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INFINITE-SERVER SYSTEMS WITH COXIAN ARRIVALS

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ABSTRACT. We consider a network of infinite-server queues where the input process is a Cox process of the following form. The arrival rate is a vector valued linear transform of a multivariate generalized (i.e., being driven by a subordinator rather than a compound Poisson process) shot-noise process. We first derive some distributional properties of the multivariate generalized shot-noise process. Then these are exploited to obtain the joint transform of the numbers of customers, at various time epochs, in a single infinite-server queue fed by the above mentioned Cox process. We also obtain transforms pertaining to the joint stationary arrival rate and queue length processes (thus facilitating the analysis of the corresponding departure process), as well as their means and covariance structure. Finally, we extend to the setting of a network of infinite-server queues.

KEYWORDS. Coxian process, M/G/∞, multivariate shot-noise process, subordinator.

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1. INTRODUCTION

In queueing theory it is commonly assumed that the customer arrival process is a Poisson process. Some recent empirical studies (see e.g. [3] for references) suggest, however, that arrival processes may exhibit *overdispersion*, i.e., the variance of the number of arrivals in an interval is larger than the corresponding mean. This has triggered research on queueing systems with overdispersed input.

The focus of the present paper is on infinite-server queues with as input process a doubly stochastic process, also known as a *Cox process*. That is a Poisson process in which the rate is not a constant; the rate process $\{\Lambda(t), t \in \mathbb{R}\}$ itself is a (nonnegative) stochastic process. As an immediate consequence of the law of total variance, Cox processes indeed are overdispersed.

Infinite-server queues with overdispersed input have various applications; one could for instance think of the number of simultaneous visitors of a popular website or online video [9, 10]. While the arrival process of visitors to the website may well be a Poisson process, its rate may jump up due to some (external) event, then decay gradually, only to jump up again because of another event. Such an example formed one of the motivations for [1, 5, 6], which are all studying infinite-server queues with an overdispersed arrival process. Before describing the main contributions of the present paper, we first provide a brief account of the existing literature.

In [5], the arrival process of the infinite-server queues is a Cox process in which the arrival rate is a *shot-noise process*. More specifically, the jumps of the shot-noise process occur according to a homogeneous Poisson process, and are i.i.d. (independent and identically distributed) with a general distribution; and between jumps, the shot-noise process decays exponentially fast. The main object of study in [5] is a feedforward network of infinite-server queues, and the main result is the joint transform of the shot-noise driven arrival rates and the numbers of customers in the queues. Also in [1] and [6] infinite-server queues are studied, but the arrival process is now a self-exciting or *Hawkes process*. Daw and Pender [1] present several interesting motivating examples. They consider deterministic jump sizes in the shot-noise process, and study in particular the $\mathrm{Hawkes}/\mathrm{Ph}/\infty$ and $\mathrm{Hawkes}/\mathrm{D}/\infty$ queues, obtaining detailed expressions for moments and auto-covariances. Koops *et al.* [6] allow generally distributed jump sizes. For the case of exponentially distributed service times, the joint Laplace- and z-transform of the Hawkes intensity and the number of customers is characterized via a partial differential equation, and that PDE is exploited to obtain recursive expressions for (joint) moments of the Hawkes intensity and the number of customers. For the case of generally distributed service times, the Hawkes process is viewed as a branching process, which allows expressing the z-transform of the number of customers in terms of the solution of a fixed-point equation.

◦ **Main goals and results.** In this paper we aim to develop a general framework for the study of infinite-server queues with as input a quite general Cox process. The modelling framework of [5] is extended in several ways; in particular, the shot-noise process is multivariate and not driven by a compound Poisson process, but by a Lévy subordinator. The main results are compact, elegant, transform expressions for joint distributions of arrival rates and numbers of customers, and properties of the departure processes, which reveal an interesting generalization of the classical Poisson results for the number-of-customers process and the departure process in the M/G/∞ queue.

∘ **Organization of the paper.** In Section 2 we introduce the multivariate shot-noise process, and we describe distributional properties of this process. Section 3 is devoted to the study of a single infinite-server queue with as input process the above described Cox process. The general case of a network of infinite-server queues is treated in Section 4. Section 5 contains some conclusions and suggestions for further research.

2. MULTIVARIATE SHOT-NOISE

In this preliminary section, we first define multivariate shot-noise and its stationary version, then we recall some basic facts on Poisson random measures, and we conclude with describing distributional properties of the multivariate shot-noise process.

∘ **Definition of multivariate shot-noise.** Let $\boldsymbol{X}(\cdot)$ be a (generalized) d-dimensional multivariate shot-noise process, which is defined as follows. Let $\boldsymbol{J}(\cdot)$ be a d-dimensional subordinator (i.e., a d-dimensional Lévy process which is non-decreasing in all components). We define the exponent of $\boldsymbol{J}(\cdot)$ by $-\eta(\cdot)$; it satisfies, for $\boldsymbol{\alpha} \in \mathbb{R}_+^d$ and with $'$ denoting transposition,

$$\eta(\boldsymbol{\alpha}) = -\log\Big(\mathbb{E}\,\mathrm{e}^{-\boldsymbol{\alpha}'\boldsymbol{J}(1)}\Big) = \boldsymbol{\alpha}'\boldsymbol{c} + \int_0^\infty (1-\mathrm{e}^{-\boldsymbol{\alpha}'\boldsymbol{x}})\nu(\mathrm{d}\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{c} \in \mathbb{R}_+^d$ and ν is an associated Lévy measure satisfying

$$\nu\Big(\big(\mathbb{R}_+^d\big)^{\mathrm{c}} \cup \{0\}\Big) = 0 \quad \text{and} \quad \int_{\mathbb{R}_+^d} (\|\boldsymbol{x}\| \wedge 1)\nu(\mathrm{d}\boldsymbol{x}) < \infty. \tag{2}$$

Also let $Q = (q_{ij})$ be a $(d \times d)$-matrix with nonnegative diagonal and nonpositive off-diagonal, and with all eigenvalues having strictly positive real parts. An example of such a matrix is $Q = (I - P')D$ where D is a positive diagonal matrix and P is a substochastic matrix satisfying $P^n \to 0$ as $n \to \infty$. Such a matrix is obtained as the negative transpose of a transition rate submatrix associated with a set of states which does not contain any closed subset of states in a time-homogeneous continuous-time Markov chain with a finite state space. A detailed motivation for studying this setting is provided in [4, Section 4].

We now introduce, for $\boldsymbol{X}(0)$ componentwise strictly positive and independent of $\boldsymbol{J}(\cdot)$, the multivariate shot-noise process $\boldsymbol{X}(t)$, cf. [4],

$$\boldsymbol{X}(t) = \mathrm{e}^{-Qt}\boldsymbol{X}(0) + \int_{(0,t]} \mathrm{e}^{-Q(t-s)}\mathrm{d}\boldsymbol{J}(s) \tag{3}$$

or alternatively, $\boldsymbol{X}(\cdot)$ is the unique (strong) solution to the stochastic integral equation

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \boldsymbol{J}(t) - Q\int_0^t \boldsymbol{X}(s)\,\mathrm{d}s\,. \tag{4}$$

The process $\boldsymbol{X}(\cdot)$ is Markovian and strictly positive (i.e., never hits or crosses any of the axes). Moreover, it has a unique stationary/ergodic/limiting distribution. This limiting distribution is identical to that of

$$\boldsymbol{X}(\infty) = \int_{(0,\infty)} \mathrm{e}^{-Qs}\,\mathrm{d}\boldsymbol{J}(s). \tag{5}$$

So far we considered $\boldsymbol{J}(t)$ for positive values of t, but we can extend $\boldsymbol{J}(\cdot)$ to the whole real line (with $\boldsymbol{J}(0) = \boldsymbol{0}$). It follows directly that

$$\boldsymbol{X}^*(t) := \int_{(-\infty,t]} \mathrm{e}^{-Q(t-s)} \mathrm{d}\boldsymbol{J}(s) \tag{6}$$

is a stationary process satisfying the same conditions that $\boldsymbol{X}$ does, that is, relations (3) and (4). From here on we assume that $\boldsymbol{X} = \boldsymbol{X}^*$.

∘ **Properties of Poisson random measures.** To proceed, we first note that it is well known that N is a Poisson random measure on some measurable space $(\mathbb{X}, \mathscr{G})$ with sigma-finite mean measure μ if and only if for any $\mathscr{G}$-measurable $f : \mathbb{X} \to \mathbb{R}_+$ we have that

$$\mathbb{E}\, \exp\Big(-\int f(s)\, \mathrm{d}N(s)\Big) = \exp\Big(\int \left(1 - \mathrm{e}^{-f(s)}\right) \mu(\mathrm{d}s)\Big)\, . \tag{7}$$

Observe that $\mathbb{E} \int f(s)\, \mathrm{d}N(s) = \int f(s)\, \mu(\mathrm{d}s)$ (which holds even if N is not Poisson). Finally, it is also known, and actually easy to check (first for indicators, then simple functions, etc.), that if $f_1(\cdot)$ and $f_2(\cdot)$ are nonnegative $\mathscr{G}$-measurable, then

$$\mathbb{E} \int f_1(s)\, \mathrm{d}N(s) \int f_2(s)\, \mathrm{d}N(s) = \int f_1(s) f_2(s) \mu(\mathrm{d}s) + \int f_1(s)\, \mu(\mathrm{d}s) \int f_2(s) \mu(\mathrm{d}s)\, . \tag{8}$$

Let $\boldsymbol{\varrho} = (\varrho_i)$ be the mean rate of growth of the subordinator $\boldsymbol{J}(\cdot)$, that is, $\varrho_i := c_i + \int_{\mathbb{R}_+^d} x_i \nu(\mathrm{d}\boldsymbol{x})$. We can also write $\varrho_i = c_i + \int_{\mathbb{R}_+} x_i \nu_i(\mathrm{d}x_i)$, using the notation

$$\nu_i(A) = \nu\Big(\mathbb{R}_+^{i-1} \times A \times \mathbb{R}_+^{d-i}\Big)\, . \tag{9}$$

It is also well known that there exists a Poisson random measure $N(\cdot, \cdot)$ on $\mathbb{R} \times \mathbb{R}_+^d$ with mean measure $\ell \otimes \nu$, where ℓ is Lebesgue measure, such that

$$J_i(t) = \begin{cases} c_i t + \displaystyle\int_{(0,t] \times \mathbb{R}_+^d} x_i\, \mathrm{d}N(s, \boldsymbol{x}), & t \geq 0, \\ c_i t - \displaystyle\int_{[t,0) \times \mathbb{R}_+^d} x_i\, \mathrm{d}N(s, \boldsymbol{x}), & t < 0, \end{cases} \tag{10}$$

where $\boldsymbol{x} = (x_1, \ldots, x_d)'$. This property entails that if we take $f_1(t, \boldsymbol{x}) = \sum_{i=1}^d g_i(s) x_i$ and $f_2(t, \boldsymbol{x}) = \sum_{i=1}^d h_i(s) x_i$, then we can immediately conclude that for each Borel $\boldsymbol{g}, \boldsymbol{h} : \mathbb{R} \to \mathbb{R}_+^d$ the following three identities hold:

$$\mathbb{E} \int_{\mathbb{R}} \boldsymbol{g}(s)'\, \mathrm{d}\boldsymbol{J}(s) = \int_{\mathbb{R}} \boldsymbol{g}(s)' \boldsymbol{\varrho}\, \mathrm{d}s, \tag{11}$$

$$\mathbb{E} \exp\Big(-\int_{\mathbb{R}} \boldsymbol{g}(s)' \mathrm{d}\boldsymbol{J}(s)\Big) = \exp\Big(-\int_{\mathbb{R}} \eta(\boldsymbol{g}(s)) \mathrm{d}s\Big) \tag{12}$$

and

$$\mathbb{C}\mathrm{ov}\Big(\int_{\mathbb{R}} \boldsymbol{g}(s)' \mathrm{d}\boldsymbol{J}(s), \int_{\mathbb{R}} \boldsymbol{h}(s)' \mathrm{d}\boldsymbol{J}(s)\Big) = \int_{\mathbb{R}} \boldsymbol{g}(s)' \Sigma \boldsymbol{h}(s) \mathrm{d}s \tag{13}$$

with

$$\Sigma := -\nabla^2 \eta(\boldsymbol{0}) = \int_{\mathbb{R}_+^d} \boldsymbol{x}\boldsymbol{x}' \nu(\mathrm{d}\boldsymbol{x}), \tag{14}$$

provided that the corresponding variances $\int_{\mathbb{R}} \boldsymbol{g}(s)'\Sigma\boldsymbol{g}(s)\mathrm{d}s$ and $\int_{\mathbb{R}} \boldsymbol{h}(s)'\Sigma\boldsymbol{h}(s)\mathrm{d}s$ are both finite. With σ_{ij} denoting the (i,j)-th coordinate of Σ, it also holds that

$$\sigma_{ij} = \int_{\mathbb{R}^d_+} x_i x_j \nu(\mathrm{d}\boldsymbol{x}) = \int_{\mathbb{R}^2_+} x_i x_j\, \nu_{ij}(\mathrm{d}x_i, \mathrm{d}x_j) \tag{15}$$

where ν_{ij} is the (i,j)-th marginal measure associated with ν (i.e., the Lévy measure associated with the (i,j)-th coordinate of $\boldsymbol{J}(\cdot)$).

◦ **Distributional properties of multivariate shot-noise.** We now point out how the d-dimensional Laplace-Stieltjes transform (LST) of $\boldsymbol{X}$ can be evaluated. To this end, we appeal to (5) and (12), so as to obtain

$$\mathbb{E}\,\mathrm{e}^{-\boldsymbol{\alpha}'\boldsymbol{X}} = \exp\left(-\int_0^\infty \eta\left(\mathrm{e}^{-Q's}\boldsymbol{\alpha}\right)\mathrm{d}s\right). \tag{16}$$

In the usual manner moments can be found from the LST. The first moment takes the following form: with $\boldsymbol{\nabla}(\boldsymbol{0})$ the vector of first derivatives,

$$\mathbb{E}\,\boldsymbol{X} = Q^{-1}\,\boldsymbol{\nabla}\eta(\boldsymbol{0}) = Q^{-1}\boldsymbol{\varrho}. \tag{17}$$

We now identify the covariance matrix of $\boldsymbol{X}$. From (15) it follows, as was also shown in [4, Thm. 5.2], that the covariance matrix of $\boldsymbol{X}$ is given by

$$\Sigma_0 = \int_0^\infty \mathrm{e}^{-Qs}\,\Sigma\,\mathrm{e}^{-Q's}\,\mathrm{d}s, \tag{18}$$

and is the unique solution of

$$Q\,\Sigma_0 + \Sigma_0\,Q' = \Sigma\,. \tag{19}$$

The next objective is to find an expression for the covariance between $\boldsymbol{X}(t)$ and $\boldsymbol{X}(t+h)$, again bearing in mind that we started the process at $-\infty$. Now, clearly

$$\boldsymbol{Y}(t,h) := \int_{(t,t+h]} \mathrm{e}^{-Q(t+h-s)}\,\mathrm{d}\boldsymbol{J}(s) \tag{20}$$

is independent of $\boldsymbol{X}(t)$, due to the independent increment property of $\boldsymbol{J}(\cdot)$. As a consequence, with

$$\boldsymbol{Z}(t,h) := \int_{(-\infty,t]} \mathrm{e}^{-Q(t+h-s)}\,\mathrm{d}\boldsymbol{J}(s), \tag{21}$$

we obtain, by splitting $\boldsymbol{X}(t+h)$ into the sum of $\boldsymbol{Y}(t,h)$ and $\boldsymbol{Z}(t,h)$, that

$$\mathbb{C}\mathrm{ov}(\boldsymbol{X}(t),\boldsymbol{X}(t+h)) = \mathbb{C}\mathrm{ov}(\boldsymbol{X}(t),\boldsymbol{Y}(t,h)+\boldsymbol{Z}(t,h)) = \mathbb{C}\mathrm{ov}(\boldsymbol{X}(t),\boldsymbol{Z}(t,h)). \tag{22}$$

Denote (for $h \geq 0$) by Σ_h a matrix of which the (i,j)-th entry is $\mathbb{C}\mathrm{ov}(X_i(t), X_j(t+h))$. It now follows from (15) that, by taking $\boldsymbol{g}(s) := \mathrm{e}^{-Q'(t-s)}x$ and $\boldsymbol{h}(s) := \mathrm{e}^{-Q'(t+h-s)}y$,

$$\boldsymbol{x}'\Sigma_h\boldsymbol{y} = \int_{-\infty}^t \boldsymbol{x}'\mathrm{e}^{-Q(t-s)}\Sigma\mathrm{e}^{-Q'(t+h-s)}\boldsymbol{y}\,\mathrm{d}s = \int_0^\infty \boldsymbol{x}'\,\mathrm{e}^{-Qs}\,\Sigma\,\mathrm{e}^{-Q'(s+h)}\boldsymbol{y}\,\mathrm{d}s \tag{23}$$

for every $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d$. It thus follows, using (18), that

$$\Sigma_h = \int_0^\infty \mathrm{e}^{-Qs}\,\Sigma\,\mathrm{e}^{-Q'(s+h)}\,\mathrm{d}s = \Sigma_0\,\mathrm{e}^{-Q'h} \tag{24}$$

Combining the above, we find the following result.

Proposition 2.1. *The ($\mathbb{R}^{2d}$-valued) covariance matrix of $(\boldsymbol{X}(t), \boldsymbol{X}(t+h))$ is given by*

$$\begin{pmatrix} \Sigma_0 & \Sigma_0\,\mathrm{e}^{-Q'h} \\ \mathrm{e}^{-Qh}\,\Sigma_0 & \Sigma_0 \end{pmatrix}. \tag{25}$$

Since Q' and $e^{-Q'h}$ commute, it also follows from (19) that

$$Q\,\Sigma_h + \Sigma_h\,Q' = \Sigma\,\mathrm{e}^{-Q'h}\,. \tag{26}$$

In fact, employing the method of proof of [4, Thm. 5.2], Σ_h is the unique solution of this equation.

Remark 2.2. In the one-dimensional case, letting σ^2 and q denote the one-dimensional versions of the matrices Σ and Q, respectively, we conclude that the covariance between $X(t)$ and $X(t+h)$ is given by $\frac{1}{2}\mathrm{e}^{-qh}\sigma^2/q$.

3. Single infinite-server queue with a Cox input process

We consider an infinite-server queue in which conditioned on $\boldsymbol{J}(\cdot)$, the arrival process is a non-homogeneous Poisson process with rate function $\Lambda(t) = \boldsymbol{a}'\boldsymbol{X}(t)$ at time t, where $\boldsymbol{a} \in \mathbb{R}^d_+$. It is throughout assumed that service times are i.i.d. and are independent of $\boldsymbol{J}(\cdot)$ (hence also of $\boldsymbol{X}(\cdot)$) and the arrival process. The service times have distribution function $F(\cdot)$, complementary distribution function $\bar{F}(\cdot)$, and in addition define $F(s,t) := F(t) - F(s)$ for $s < t$.

3.1. **Transform of queue lengths and numbers of arrivals.** Our first objective is to derive the transform of queue lengths at various points in time, jointly with the number of arrivals in corresponding intervals.
To this end, we consider $n \in \mathbb{N}$ time intervals, say $(t_0, t_1]$ up to $(t_{n-1}, t_n]$ (where it is assumed that $t_{i-1} < t_i$ for $i = 1, \ldots, n$). Our goal is to establish the joint transform of the queue lengths L_i at each of the t_i and the numbers of arrivals A_i in each of the intervals $(t_{i-1}, t_i]$, i.e., we shall compute the transform, for $\boldsymbol{w} \in \mathbb{R}^{n+1}$ and $\boldsymbol{z} \in \mathbb{R}^n$,

$$\Psi(\boldsymbol{w}, \boldsymbol{z}) := \mathbb{E}\left(\prod_{i=0}^{n} w_i^{L_i} \cdot \prod_{i=1}^{n} z_i^{A_i}\right). \tag{27}$$

Notice that a job arriving in the interval $(t_{i-1}, t_i]$ contributes to A_i (and does not contribute to any of the other A_j), and potentially contributes to L_i up to L_n. More precisely, supposing that the job arrives at $s \in (t_{i-1}, t_i]$, with probability $F(t_j - s, t_{j+1} - s)$ it contributes to L_i up to L_j, for $j \in \{i, \ldots, n\}$ and defining $t_{n+1} := \infty$. Conditional on $\Lambda(\cdot) = \lambda(\cdot)$ this concerns a Poisson contribution with parameter

$$\int_{t_{i-1}}^{t_i} \lambda(s) F(t_j - s, t_{j+1} - s)\,\mathrm{d}s; \tag{28}$$

conditional on $\Lambda(\cdot) = \lambda(\cdot)$ all these contributions are independent. In addition we recall that the probability generating function (evaluated in z) of a Poisson random variable with parameter λ equals $\exp(\lambda(z-1))$. Combining the above elements, we obtain

$$\Psi(\boldsymbol{w}, \boldsymbol{z}) = \mathbb{E}\exp\left(\int_{-\infty}^{t_n} \Lambda(s)\psi(s \,|\, \boldsymbol{w}, \boldsymbol{z})\,\mathrm{d}s\right), \tag{29}$$

where, with $t_{-1} := -\infty$,

$$\psi(s \,|\, \boldsymbol{w}, \boldsymbol{z}) = \sum_{i=0}^{n} \psi_i(s \,|\, \boldsymbol{w}, \boldsymbol{z}) 1_{\{s \in (t_{i-1}, t_i]\}}; \tag{30}$$

the individual $\psi_i(s \,|\, \boldsymbol{w}, \boldsymbol{z})$ are defined by

$$\begin{aligned} \psi_0(s \,|\, \boldsymbol{w}, \boldsymbol{z}) &:= \sum_{j=0}^{n} F(t_j - s, t_{j+1} - s) \left(\prod_{i=0}^{j} w_i - 1 \right) \\ &= \sum_{j=0}^{n} F(t_j - s, t_{j+1} - s) \prod_{i=0}^{j} w_i + F(t_0 - s) - 1 \end{aligned} \tag{31}$$

for $i = 0$, and

$$\begin{aligned} \psi_i(s \,|\, \boldsymbol{w}, \boldsymbol{z}) &:= \sum_{j=i}^{n} F(t_j - s, t_{j+1} - s) \left(z_i \prod_{k=i}^{j} w_k - 1 \right) \\ &= \sum_{j=i}^{n} F(t_j - s, t_{j+1} - s) z_i \prod_{k=i}^{j} w_k + F(t_i - s) - 1 \end{aligned} \tag{32}$$

for $i \in \{1, \ldots, n\}$.
The representation (29) generally holds, i.e., for any non-negative arrival rate process $\Lambda(\cdot)$. For the case of multivariate shot-noise, however, the expression can be made more explicit. To this end, we substitute

$$\Lambda(s) = \boldsymbol{a}' \boldsymbol{X}(s) = \boldsymbol{a}' \int_{-\infty}^{s} \mathrm{e}^{-Q(s-r)} \mathrm{d}\boldsymbol{J}(r). \tag{33}$$

By applying Equation (16), we obtain that (29) equals

$$\begin{aligned} \Psi(\boldsymbol{w}, \boldsymbol{z}) &= \mathbb{E} \exp\left(\boldsymbol{a}' \int_{-\infty}^{t_n} \int_{-\infty}^{s} \mathrm{e}^{-Q(s-r)} \mathrm{d}\boldsymbol{J}(r) \, \psi(s \,|\, \boldsymbol{w}, \boldsymbol{z}) \, \mathrm{d}s \right) \\ &= \mathbb{E} \exp\left(\boldsymbol{a}' \int_{-\infty}^{t_n} \int_{r}^{t_n} \mathrm{e}^{-Q(s-r)} \psi(s \,|\, \boldsymbol{w}, \boldsymbol{z}) \, \mathrm{d}s \, \mathrm{d}\boldsymbol{J}(r) \right) \\ &= \exp\left(- \int_{-\infty}^{t_n} \eta \left(- \int_{r}^{t_n} \mathrm{e}^{-Q'(s-r)} \, \psi(s \,|\, \boldsymbol{w}, \boldsymbol{z}) \mathrm{d}s \, \boldsymbol{a} \right) \mathrm{d}r \right). \end{aligned} \tag{34}$$

We have thus established the following result.

Theorem 3.1. *For any* $\boldsymbol{w} \in \mathbb{R}^{n+1}$ *and* $\boldsymbol{z} \in \mathbb{R}^{n}$,

$$\Psi(\boldsymbol{w}, \boldsymbol{z}) = \exp\left(- \int_{-\infty}^{t_n} \eta \left(- \int_{r}^{t_n} \mathrm{e}^{-Q'(s-r)} \, \psi(s \,|\, \boldsymbol{w}, \boldsymbol{z}) \mathrm{d}s \, \boldsymbol{a} \right) \mathrm{d}r \right). \tag{35}$$

Interestingly, the above result directly enables us to describe the distribution of the number of departures in all intervals. Let D_i be the number of departures in $(t_{i-1}, t_i]$. Then we would like to evaluate

$$\Phi(\boldsymbol{v}) := \mathbb{E} \left(\prod_{i=1}^{n} v_i^{D_i} \right). \tag{36}$$

Now observe that $L_i = L_{i-1} + A_i - D_i$, and hence $D_i = L_{i-1} - L_i + A_i$. As a consequence,

$$\Phi(\boldsymbol{v}) = \mathbb{E}\left(\prod_{i=1}^n v_i^{L_{i-1}-L_i+A_i}\right) = \mathbb{E}\left(v_1^{L_0} \cdot \prod_{i=1}^{n-1}\left(\frac{v_{i+1}}{v_i}\right)^{L_i} \cdot v_n^{-L_n} \cdot \prod_{i=1}^n v_i^{A_i}\right) \tag{37}$$

With $w_1(\boldsymbol{v}) := v_1$, $w_i(\boldsymbol{v}) := v_{i+1}/v_i$ for $i \in \{1, \dots, n-1\}$ and $w_n(\boldsymbol{v}) := v_n^{-1}$, we thus find the following result.

Proposition 3.2. *For any* $\boldsymbol{v} \in \mathbb{R}^n$,

$$\Phi(\boldsymbol{v}) = \Psi(\boldsymbol{w}(\boldsymbol{v}), \boldsymbol{v}). \tag{38}$$

3.2. **Explicit calculations.** Let $L(t)$ be the number of customers present at time t. In this subsection we compute (i) mean and variance of $L(0)$, (ii) the joint transform of $\Lambda(0)$ and $L(0)$, and (iii) the covariance between $L(0)$ and $L(t)$ for some $t > 0$. As before, we assume that the process started at $-\infty$, so that it displays stationary behavior at time 0 (and consequently at t as well). In principle (factorial) moments can be derived by differentiating $\Psi(\boldsymbol{w}, \boldsymbol{z})$ suitably often and plugging in $\boldsymbol{w} = \boldsymbol{1}$ and $\boldsymbol{z} = \boldsymbol{1}$, but elegant direct arguments can be given, as we show now.
We first introduce some notation. Define β as the mean service time:

$$\beta = \int_0^\infty \bar{F}(s)\,\mathrm{d}s. \tag{39}$$

The density of the residual service time B_e is given by $f_\mathrm{e}(s) := \bar{F}(s)/\beta$.

∘ **Mean and variance of** $\Lambda(0)$ **and** $L(0)$**.** It is easily checked that $\mathbb{E}\,\Lambda(0) = \lambda := \boldsymbol{a}'\,Q^{-1}\,\boldsymbol{\varrho}$ and $\mathbb{V}\mathrm{ar}\,\Lambda(0) = \boldsymbol{a}'\,\Sigma_0\,\boldsymbol{a}$. We therefore now concentrate on the mean and variance of $L(0)$. Let $\mathscr{P}(\mu)$ denote a Poisson random variable with parameter μ. It is straightforward that

$$\mathbb{E}\,L(0) = \mathbb{E}\left(\mathscr{P}\left(\int_{-\infty}^0 \Lambda(s)\bar{F}(-s)\,\mathrm{d}s\right)\right) = \mathbb{E}\left(\int_{-\infty}^0 \Lambda(s)\bar{F}(-s)\,\mathrm{d}s\right). \tag{40}$$

We thus find, with $\boldsymbol{J}$ shorthand notation for the entire path of the process $\boldsymbol{J}(\cdot)$,

$$\begin{aligned} \mathbb{E}[L(0)\,|\,\boldsymbol{J}] &= \int_{-\infty}^0 \Lambda(s)\bar{F}(-s)\,\mathrm{d}s = \int_0^\infty \Lambda(-s)\bar{F}(s)\,\mathrm{d}s \\ &= \beta\int_0^\infty \Lambda(-s)f_\mathrm{e}(s)\,\mathrm{d}s = \beta\,\mathbb{E}[\Lambda(-B_\mathrm{e})\,|\,\boldsymbol{J}]. \end{aligned} \tag{41}$$

For later use, we rewrite this expression as

$$\begin{aligned} \beta\,\mathbb{E}[\Lambda(-B_\mathrm{e})\,|\,\boldsymbol{J}] &= \beta\,\mathbb{E}[\boldsymbol{a}'\boldsymbol{X}(-B_\mathrm{e})\,|\,\boldsymbol{J}] = \beta\,\boldsymbol{a}'\,\mathbb{E}[X(-B_\mathrm{e})\,|\,\boldsymbol{J}] \\ &= \beta\int_{-\infty}^0 \left(\mathbb{E}\,\boldsymbol{a}'\,\mathrm{e}^{-Q(-B_\mathrm{e}-s)}1_{\{s\leq -B_\mathrm{e}\}}\right)\mathrm{d}\boldsymbol{J}(s) \\ &\stackrel{\mathrm{d}}{=} \beta\int_0^\infty \left(\mathbb{E}\,\mathrm{e}^{-Q'(s-B_\mathrm{e})}\boldsymbol{a}\,1_{\{B_\mathrm{e}\leq s\}}\right)'\mathrm{d}\boldsymbol{J}(s) \end{aligned} \tag{42}$$

where the last step is due to time-reversibility of $\boldsymbol{J}(\cdot)$.
Applying (11), we obtain

$$\mathbb{E}\,L(0) = \beta\,\mathbb{E}\,\Lambda(0) = \lambda\beta. \tag{43}$$

Now we move to computing $\mathbb{V}\mathrm{ar}\, L(0)$. Due to the fact that $L(0)$ is mixed Poisson, we have that $\mathbb{V}\mathrm{ar}(L(0)\,|\,\boldsymbol{J}) = \mathbb{E}(L(0)\,|\,\boldsymbol{J})$ and thus $\mathbb{E}\,\mathbb{V}\mathrm{ar}(L(0)\,|\,\boldsymbol{J}) = \mathbb{E}\,L(0) = \lambda\beta$. It thus follows with the law of total variance that

$$\begin{aligned}\mathbb{V}\mathrm{ar}\, L(0) &= \mathbb{E}\,\mathbb{V}\mathrm{ar}\,(L(0)\,|\,\boldsymbol{J}) + \mathbb{V}\mathrm{ar}\,(\mathbb{E}[L(0)\,|\,\boldsymbol{J}])\\ &= \lambda\beta + \mathbb{V}\mathrm{ar}(\mathbb{E}(\Lambda(-B_{\mathrm{e}})\,|\,\boldsymbol{J}))\beta^2\\ &= \lambda\beta + \beta^2\int_0^\infty \boldsymbol{a}'\Big(\mathbb{E}\,\mathrm{e}^{-Q(s-B_{\mathrm{e}})}1_{\{B_{\mathrm{e}}\le s\}}\Big)\Sigma\Big(\mathbb{E}\,\mathrm{e}^{-Q'(s-B_{\mathrm{e}})}1_{\{B_{\mathrm{e}}\le s\}}\Big)\boldsymbol{a}\,\mathrm{d}s \end{aligned} \tag{44}$$

which follows from (43) and (15). Expression (44) can be substantially simplified, which we do later in this section. Observe that $\mathbb{V}\mathrm{ar}\, L(0) \ge \mathbb{E}\, L(0)$, reflecting the fact that Coxian arrival rates lead to overdispersed arrival processes; see e.g. [3, 5].

◦ **Joint transform of $\Lambda(0)$ and $L(0)$.** Here the goal is to determine $\mathbb{E}\,\mathrm{e}^{-v\Lambda(0)}\,w^{L(0)}$. By arguments similar to those used in Section 3,

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{-v\Lambda(0)}\,w^{L(0)} &= \mathbb{E}\exp\bigg((w-1)\int_{-\infty}^0 \Lambda(s)\bar{F}(-s)\mathrm{d}s - v\,\Lambda(0)\bigg)\\ &= \mathbb{E}\exp\bigg((w-1)\boldsymbol{a}'\int_{-\infty}^0 X(s)\bar{F}(-s)\mathrm{d}s - v\,\boldsymbol{a}'\boldsymbol{X}(0)\bigg). \end{aligned} \tag{45}$$

Now observe that

$$\begin{aligned}\int_{-\infty}^0 X(s)\bar{F}(-s)\mathrm{d}s &= \int_{-\infty}^0\int_{-\infty}^s \mathrm{e}^{-Q(r-s)}\,\mathrm{d}\boldsymbol{J}(r)\,\bar{F}(-s)\,\mathrm{d}s\\ &= \int_{-\infty}^0\int_r^0 \mathrm{e}^{-Q(r-s)}\bar{F}(-s)\mathrm{d}s\,\mathrm{d}\boldsymbol{J}(r)\\ &= \int_0^\infty\int_0^s \mathrm{e}^{-Q(s-r)}\bar{F}(r)\mathrm{d}r\,\mathrm{d}\boldsymbol{J}(s), \end{aligned} \tag{46}$$

so that

$$\mathbb{E}\,\mathrm{e}^{-v\Lambda(0)}\,w^{L(0)} = \mathbb{E}\exp\bigg(\boldsymbol{a}'\int_0^\infty \Omega(v,w,s)\,\mathrm{d}\boldsymbol{J}(s)\bigg), \tag{47}$$

with

$$\begin{aligned}\Omega(v,w,s) &:= (w-1)\int_0^s \mathrm{e}^{-Q(s-r)}\bar{F}(r)\,\mathrm{d}r - v\,\mathrm{e}^{-Qs}\\ &= (w-1)\beta\,\mathbb{E}\,\mathrm{e}^{-Q(s-B_{\mathrm{e}})}1_{\{B_{\mathrm{e}}\le s\}} - v\,\mathrm{e}^{-Qs}. \end{aligned} \tag{48}$$

Applying (12), we thus obtain

$$\mathbb{E}\,\mathrm{e}^{-v\Lambda(0)}\,w^{L(0)} = \exp\bigg(-\int_0^\infty \eta\big(-\Omega(v,w,s)'\boldsymbol{a}\big)\mathrm{d}s\bigg). \tag{49}$$

It is possible to apply this joint transform to the computation of moments of $L(0)$ and $\Lambda(0)$, as well as their mixed moments, but lower moments can typically be computed by direct arguments. The mean and variance of $\Lambda(0)$ and $L(0)$ have been identified above. We therefore now focus on the covariance between $\Lambda(0)$ and $L(0)$. Since $\mathbb{E}[\Lambda(0)\,|\,\boldsymbol{J}] =$

$\Lambda(0)$ (as $\boldsymbol{X}(\cdot)$ is a functional of $\boldsymbol{J}(\cdot)$) and since $B_{\rm e}$ is independent of J, we find

$$\begin{aligned}\mathbb{C}{\rm ov}(\Lambda(0),L(0)) &= \mathbb{C}{\rm ov}(\mathbb{E}[L(0)\,|\,\boldsymbol{J}],\Lambda(0)) = \mathbb{C}{\rm ov}(\mathbb{E}[\Lambda(-B_{\rm e})\,|\,\boldsymbol{J}],\Lambda(0))\\ &= \mathbb{C}{\rm ov}(\Lambda(-B_{\rm e}),\Lambda(0)) = \mathbb{E}\,\mathbb{C}{\rm ov}(\Lambda(-B_{\rm e}),\Lambda(0)\,|\,B_{\rm e})\\ &= \boldsymbol{a}'\,\mathbb{E}\,\Sigma_{B_{\rm e}}\,\boldsymbol{a} = \boldsymbol{a}'\,\Sigma_0\,\mathbb{E}\,{\rm e}^{-Q'B_{\rm e}}\,\boldsymbol{a}\,,\end{aligned} \tag{50}$$

where we employ the fact that the covariance matrix between $\boldsymbol{X}(t)$ and $\boldsymbol{X}(t+h)$, which is the same as that between $\boldsymbol{X}(t-h)$ and $\boldsymbol{X}(t)$, is given by $\Sigma_h = \Sigma_0\,{\rm e}^{-Q'h}$.

∘ **Covariance between $L(0)$ and $L(t)$.** The starting point is the law of total covariance:

$$\mathbb{C}{\rm ov}\,(L(0),L(t)) = \mathbb{E}\,\mathbb{C}{\rm ov}\,(L(0),L(t)\,|\,\boldsymbol{J}) + \mathbb{C}{\rm ov}(\mathbb{E}[L(0)\,|\,\boldsymbol{J}],\mathbb{E}[L(t)\,|\,\boldsymbol{J}]). \tag{51}$$

We evaluate the two terms separately. The first term can be rewritten as follows. Let $C_1(t)$ denote the number of customers that arrive before time 0 and depart in $(0,t]$; $C_2(t)$ is the number of customers that arrive before time 0 and depart after t; finally, $C_3(t)$ is the number of customers that arrive in $(0,t]$ and depart after t. Evidently, due to the conditional independence of these three quantities,

$$\begin{aligned}\mathbb{C}{\rm ov}\,(L(0),L(t)\,|\,\boldsymbol{J}) &= \mathbb{C}{\rm ov}\,(C_1(t)+C_2(t),C_2(t)+C_3(t)\,|\,\boldsymbol{J})\\ &= \mathbb{V}{\rm ar}\,(C_2(t)|\,J) = \mathbb{E}(C_2(t)\,|\,\boldsymbol{J}),\end{aligned} \tag{52}$$

with, mimicking the above arguments,

$$\mathbb{E}(C_2(t)\,|\,\boldsymbol{J}) \stackrel{\rm d}{=} \beta\int_0^\infty {\rm E}\Big({\rm e}^{-Q'(s+t-B_{\rm e})}\boldsymbol{a}\,1_{\{t<B_{\rm e}\le t+s\}}\Big)\,{\rm d}\boldsymbol{J}(s), \tag{53}$$

where the last equality is due to the fact that the conditional distribution of $C_2(t)$ given $\boldsymbol{J}$ is Poisson. Thus, with $F_{\rm e}(\cdot)$ denoting the distribution function of $B_{\rm e}$ and $\bar{F}_{\rm e}(\cdot)$ the corresponding complementary distribution function,

$$\begin{aligned}\mathbb{E}\,\mathbb{C}{\rm ov}\,(L(0),L(t)\,|\,\boldsymbol{J}) &= \beta\int_0^\infty \mathbb{E}\Big(\boldsymbol{a}'{\rm e}^{-Q(s+t-B_{\rm e})}\boldsymbol{\varrho}1_{\{t<B_{\rm e}\le t+s\}}\Big)\,{\rm d}s\\ &= \beta\,\mathbb{E}\bigg(\int_{B_{\rm e}-t}^\infty\Big(\boldsymbol{a}'{\rm e}^{-Q(s-(B_{\rm e}-t))}\boldsymbol{\varrho}\Big){\rm d}s\,1_{\{B_{\rm e}>t\}}\bigg)\\ &= \beta\,\mathbb{E}\bigg(\int_0^\infty \boldsymbol{a}'{\rm e}^{-Qs}\,{\rm d}s\,1_{\{B_{\rm e}>t\}}\bigg)\boldsymbol{\varrho}\\ &= \beta\,\boldsymbol{a}'Q^{-1}\,\boldsymbol{\varrho}\,\mathbb{P}(B_{\rm e}>t) = \lambda\beta\bar{F}_{\rm e}(t)\,.\end{aligned} \tag{54}$$

Next, we move to the second term. To this end, we first recall that

$$\mathbb{E}[L(0)\,|\,\boldsymbol{J}] = \int_0^\infty \mathbb{E}\Big({\rm e}^{-Q'(s-B_{\rm e})}\boldsymbol{a}\,1_{\{B_{\rm e}\le s\}}\Big){\rm d}\boldsymbol{J}(s), \tag{55}$$

$$\mathbb{E}[L(t)\,|\,\boldsymbol{J}] = \int_{-t}^\infty \mathbb{E}\Big({\rm e}^{-Q'(s+t-B_{\rm e})}\boldsymbol{a}\,1_{\{B_{\rm e}\le s+t\}}\Big){\rm d}\boldsymbol{J}(s)\,. \tag{56}$$

As $\boldsymbol{J}$ has independent increments, when considering the covariance between $\mathbb{E}[L(0)\mid \boldsymbol{J}]$ and $\mathbb{E}[L(t)\mid \boldsymbol{J}]$, we can restrict ourselves to integrating over $s>0$ only; more concretely,

$$\begin{aligned}(57)\qquad &\mathbb{C}\mathrm{ov}(\mathbb{E}[L(0)\mid \boldsymbol{J}],\mathbb{E}[L(t)\mid \boldsymbol{J}])\\ &=\beta^2\,\mathbb{C}\mathrm{ov}\Big(\int_0^\infty \mathbb{E}\Big(e^{-Q'(s-B_{\mathrm{e}})}\boldsymbol{a}\,1_{\{B_{\mathrm{e}}\le s\}}\Big)\mathrm{d}\boldsymbol{J}(s),\\ &\qquad\qquad \int_0^\infty \mathrm{E}\Big(\mathrm{e}^{-Q'(s+t-B_{\mathrm{e}})}\boldsymbol{a}\,1_{\{B_{\mathrm{e}}\le s+t\}}\Big)\mathrm{d}\boldsymbol{J}(s)\Big)\\ &=\beta^2\int_0^\infty \mathbb{E}\Big(\boldsymbol{a}'\,e^{-Q(s-B_{\mathrm{e}})}1_{\{B_{\boldsymbol{e}}\le s\}}\Big)\Sigma\,\mathbb{E}\Big(\mathrm{e}^{-Q'(s+t-B_{\mathrm{e}})}\,\boldsymbol{a}\,1_{\{B_{\mathrm{e}}\le s+t\}}\Big)\mathrm{d}s\,,\end{aligned}$$

where the last step follows by using (15).

As it turns out, the last formula (as well as (44)) may be simplified, as follows. Let $B_{\mathrm{e},1}$ and $B_{\mathrm{e},2}$ be two i.i.d. copies of B_{e}. Recalling (19) and denoting $x^+ := x\vee 0$ and $x^- := -x\wedge 0 = (-x)^+$, (58) can be rewritten as

$$\begin{aligned}(58)\quad &\beta^2\boldsymbol{a}'\int_0^\infty \mathbb{E}\Big(e^{-Q(s-B_{\mathrm{e}})}1_{\{B_{\mathrm{e}}\le s\}}\Big)\Sigma\,\mathbb{E}\Big(e^{-Q'(s+t-B_{\mathrm{e}})}1_{\{B_{\mathrm{e}}\le s+t\}}\Big)\mathrm{d}s\,\boldsymbol{a}\\ &=\beta^2\boldsymbol{a}'\,\mathbb{E}\left(\int_{B_{\mathrm{e},1}\vee(B_{\mathrm{e},2}-t)}^\infty \mathrm{e}^{-Q(s-B_{\mathrm{e},1})}\Sigma\,\mathrm{e}^{-Q'(s+t-B_{\mathrm{e},2})}\mathrm{d}s\right)\boldsymbol{a}\\ &=\beta^2\boldsymbol{a}'\,\mathbb{E}\left(\mathrm{e}^{-Q(B_{\mathrm{e},1}-B_{\mathrm{e},2}+t)^-}\left(\int_0^\infty \mathrm{e}^{-Qs}\Sigma\,\mathrm{e}^{-Q's}\mathrm{d}s\right)\mathrm{e}^{-Q'(B_{\mathrm{e},1}-B_{\mathrm{e},2}+t)^+}\right)\boldsymbol{a}\\ &=\beta^2\boldsymbol{a}'\,\mathbb{E}\Big(\mathrm{e}^{-Q(B_{\mathrm{e},1}-B_{\mathrm{e},2}+t)^-}\Sigma_0\,\mathrm{e}^{-Q'(B_{\mathrm{e},1}-B_{\mathrm{e},2}+t)^+}\Big)\boldsymbol{a}\,,\end{aligned}$$

where in the last equality (18) has been used. For any $s\in\mathbb{R}$ we have that, recalling Equation (24),

$$(59)\qquad \boldsymbol{a}'\,\mathrm{e}^{-Qs^-}\,\Sigma_0\,\mathrm{e}^{-Q's^+}\boldsymbol{a}=\boldsymbol{a}'\,\Sigma_0\,\mathrm{e}^{-Q'|s|}\boldsymbol{a}=\boldsymbol{a}'\,\Sigma_{|s|}\,\boldsymbol{a}\,.$$

Thus, denoting by $R(t)$ the autocorrelation function (with $R(0)=1$), then adding the two terms yields

$$(60)\qquad R(t)\cdot\mathbb{V}\mathrm{ar}\,L(0)=\mathbb{C}\mathrm{ov}(L(0),L(t))=\lambda\beta\bar{F}_{\mathrm{e}}(t)+\beta^2\,\boldsymbol{a}'\,\mathbb{E}\big(\Sigma_{|B_{\mathrm{e},1}-B_{\mathrm{e},2}+t|}\big)\boldsymbol{a}\,.$$

In particular, when $t=0$, (60) provides us with a more simplified expression for $\mathbb{V}\mathrm{ar}\,L(0)$. The density of $B_{\mathrm{e},1}-B_{\mathrm{e},2}$ is clearly symmetric around zero and is given by

$$(61)\qquad g(x)=\int_0^\infty f_{\mathrm{e}}(y)f_{\mathrm{e}}(y+|x|)\,\mathrm{d}y\,.$$

Since $f_{\mathrm{e}}(\cdot)=\beta^{-1}\bar{F}(\cdot)$ is non-increasing on $[0,\infty)$, this implies that $g(\cdot)$ is unimodal.

We end this subsection by considering the 1-dimensional case. Since both $h(x):=\mathrm{e}^{-q|x|}$ ($q>0$) and $g(x)$ are symmetric and unimodal (around zero), it follows by [11] that so is their convolution. Alternatively, this follows also from [2] as $h(\cdot)$ is also log-concave. This implies that

$$\begin{aligned}(62)\qquad \mathbb{E}\,\mathrm{e}^{-q|B_{\mathrm{e},1}-B_{\mathrm{e},2}+t|}&=\int_{-\infty}^\infty \mathrm{e}^{-q|x+t|}g(x)\,\mathrm{d}x=\int_{-\infty}^\infty \mathrm{e}^{-q|-x+t|}g(-x)\,\mathrm{d}x\\ &=\int_{-\infty}^\infty \mathrm{e}^{-q|t-x|}g(x)\,\mathrm{d}x\end{aligned}$$

is unimodal with a mode at zero, hence non-increasing on $[0,\infty)$ and thus so is $R(\cdot)$. A similar result holds if $\boldsymbol{a}$ is an eigenvector associated with a real-valued (positive) eigenvalue q of Q' since then $\boldsymbol{a}'\,\Sigma_0\,\mathrm{e}^{-Q'|x|}\,\boldsymbol{a} = \boldsymbol{a}'\,\Sigma_0\,\boldsymbol{a}\,\mathrm{e}^{-q|x|}$. However, in general, even though $\boldsymbol{a}'\,\Sigma_0\,\mathrm{e}^{-Q'|x|}\,\boldsymbol{a}$ is a symmetric function, it is not clear if it is decreasing on $[0,\infty)$ or if it is log-concave. However, since it vanishes as $|x|\to\infty$, it is clear that $R(t)$ vanishes as $t\to\infty$.

4. The network case

In this section we consider networks of infinite-server queues with a Cox input process. There are n queues, with the arrival rate of queue k being a non-homogeneous Poisson process with rate function $\Lambda_k(t) = \boldsymbol{a}_k'\boldsymbol{X}(t)$. This is the same as defining the vector of arrival processes to be $\boldsymbol{\Lambda}(t) = A\boldsymbol{X}(t)$ for some matrix A with rows $\boldsymbol{a}_k'$. The $\boldsymbol{X}(t)$ process is the same multivariate shot-noise process as before. Notice that in this construction the arrival processes at the various queues are potentially dependent.

Define by $p_{km}(t)$ the probability that a job entering at queue k at time 0 is at queue m at time t; likewise, $p_{k0}(t)$ is the probability that a job entering at queue k at time 0 has left the network by time t. In the case of exponentially distributed service times and probabilistic routing, these $p_{km}(t)$'s can be computed more explicitly (relying on the machinery developed for phase-type distributions).

4.1. **Joint queue-length.** We focus on analyzing the stationary joint queue-length distribution. In principle, virtually all quantities studied in the previous section can be derived again, at the expense of introducing rather heavy notation.

In this section, we let K_m denote the stationary queue length at node $m\in\{1,\ldots,n\}$. Our objective is to compute the joint probability generating function

$$\Pi(\boldsymbol{w}) = \mathbb{E}\left(\prod_{m=1}^{n} w_m^{K_m}\right). \tag{63}$$

Using the same arguments as in the previous section, we conclude that K_m has a mixed-Poisson distribution. More precisely, K_m has Poisson distribution with (random) parameter

$$\int_{-\infty}^{0}\sum_{k=1}^{n}\Lambda_k(s)p_{km}(-s)\,\mathrm{d}s. \tag{64}$$

It follows that

$$\begin{aligned}\Pi(\boldsymbol{w}) &= \mathbb{E}\exp\left(\sum_{m=1}^{n}\int_{-\infty}^{0}\sum_{k=1}^{n}\Lambda_k(s)\,p_{km}(-s)\,\mathrm{d}s\,(w_m-1)\right)\\ &= \mathbb{E}\exp\left(\sum_{m=1}^{n}\int_{-\infty}^{0}\sum_{k=1}^{n}\boldsymbol{a}_k'\boldsymbol{X}(s)\,p_{km}(-s)\,\mathrm{d}s\,(w_m-1)\right).\end{aligned} \tag{65}$$

Recalling that

$$\boldsymbol{X}(s) = \int_{-\infty}^{s}\mathrm{e}^{-Q(s-r)}\mathrm{d}\boldsymbol{J}(r), \tag{66}$$

we obtain that

$$\Pi(\boldsymbol{w}) = \mathbb{E}\exp\left(\sum_{k=1}^{n} \boldsymbol{a}_k' \int_{-\infty}^{0}\int_{r}^{0} \mathrm{e}^{-Q(s-r)}\,\pi_k(s\,|\,\boldsymbol{w})\,\mathrm{d}s\,\mathrm{d}\boldsymbol{J}(r)\right), \tag{67}$$

with

$$\pi_k(s\,|\,\boldsymbol{w}) := \sum_{m=1}^{n} p_{km}(-s)\,(w_m-1) = \sum_{m=1}^{n} p_{km}(-s)\,w_m + p_{k0}(-s) - 1. \tag{68}$$

This leads to the following result.

Theorem 4.1. *For any* $\boldsymbol{w}\in\mathbb{R}^n$,

$$\Pi(\boldsymbol{w}) = \exp\left(-\int_{-\infty}^{0}\eta\left(-\sum_{k=1}^{n}\int_{r}^{0}\mathrm{e}^{-Q'(s-r)}\,\pi_k(s\,|\,\boldsymbol{w})\,\mathrm{d}s\,\boldsymbol{a}_k\right)\mathrm{d}r\right). \tag{69}$$

4.2. **Example.** In this illustrative example we consider a two-node tandem system in which the service times at both nodes are exponential with parameter $\kappa>0$. We assume for ease that there is only input at the first queue. It is immediate that, for $t\geqslant 0$,

$$p_{11}(t) = \mathrm{e}^{-\kappa t},\quad p_{12}(t) = \kappa t\,\mathrm{e}^{-\kappa t},\quad p_{10} = 1-\mathrm{e}^{-\kappa t}-\kappa t\,\mathrm{e}^{-\kappa t}; \tag{70}$$

use that the sojourn time in the system is Erlang(2). We find that

$$\pi_1(t\,|\,w_1,w_2) = (w_1-1)\,\mathrm{e}^{\kappa t} - (w_2-1)\,\kappa t\,\mathrm{e}^{\kappa t}. \tag{71}$$

Assume for ease that $d=1$. We choose $a_1=1$ (whereas $a_2=0$, as we assumed no external input to the second queue). As a consequence, we have that $\Lambda_1(t)=X(t)$ and $\Lambda_2(t)=0$, where $X(t)$ can be represented as

$$X(t) = \int_{(-\infty,t]} \mathrm{e}^{-q(t-s)}\,\mathrm{d}J(s), \tag{72}$$

for some $q>0$ and a (scalar) Lévy subordinator $J(\cdot)$ (with exponent $-\eta(\cdot)$).
Appealing to Thm. 4.1,

$$\Pi(w_1,w_2) = \mathbb{E}\big[w_1^{K_1}w_2^{K_2}\big] = \exp\left(-\int_{-\infty}^{0}\eta\left(-\int_{r}^{0}\mathrm{e}^{-q(s-r)}\pi_1(s\,|\,w_1,w_2)\,\mathrm{d}s\right)\mathrm{d}r\right). \tag{73}$$

To evaluate this expression, observe that

$$\begin{aligned}\int_{r}^{0}\mathrm{e}^{-q(s-r)}\pi_1(s\,|\,w_1,w_2)\,\mathrm{d}s &= \int_{r}^{0}\mathrm{e}^{-q(s-r)}((w_1-1)\,\mathrm{e}^{\kappa s}-(w_2-1)\,\kappa s\,\mathrm{e}^{\kappa s})\mathrm{d}s\\ &= (w_1-1)\frac{\mathrm{e}^{qr}-\mathrm{e}^{\kappa r}}{\kappa-q} + (w_2-1)\left(\frac{\kappa r\,\mathrm{e}^{\kappa r}}{\kappa-q}+\frac{\kappa(\mathrm{e}^{qr}-\mathrm{e}^{\kappa r})}{(\kappa-q)^2}\right).\end{aligned} \tag{74}$$

Now suppose that $J(\cdot)$ corresponds to a Gamma process [7, Section 1.2.4] with (without loss of generality) rate and shape parameters both equal to 1, i.e.,

$$\eta(\alpha) = -\log\left(\frac{1}{1+\alpha}\right) = \log(1+\alpha). \tag{75}$$

The probability generating function of the queue length in the first queue is therefore

$$\Pi(w,1) = \exp\left(-\int_{-\infty}^{0}\log\left(1+(1-w)\frac{\mathrm{e}^{qr}-\mathrm{e}^{\kappa r}}{\kappa-q}\right)\mathrm{d}r\right). \tag{76}$$

Using the Taylor expansion of the logarithm, the exponent of this expression can be rewritten as

$$\int_{-\infty}^{0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(v(w)\,({\rm e}^{qr}-{\rm e}^{\kappa r}))^n{\rm d}r, \tag{77}$$

with $v(w):=(1-w)/(\kappa-q)$. Relying on the binomium, this equals

$$\int_{-\infty}^{0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(v(w))^n\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\,{\rm e}^{qrk+\kappa r(n-k)}\right){\rm d}r. \tag{78}$$

Swapping the order of the summations and the integral, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n}(v(w))^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{qk+\kappa(n-k)}. \tag{79}$$

We thus end up with

$$\mathbb{E}\big[w^{K_1}\big]=\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(v(w))^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{qk+\kappa(n-k)}\right). \tag{80}$$

A similar procedure yields the probability generating function of the second queue, but the resulting expressions are slightly more complicated.

5. Conclusions and suggestions for further research

In this paper we have considered a network of infinite-server queues where the input process is a Cox process that allows much modelling flexibility; the arrival rate is represented as a linear combination of the components of a multivariate generalized shot-noise process. We have derived some distributional properties of the multivariate shot-noise process, subsequently exploiting them to obtain the joint transform of the numbers of customers, at consecutive time epochs, in an infinite-server queue with as input process such a Cox process. We have also derived the joint steady-state transform of the vectors of arrival rate and queue length, as well as their means and covariance structure, and we have studied the departure process from the queue. Finally, we extended our analysis to the setting of a network of infinite-server queues, allowing the arrival processes at the various queues to be dependent Cox processes.

In a future study, we intend to investigate various related aspects, including (i) develop a recursive scheme that will allow us to obtain higher-order moments of $(\Lambda(t), L(t))$, (ii) derive asymptotics of the queue length process, under assumptions regarding the tail behavior of the shot-noise process, and (iii) study the heavy-traffic behavior of the queue-length process. It could also be investigated to what extent the assumption of exponential decay in the shot-noise process can be relaxed.

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