Gibbsianness related to minimisers of a large deviation rate function

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Motivation: Gibbs-non-Gibbs transitions on the lattice

\( \mu_0 \) Gibbs measure
\( \mu_t \) evolved measure by dynamics over \( t > 0, (\mu_t = \mu_0 S_t) \)

E.g. \( \mu_0 \) Ising model (on \( \{-1, 1\}^{\mathbb{Z}^2} \))
E.g. spin flips \(-1 \Rightarrow 1 \) & \( 1 \Rightarrow -1 \) with certain rate

\[
\begin{array}{cccccc}
+ & + & + & + & + & - \\
+ & + & - & + & + & + \\
+ & - & - & - & + & - \\
+ & - & - & + & + & - \\
- & - & - & + & + & - \\
- & + & + & + & + & + \\
\end{array}
\begin{array}{cccccc}
+ & + & + & + & + & - \\
- & + & - & + & - & + \\
+ & - & - & - & - & + \\
+ & + & - & - & + & - \\
- & - & + & - & + & + \\
- & + & + & - & + & + \\
\end{array}
\begin{array}{cccccc}
+ & + & + & - & + & + \\
- & + & - & + & - & + \\
+ & - & + & - & + & - \\
+ & - & - & - & - & + \\
+ & - & + & - & + & - \\
- & + & + & - & + & + \\
\end{array}
\]

**Question** Is \( \mu_t \) Gibbs? For which \( t \)?
A **mean-field** system describes countably many spins
- no spatial structure
- **equal interaction** between spins

As initial system we consider the mean field system \((\mu_n,0)_{n \in \mathbb{N}}\)

\[
\mu_n,0 \propto e^{-nV \circ m_n(x)} \, d\lambda^n(x) \quad \text{(on } \mathbb{R}^n),
\]

where \(m_n(x) = \frac{x_1 + \cdots + x_n}{n}\) called magnetisation of \(x\),
\(\lambda \sim \mathcal{N}(0,1)\)

How to describe a “probability” for the infinite number of spins?
Mean-field Gibbsianness

For all $x_2, \ldots, x_n \in \mathbb{R}$ with $m_{n-1}(x_2, \ldots, x_n) = \alpha$:

$$\mu_{n,0} \left( \cdot | x_2, \ldots, x_n \right) \propto e^{-nV \left( \frac{x_n}{n} + \frac{n-1}{n} \alpha \right)} \, d\mu_{\mathcal{N}(0,1)}(x).$$

sequentially Gibbs $\approx$ asymptotic version of this:

**Definition**

$(\mu_{n,0})_{n \in \mathbb{N}}$ is sequentially Gibbs if $\forall \alpha \in \mathbb{R} \exists$ probability $\gamma_\alpha$ s.t.

$$m_{n-1}(x_2^n, \ldots, x_n^n) \to \alpha \quad \implies \quad \mu_{n,0} \left( \cdot | x_2^n, \ldots, x_n^n \right) \to \gamma_\alpha.$$

**Theorem**

If $V \in C^1(\mathbb{R}, [0, \infty))$, then $(\mu_{n,0})_{n \in \mathbb{N}}$ is sequentially Gibbs with

$$\gamma_\alpha \propto e^{-x_1 V'(\alpha)} \, d\mu_{\mathcal{N}(0,1)}(x_1),$$
Brownian motions for the mean-field system

Evolve \((\mu_n, 0)_{n \in \mathbb{N}}\) by independent Brownian motions to \((\mu_n, t)_{n \in \mathbb{N}}\).

**Theorem**

If \(V \in C^1(\mathbb{R}, [0, \infty))\), \(t > 0\), then \((\mu_n, t)_{n \in \mathbb{N}}\) is sequentially Gibbs \(\iff\) \(\Psi_{t, \alpha} : r \mapsto V(r) + \frac{r^2}{2} + \frac{(r-\alpha)^2}{2t}\) has a unique global minimiser for all \(\alpha\).

\(X_1, X_2, \ldots\) : initial coordinates
\(Y_1, Y_2, \ldots\) : final/evolved coordinates

\[ \mathbb{P}(m_n(X_1, \ldots, X_n) \approx x | m_n(Y_1, \ldots, Y_n) = \alpha) \approx e^{-n(\Psi_{t, \alpha}(x) - C_{t, \alpha})} \]

\(\Psi_{t, \alpha}\) (up to constant) is the LDP-rate function of the magnetisation \((m_n(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n})\) at time 0 given the magnetisation at time \(t\) equals \(\alpha\).
Second difference quotient and global minimisers

The second difference quotient determines whether \( \Psi_{t,\alpha} \) has unique global minimisers for all \( \alpha \).

\[
\Phi_2 V(x, y, z) = \frac{1}{z - x} \left( \frac{V(z) - V(y)}{z - y} - \frac{V(y) - V(x)}{y - x} \right) \quad (x < y < z).
\]

**Theorem (Summary)**

Let \( V \in C^1(\mathbb{R}, [0, \infty)) \). Then \((\mu_{n,0})_{n \in \mathbb{N}}\) is sequentially Gibbs and for \( t > 0 \) TFAE:

(a) \((\mu_{n,t})_{n \in \mathbb{N}}\) is sequentially Gibbs.

(b) \( \Psi_{t,\alpha} : r \mapsto V(r) + \frac{r^2}{2} + \frac{(r-\alpha)^2}{2t} \) has a unique global minimiser for all \( \alpha \).

(c) \( \Phi_2 V > -\frac{1+t}{2t} \).
Possible scenarios for Gibbsianness \( (\mu_{n,t})_{n \in \mathbb{N}} \)

Think of \( \Phi_2 V \) as \( V'' \).

1. \( \Phi_2 V \geq -\frac{1}{2} \) (Gibbs)

2. \( \Phi_2 V \geq -M \) for some \( M > \frac{1}{2} \) (Gibbs, non-Gibbs)

3. \( \Phi_2 V \) is not bounded from below (non-Gibbs)
Mean-field systems (level-2)

Level-2 mean-field system: \((\rho_n)_{n \in \mathbb{N}}\) with \(\rho_n\) a probability measure on \(\mathcal{X}^n\),

\[
\rho_n \propto e^{-nF_n \circ L_n(x)} \, d\lambda^n(x)
\]

\(L_n(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}\) empirical distribution of \(x\)

Whenever \(F(\zeta) = V(\int z \, d\zeta(z))\), as \(m_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i = \int z \, d[L_n(x)](z)\)

\[
\rho_n \propto e^{-nV \circ m_n(x)} \, d\lambda^n(x)
\]

**Definition**

\((\rho_n)_{n \in \mathbb{N}}\) is **sequentially Gibbs** if \(\forall \, \zeta \in \mathcal{P}_c(\mathcal{X}) \ \exists \) probability \(\gamma_\zeta\) s.t.

\[
L_{n-1}(x_2^n, \ldots, x_n^n) \overset{*}{\rightarrow} \zeta \quad \Rightarrow \quad \mu_{n,0}(\cdot | x_2^n, \ldots, x_n^n) \rightarrow \gamma_\zeta.
\]
Initial Gibb 반환 in level-2

With $F : \mathcal{P}_c(\mathcal{X}) \to [0, \infty)$ and

$$\mu_{n,0} \propto e^{-nF \circ L_n(x)} \, d\lambda^n(x)$$

We obtain an analogous statement for sequentially Gibbs.

**Theorem**

If $F$ is "$C^1$" then $(\mu_{n,0})_{n \in \mathbb{N}}$ is sequentially Gibbs with

$$\gamma_\zeta \propto e^{-\delta V(\zeta, \delta x_1)} \, d\lambda(x_1).$$

$\delta V(\zeta, \delta x)$ sort of directional derivative of $V$ at $\zeta$ in the direction of $\delta x$. 
Unique minimiser implies Gibbs

\[ P : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \rightarrow [0, 1] \] transformation kernel from \( \mathcal{X} \) to \( \mathcal{Y} \),
\[ \mu_{n,t} = \mu_{n,0} P^n, \] transformed measure with independent transformations.
\( X_1, X_2, \ldots : \) initial coordinates
\( Y_1, Y_2, \ldots : \) final/evolved coordinates

**Theorem**

Suppose that for all \( \zeta \in \mathcal{P}_c(\mathcal{Y}) \) there exists a rate function \( I_\zeta \) such that for all \( \zeta_n \xrightarrow{*} \zeta \) we have the large deviation principle

\[
P(L_n(X_1, \ldots, X_n) \approx \xi | L_n(Y_1, \ldots, Y_n) = \zeta_n) \approx e^{-n I_\zeta(\xi)}.
\]

Then (a) implies (b):

(a) \( I_\zeta \) has a **unique global minimiser** for all \( \zeta \in \mathcal{P}_c(\mathcal{Y}) \).

(b) \( (\mu_{n,t})_{n \in \mathbb{N}} \) is sequentially Gibbs.

Such LDP exists in case \( \mathcal{X} \) and \( \mathcal{Y} \) are finite.