(j_{x,x+1}) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta)$$
II. BOLTZMANN-GIBBS PRINCIPLES (BG)

• Remember:

\[ Y_t^n(\varphi) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \varphi \left( \frac{x}{n} \right) (\eta_{tn^2}(x) - \rho) \]

\[ \Rightarrow M_t^n(\varphi) = Y_t^n(\varphi) - Y_0^n(\varphi) - \int_0^t n^2 \mathcal{L}(Y_s^n(\varphi)) \, ds \]

• Density current:

▷ Conservation law:

\[ \mathcal{L} (\eta(x)) = -\nabla (j_{x,x+1}) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta) \]

▷ Integral part:

\[ \int_0^t n^2 \mathcal{L}(Y_s^n(\varphi)) \, ds = \frac{n}{\sqrt{n}} \int_0^t \sum_{x \in \mathbb{Z}} \varphi' \left( \frac{x}{n} \right) j_{x,x+1}(\eta_{sn^2}) \, ds \]
• Decomposition of the current:

\[
j_{x,x+1}(\eta) = \nabla(h_x) + \frac{b}{2n\gamma} \left\{ \eta(x)\eta(x+1) + \eta(x)\eta(x+1)\eta(x-1) + \ldots \right\}
\]

gradient \hspace{2cm} \text{polynomial} \hspace{2cm} \text{degree 3}
• Decomposition of the current:

\[ j_{x,x+1}(\eta) = \nabla h_x + \frac{b}{2n\gamma} P_x(\eta) \]

- gradient
- polynomial
• Decomposition of the current:

\[
j_{x,x+1}(\eta) = \nabla(h_x) + \frac{b}{2n\gamma} P_x(\eta)
\]

1. **Gradient part \( \nabla(h_x) \): First-order BG

• Second integration by part:

\[
\frac{1}{\sqrt{n}} \int_0^t \sum_{x \in \mathbb{Z}} \varphi''\left(\frac{x}{n}\right) \left\{ \eta_{sn^2}(x) - \rho \right\} ds = \int_0^t \gamma_s^n(\varphi'') ds
\]

close the equation
• Decomposition of the current:

\[
j_{x,x+1}(\eta) = \nabla(h_x) + \frac{b}{2n\gamma} P_x(\eta)
\]

1. **Gradient part \( \nabla(h_x) \): First-order BG

• Second integration by part:

\[
\frac{1}{\sqrt{n}} \int_0^t \sum_{x \in \mathbb{Z}} \varphi'' \left( \frac{x}{n} \right) \left\{ \eta_{sn^2}(x) - \rho \right\} ds = \int_0^t \gamma^n(\varphi'') ds
\]

• The asymmetry disappears when \( \gamma > \frac{1}{2} \):

\[
\mathbb{E}_\rho \left[ \left( \int_0^t b \sqrt{n} \sum_{x \in \mathbb{Z}} \varphi' \left( \frac{x}{n} \right) P_x(\eta_{sn^2}) ds \right)^2 \right] \xrightarrow{n \to \infty} 0.
\]
• Decomposition of the current:

\[
j_{x,x+1}(\eta) = \nabla(h_x) + \frac{b}{2n\gamma} P_x(\eta)
\]

1. **Gradient part** \( \nabla(h_x) $$:$$ **First-order BG**

• **Second integration by part:**

\[
\frac{1}{\sqrt{n}} \int_0^t \sum_{x \in \mathbb{Z}} \varphi''\left(\frac{x}{n}\right) \{\eta_{sn^2}(x) - \rho\} \ ds = \int_0^t \gamma_s^n(\varphi'') \ ds
\]

• **The asymmetry disappears when** \( \gamma > \frac{1}{2} $$:

\[
\mathbb{E}_\rho \left[ \left( \int_0^t b\sqrt{n} \sum_{x \in \mathbb{Z}} \varphi'\left(\frac{x}{n}\right) P_x(\eta_{sn^2}) \ ds \right)^2 \right] \xrightarrow{n \to \infty} 0.
\]

**Not true for** \( \gamma = \frac{1}{2} $$!!

In other words, in $\mathcal{Y}_t^n(\cdot)$ replace

$$
\bar{\eta}(x)\bar{\eta}(x+1) \quad \text{with} \quad (\bar{\eta}^\ell(x))^2 \quad 1 \ll \ell \ll n
$$

where

$$
\bar{\eta}(x) = \eta(x) - \rho, \quad \bar{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} (\eta(y) - \rho).
$$
2. POLYNOMIALS: SECOND-ORDER BG [Gonçalves-Jara 2014]

In other words, in $\mathcal{Y}_t^n(\cdot)$ replace

$$\overline{\eta}(x) \overline{\eta}(x + 1) \quad \text{with} \quad (\overline{\eta}^\ell(x))^2$$

where

$$\overline{\eta}(x) = \eta(x) - \rho, \quad \overline{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} (\eta(y) - \rho).$$
2. **Polynomials: Second-order BG** [Gonçalves-Jara 2014]

In other words, in $\gamma_t^n(\cdot)$ replace

\[
\bar{\eta}(x)\bar{\eta}(x+1) \quad \text{with} \quad \left(\bar{\eta}^\ell(x)\right)^2 \quad \ell = \varepsilon n
\]

where

\[
\bar{\eta}(x) = \eta(x) - \rho, \quad \bar{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} (\eta(y) - \rho).
\]

- **Proof of** [Gonçalves, Jara, Sethuraman 2015]
  - Rates bounded away from 0
2. **Polynomials: Second-order BG** [Gonçalves-Jara 2014]

In other words, in $\mathcal{Y}_t^n(\cdot)$ replace

$$\overline{\eta}(x)\overline{\eta}(x + 1) \quad \text{with} \quad (\overline{\eta}^\ell(x))^2 \quad \ell = \varepsilon n$$

where

$$\overline{\eta}(x) = \eta(x) - \rho, \quad \overline{\eta}^\ell(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} (\eta(y) - \rho).$$

- **Proof of** [Gonçalves, Jara, Sethuraman 2015]
  - Rates bounded away from 0
  - Multiscale analysis using the **spectral gap**
2. **Polynomials: Second-order BG**  [Gonçalves-Jara 2014]

In other words, in $\gamma^n_t(\cdot)$ replace

\[
\overline{\eta}(x)\overline{\eta}(x+1) \quad \text{with} \quad (\overline{\eta}^{\ell}(x))^2 \quad \ell = \epsilon n
\]

where

\[
\overline{\eta}(x) = \eta(x) - \rho, \quad \overline{\eta}^{\ell}(x) = \frac{1}{\ell} \sum_{y=x+1}^{x+\ell} (\eta(y) - \rho).
\]

- **Proof of [Gonçalves, Jara, Sethuraman 2015]**
  - Rates bounded away from 0
  - Multiscale analysis using the *spectral gap*
  - Proof for every *local function*
Limiting process for $\gamma = \frac{1}{2}$

**Stochastic Burgers Equation**

\[
\{Y^n_t(\cdot) ; t \in [0,T]\} \text{ converges to the unique energy solution of}

\[dY_t = D(\rho) \Delta Y_t \, dt - 2b \nabla (Y^2_t) \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t).\]

• Energy solutions: [Gonçalves, Jara, Gubinelli 2014]
• Uniqueness result: [Gubinelli, Perkowski 2015]
• New proof of the second-order BG without spectral gap: [Franco, Gonçalves, S. 2016]
• New applications to different models: \(\rightarrow\) Hamiltonian oscillators, \(\rightarrow\) Exclusion processes with slow bonds (Talk of Patricia Gonçalves)
Limiting process for $\gamma = \frac{1}{2}$

### Stochastic Burgers Equation

The process $\{ \mathcal{Y}_t^n(\cdot) ; t \in [0,T] \}$ converges to the unique energy solution of

$$d\mathcal{Y}_t = D(\rho) \Delta \mathcal{Y}_t \, dt - 2b \nabla (\mathcal{Y}_t^2) \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t).$$

- **Energy solutions**: [Gonçalves, Jara, Gubinelli 2014]
Limiting process for $\gamma = \frac{1}{2}$

**Stochastic Burgers Equation**

$\{\mathcal{Y}_t^n(\cdot); t \in [0, T]\}$ converges to the *unique energy solution* of

$$d\mathcal{Y}_t = D(\rho) \Delta \mathcal{Y}_t \, dt - 2b \nabla (\mathcal{Y}_t^2) \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t).$$

- **Energy solutions**: [Gonçalves, Jara, Gubinelli 2014]
- **Uniqueness result**: [Gubinelli, Perkowski 2015]
Limiting process for $\gamma = \frac{1}{2}$

Stochastic Burgers Equation

\[ \{ \mathcal{Y}_t^n(\cdot) ; t \in [0, T] \} \] converges to the unique energy solution of

\[
d\mathcal{Y}_t = D(\rho) \Delta \mathcal{Y}_t \, dt - 2b \nabla (\mathcal{Y}_t^2) \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t).
\]

- **Energy solutions:** [Gonçalves, Jara, Gubinelli 2014]
- **Uniqueness result:** [Gubinelli, Perkowski 2015]
- **New proof of the second-order BG without spectral gap:** [Franco, Gonçalves, S. 2016]
Limiting process for $\gamma = \frac{1}{2}$

Stochastic Burgers Equation

$\{Y^n_t(\cdot); t \in [0, T]\}$ converges to the unique energy solution of

$$dY_t = D(\rho) \Delta Y_t \, dt - 2b \nabla (Y_t^2) \, dt + \sqrt{2\chi(\rho)D(\rho)} \nabla (dB_t).$$

- **Energy solutions**: [Gonçalves, Jara, Gubinelli 2014]
- **Uniqueness result**: [Gubinelli, Perkowski 2015]
- **New proof of the second-order BG without spectral gap**: [Franco, Gonçalves, S. 2016]
- **New applications to different models**:
  - Hamiltonian oscillators
Limiting process for $\gamma = \frac{1}{2}$

Stochastic Burgers Equation

$\{ \mathcal{Y}^n_t(\cdot) ; t \in [0, T] \}$ converges to the unique energy solution of

$$d\mathcal{Y}_t = D(\rho) \Delta \mathcal{Y}_t \, dt - 2b \nabla (\mathcal{Y}^2_t) \, dt + \sqrt{2c(\rho)D(\rho)} \nabla (dB_t).$$

- **Energy solutions**: [Gonçalves, Jara, Gubinelli 2014]
- **Uniqueness result**: [Gubinelli, Perkowski 2015]
- **New proof of the second-order BG without spectral gap**: [Franco, Gonçalves, S. 2016]
- **New applications to different models**:
  - Hamiltonian oscillators
  - Exclusion processes with slow bonds
    ($\rightarrow$ Talk of Patricia Gonçalves)
Thank you for your attention!