EURANDOM PREPRINT SERIES

2021-002

March 22, 2021

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ISSN 1389-2355

Peer-to-Peer Lending: a Growth-Collapse Model and its Steady-State Analysis

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March 22, 2021

Abstract

We present a stochastic growth-collapse model for the capital process of a peer-to-peer lending platform. New lenders arrive according to a compound Poisson-type process with a state-dependent intensity function; the growth of the lending capital is from time to time interrupted by partial collapses whose arrival intensities and sizes are also state-dependent. In our model the steady-state probability distribution of the capital level administered via the platform is a key performance measure, because the brokerage fee is a fixed (small) fraction of it. In the case of exponentially distributed upward jumps we derive an explicit expression for its probability density, for quite general arrival rates of upward and downward jumps and for certain collapse mechanisms. In the case of generally distributed upward jumps, we derive an explicit expression for the Laplace transform of the steady-state cash level density in various special cases. An alternative model featuring up and down periods and a shot noise mechanism for the downward evolution is also analyzed in steady state.

1 Introduction

The goals of this paper are (i) to present a stochastic capital management model for peer-to-peer (P2P) lending, and (ii) to perform a steady-state analysis of the capital level in that model.

P2P lending is the practice of lending money through online services that match lenders with borrowers. The P2P lending company offers a platform where people, or businesses, can lend directly to other people or businesses without the need for a bank as a middleman. It takes brokerage fees for providing the match-making platform (and for doing a credit-check of the borrowers). Compared to investment and savings products from banks, borrowers can borrow money at lower interest rates, and lenders can earn higher returns. Although the P2P lending company applies a strict screening system, it may happen that a borrower is in arrears. If that happens, the company initiates collection procedures against the debtor.

P2P lending started in 2005 in the UK, shortly thereafter followed by the US, and became more popular after the 2007/2008 financial crisis when banks refused to increase their loan

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portfolios. Nowadays there are numerous P2P platforms all over the world; P2P lending is rapidly gaining recognition among rule-makers and regulators.
A distinguishing feature is that the money of the lenders is transferred to a trust account from where the borrowing is carried out. The repayments are also transferred directly to the trust account whose existence ensures that investor's deposits do not mingle with the money of the P2P lending company, so that the lenders are protected in case of an insolvency of that company. Another distinguishing feature is that there is no direct relation between lender and borrower; every deposit is typically divided into much smaller parts among many borrowers. A principal difference between banks and P2P systems is the economic fact that depositors become lenders whose profits are *not* exposed to the usual fluctuations in the capital market, so that the cash level process has relatively little volatility; P2P systems are exposed only to the exceptional volatility caused by a severe crash in which many borrowers go bankrupt. Examples of such crashes are the dot.com bubble in 2000, the subprime crisis in 2008 that started with the crash of Lehman Brothers, and the corona crisis of 2020. In these cases, the stock market crash was accompanied by bankruptcies in the entire business sector of small and medium businesses and even large companies. In other words, the crisis hit the entire capital market, since all those economic entities could not meet their financial obligations.

The P2P cash management model. In our model we investigate the temporal evolution of the total amount of money transferred to the P2P company by lenders. We assume that all lenders want long-term investments so that every deposit is virtually forever available for lending out by the system. The jumps are due to new arrivals of lenders. There is an infinite demand by borrowers, so that all money is always lent to borrowers. Their back payments are split in two streams. The pure interest on the residual debt goes to the lenders (is not reborrowed). The part that gradually repays the loan goes to the system and is immediately given to new borrowers (sufficient demand is always available). This way the total amount of borrowed money is always equal to the total amount invested by the lenders. The lenders are satisfied with the continuous stream of interest on their investment that they are receiving. The P2P company receives a small fraction of the incoming pure interest as its brokerage fee, which constitutes its profit. The upward evolution of the cash level is only interrupted by the effects of a severe economic crash (examples were mentioned above) which causes a significant portion of the borrowers to default. Note that the P2P lending company's cash level is virtual in the sense that the money is never in the company's account but always lent out in full.

We model the content (cash) level as a stochastic process that fluctuates over time. It jumps upward and downward at random times, and stays constant in between jumps. A jump upward occurs whenever a lender deposits a new amount in the account. The jumps downward represent the effects of crises. In such a case, the market partially collapses and a random portion of the capital is lost. We shall model this via downward jumps of the content level by a random fraction that is proportional to that content level. Generally speaking (a detailed model description is provided in Section 2), we assume that arrivals of deposits and of crises occur according to independent Poisson processes, whose rates are allowed to depend on the current cash level. In the main model of the paper, we do not take into account the following two possibilities: (i) a lender decides to withdraw part of her deposit; (ii) an individual borrower is not able to repay her loan. In Remark 6 we briefly outline how (i) could be taken into account in our model analysis; (ii) could be a problem for further research.

Main results. Our main focus is on the steady-state probability distribution of the cash level. This is the key performance measure, because the brokerage fee of the P2P lending company is a fixed portion of it. In the case of exponentially distributed upward jumps we derive an explicit expression for the stationary cash level probability density, for quite general arrival rates of upward and downward jumps, and for a proportionality function $h(x) = x^a$, with $0 < x < 1$ and $a > 0$, for the downward jumps. In the case of generally distributed upward jumps, we derive an explicit expression for the Laplace transform of the steady-state cash level density in two cases: (i) constant jump rates and $h(x) = x^a$, and (ii) the ratio of intensities of upward and downward jumps is inversely proportional to x, and $h(x) = x$ (jumps downward are uniformly proportional to the just-before-crash cash level).
We also obtain the Laplace transform of the stead-state cash level in a model variant in which there is a background process that alternates between two period types. During *up* periods the cash level grows, according to a compound Poisson process, and stays constant in between jumps; during *down* periods (recessions) it grows according to another compound Poisson process, but in between jumps it decreases gradually, with a speed that is proportional to the cash level.

Related literature. The economic research has focussed on descriptively studying the real-life determinants of online P2P lending and borrowing practices (see e.g. the survey articles by Bachmann et al. [4] and Chen and Han [13] as well as Au et al. [3]). To the best of our knowledge this paper provides the first attempt toward a mathematical analysis of P2P systems by means of stochastic models. The models presented here bear a similarity to models in disciplines like storage theory, insurance risk and queueing theory. Some key papers on storage processes with a non-constant release rate are those of Gaver and Miller [15], Harrison and Resnick [16] and Brockwell et al. [12]. We also refer to [16] for an insightful discussion of the stability condition of storage processes with state-dependent release rate. Bekker et al. [5] consider a class of queueing models in which both the arrival rate and the service speed may be workload dependent. We further refer to [8] for the analysis of a large class of storage processes in which the rate at which storage increases or decreases is an affine function of the current storage level, while also upward and downward jumps are allowed. That paper also considers related – in some cases dual – insurance risk models, and contains many references. The downward jumps in our model also occur in the literature on so-called growth-collapse models; see, e.g., [7, 10, 17].

The paper is organized as follows. The main model under consideration is described in Section 2. The case of exponentially distributed upward jumps and very general arrival rate functions of deposits and crises is analyzed in Section 3, while Section 4 is devoted to the case of generally distributed upward jumps. Section 5 studies the model variant in which up and down periods alternate.

2 Model description

In this section we present the model under consideration, and we discuss the stability condition. We describe the capital (content level) of the cash management system of a P2P lending company as a stochastic process $\{\mathbf{V}(t),\ t \geq 0\}$ that evolves in the following way. It jumps upward and downward at random times, and stays constant in between jumps. Jumps upward (due to deposits) occur according to a Poisson process, with state-dependent rate function $\lambda(w)$ when the capital equals w. Successive jump sizes are independent, identically distributed (i.i.d.) integrable random variables, generically denoted by $\mathbf{G}$, with distribution function $G(\cdot)$ and Laplace-Stieltjes transform (LST) $\gamma(\cdot)$. The upward jump sizes are assumed to have a continuous density and finite mean.
Jumps downward (due to crises) occur according to a Poisson process, with state-dependent rate $\eta(w)$ when the capital equals w. They occur independently of the process of upward jumps. A special feature of the model is that the sizes of downward jumps depend on their starting level w, via a function $h(\cdot)$: If the capital level is w just before a downward jump, then the probability to jump to a level lower than x is given by $h(\frac{x}{w})$, for any $0 < x < w$. Mathematically, this can be viewed as a multiplication of the just-before-crash level by a random variable with distribution function $h(\cdot)$ on $[0, 1]$. We shall restrict ourselves in this paper to the choice $h(y) = y^a$ with $0 < y < 1$ and $a > 0$. Observe that the case $a = 1$ corresponds to jumps downward from level w that are uniformly distributed on $(0, w)$; furthermore observe that for these $h(\cdot)$ level 0 is never reached. The latter property also holds for any other $a > 0$.
The cash level remains constant between jumps, because the back payments of the borrowers are split into pure interest for the residual debt and repayments of the loan: the first stream goes to the lender except for a proportional fee that goes to the P2P company, while the second stream remains in the P2P system (immediately used for further lending), so that the deposits of all lenders remain unchanged.

Let us first discuss the *question of stability of* $\mathbf{V} = \{\mathbf{V}(t),\ t \geq 0\}$. The lack-of-memory property of the two underlying jump processes together with the proportionality feature of the downward jumps imply that $\mathbf{V}$ is a Markov process. Under our assumptions it is readily seen that the transition densities of $\mathbf{V}$ are jointly continuous in their variables and there is an open set on which they are bounded away from zero. Therefore the theory of Harris recurrence for Markov processes with continuous time and with state space $[0, \infty)$ can be applied (the standard results on Harris recurrence needed here can e.g. be found in Section VII.3 of Asmussen (2003); see in particular Example 3.1 and Proposition 3.8). $\mathbf{V}$ can be reconstructed as a Harris process as follows. We use Example 3.8 of Asmussen (2003) where without restriction of generality we can take $R = S = [0, 1)$ and an arbitrary $r > 0$. Then we can fix an $\epsilon > 0$ such that the transition probability measure $B \mapsto P(\mathbf{V}(t) \in B|\mathbf{V}(0) = x) = P^r(x, B)$ dominates the measure $B \mapsto \epsilon l(B \cap [0, 1))$, l being the Lebesgue measure. Now we can construct a process equivalent to $\mathbf{V}$ as follows. It is equal to $\mathbf{V}$ until the time τ at which the first jump from $[1, \infty)$ to $[0, 1)$ occurs. Then with probability ϵ the process starts a new cycle at time $\tau + r$ with the Lebesgue measure on $[0, 1)$ as restarting distribution, while with probability $1 - \epsilon$ it continues at time $\tau + r$ with the restarting distribution

$$B \mapsto (1 - \epsilon)^{-1}(P^r(\mathbf{V}(\tau), B) - \epsilon l(B)).$$

Finally, the missing intermediate piece $(\mathbf{V}(t))_{\tau<t<\tau+r}$ is constructed by using the conditional distribution of $(\mathbf{V}(t))_{0<t<r}$ given that the boundary values $\mathbf{V}(0)$ and $\mathbf{V}(r)$ are equal to the already constructed values of $\mathbf{V}(\tau)$ and $\mathbf{V}(\tau+r)$, respectively.
Continuing with this construction cycle per cycle, this provides $\mathbf{V}$ with a regenerative structure. Hence, $\mathbf{V}$ is positive recurrent if the cycle lengths and the accumulated increments during cycles have a finite expectation. For this it is sufficient that the following condition holds.

Condition SC. $\eta(w) > \eta_0$ for some $\eta_0 > 0$ and all $w \geq 1$, and $\lambda(w)/\eta(w) < c < \infty$ for some $c > 0$ and all $w \geq 1$.

To see this, note that in this case the expected time between any two successive partial collapses is smaller than $1/\eta_0$ and the corresponding expected accumulated increase is smaller than $m(1+c)$, because after any upward jump the probability that the next jump will be a collapse is greater than $1/(1+c)$ so that the number of upward jumps before a downward jump is geometrically distributed. Now denote by $\mathbf{I}_n$ the increase between the $(n-1)$st and the nth collapse and by $\mathbf{B}_n$ the proportionality factor corresponding to the nth collapse. Then the cash level just before the nth collapse is given by

$$\mathbf{C}_n = \mathbf{I}_n + \mathbf{B}_{n-1}\mathbf{I}_{n-1} + ... + \mathbf{B}_{n-1}\cdots\mathbf{B}_1\mathbf{I}_1, \quad n > 1,$$

and $\mathbf{C}_1 = \mathbf{I}_1$. Note that $\mathbf{B}_{n-1}\cdots\mathbf{B}_j$ is independent of $\mathbf{I}_j$ for every j and that the $\mathbf{B}_n$ are i.i.d. and have the common distribution function h on (0,1) so that their common mean is smaller than 1. Since the $\mathbf{I}_n$ have uniformly bounded means, it follows that

$$K = \sup_n E(\mathbf{C}_n) < \infty,$$

and thus

$$\sup_n P(\mathbf{C}_n \geq 2K) \leq \sup_n E(\mathbf{C}_n)/2K = 1/2.$$

Hence, as long as $\mathbf{C}_n < 2K$, the probability that level 1 is downcrossed at the nth jump downwards is bounded from below by $P(\mathbf{B}_n < 1/(2K)) = h(1/(2K))$, and the probability that $\mathbf{C}_n < 2K$ is at least 1/2. Thereafter, a randomization takes place which leads with probability ϵ to the beginning of a new cycle. Therefore, the cycle lengths have finite mean.
It follows that the steady-state distribution of $\mathbf{V}$ exists when Condition SC holds. We denote the steady-state capital level by $\mathbf{V}_e$ and its density by $f(\cdot)$. In the next two sections we aim to determine $f(\cdot)$ for a number of choices of $\lambda(\cdot)$, $\eta(\cdot)$, $G(\cdot)$ and a.

3 Case I: Exponential jumps upward

In this section we restrict ourselves to the case of exponentially distributed upward jumps with mean $1/\mu$: $\mathbb{P}(\mathbf{G} < w) = 1 - \mathrm{e}^{-\mu w}$, $w > 0$. That restriction will allow us to derive the density $f(\cdot)$ of the steady-state capital level $\mathbf{V}_e$, without having to resort to LSTs and while allowing quite general arrival rate functions of deposits and crises. We use the level-crossing technique (cf. Brill [11] and Cohen [14]), which states that, in equilibrium, the rate of upcrossing any level $x > 0$ should equal the rate of downcrossing that level. This results in the following integral equation, which we first formulate for *generally* distributed upward jumps and a non-specified function $h(\cdot)$:

$$\int_{w=0}^{x} \lambda(w)(1-G(x-w))f(w)\mathrm{d}w = \int_{w=x}^{\infty} h(\frac{x}{w})\eta(w)f(w)\mathrm{d}w, \quad x>0. \tag{1}$$

The lefthand side of (1) represents the rate to upcross level x, and the righthand side represents the rate to downcross level x. Indeed, when the capital equals $w<x$, then the probability to upcross level x in the next $\mathrm{d}w$ time units equals $\lambda(w)\mathrm{d}w$ times the probability $1-G(x-w)$ that a jump from level w is larger than $x-w$ (we ignore $\mathrm{o}(w)$ contributions); and when the capital equals $w>x$, then the probability to downcross level x in the next $\mathrm{d}w$ time units equals $\eta(w)\mathrm{d}w$ times the probability that a downward jump from level w is larger than $w-x$, i.e., that the capital reduction factor exceeds w/x, i.e., the proportionality factor is less than x/w, and this probability equals $h(x/w)$. In principle there also should be a term in the lefthand side that represents jumps from level 0 that upcross x. However, our choice of $h(\cdot)$ will exclude the possibility that level zero is reached.
The choice $G(x)=1-\mathrm{e}^{-\mu x}$ and $h(y)=y^a$ reduces (1) to

$$\int_{w=0}^{x} \lambda(w)\mathrm{e}^{-\mu(x-w)}f(w)\mathrm{d}w = \int_{w=x}^{\infty} (\frac{x}{w})^a\eta(w)f(w)\mathrm{d}w, \quad x>0. \tag{2}$$

Introduce, for $w>0$, $z(w):=\frac{\eta(w)}{w^a}f(w)$ and $R(w):=\frac{\lambda(w)}{\eta(w)}$. Then (2) becomes

$$\int_{w=0}^{x} w^aR(w)\mathrm{e}^{\mu w}z(w)\mathrm{d}w = x^a\mathrm{e}^{\mu x}\int_{w=x}^{\infty} z(w)\mathrm{d}w, \quad x>0. \tag{3}$$

Differentiation with respect to x yields, after division by $\mathrm{e}^{\mu x}$:

$$x^aR(x)z(x) = (\mu x^a + a x^{a-1})\int_{w=x}^{\infty} z(w)\mathrm{d}w - x^a z(x), \tag{4}$$

and hence

$$(R(x)+1)z(x) = (\mu+\frac{a}{x})\int_{w=x}^{\infty} z(w)\mathrm{d}w. \tag{5}$$

Differentiating once more, and using (5) to eliminate the integral, gives

$$(R(x)+1)z'(x) = -[R'(x)+\mu+\frac{a}{x}+\frac{a}{\mu x^2+ax}(R(x)+1)]z(x), \tag{6}$$

so

$$\frac{z'(x)}{z(x)} = -\frac{a}{\mu x^2+ax} - \frac{R'(x)+\mu+a/x}{R(x)+1}. \tag{7}$$

Its solution is easily seen to be

$$z(x) = C(1+\frac{a}{\mu x})\frac{1}{R(x)+1}\exp\Big(-\int^{x}\frac{\mu y+a}{y(R(y)+1)}\mathrm{d}y\Big), \tag{8}$$

with C a constant. Notice that we do not yet specify the lower integration bound in the integral in (8). We need to take into account the possibility that $y=0$ is a singularity of its integrand. The integral has no other singularities, because $R(y)>0$ for all $y>0$. In conclusion we have the following result.

Theorem 3.1. *The stationary cash level density for the case of exp(μ) distributed upward jumps, upward jump rate $\lambda(x)$, downward jump rate $\eta(x)$ and proportionality function $h(x) = x^a$ is given by*

$$f(x) = C(x^a + \frac{a}{\mu}x^{a-1})\frac{1}{\lambda(x)+\eta(x)}\exp\Big(-\int^x \frac{\mu y + a}{y(R(y)+1)}\mathrm{d}y\Big), \quad x > 0. \tag{9}$$

The constant C is determined by the fact that the integral of the cash level density equals one: $\int_0^\infty f(x)\mathrm{d}x = 1$.

Special cases.
(i) If $R(x) = \lambda(x)/\eta(x) = r$, then

$$\begin{aligned} f(x) &= \frac{C}{r+1}(x^a + \frac{a}{\mu}x^{a-1})\frac{1}{\eta(x)}\exp\Big(-\int^x \frac{\mu y + a}{y(r+1)}\mathrm{d}y\Big) \\ &= \frac{C}{r+1}(x^a + \frac{a}{\mu}x^{a-1})\frac{1}{\eta(x)}x^{-\frac{a}{r+1}}\mathrm{e}^{-\frac{\mu}{r+1}x}, \quad x > 0. \end{aligned} \tag{10}$$

Observe that, when $\eta(x)$ is a constant, say η (and hence $\lambda(x)$ also is a constant), this is a mixture of the two Gamma densities Gamma($a\frac{r}{r+1}+1, \frac{\mu}{r+1}$) and Gamma($a\frac{r}{r+1}, \frac{\mu}{r+1}$). A straightforward calculation gives the constant C:

$$C = \frac{\mu}{a\eta}\frac{(\frac{\mu}{r+1})^{a\frac{r}{r+1}}}{\Gamma(a\frac{r}{r+1})}. \tag{11}$$

Notice that the resulting expression for $f(x)$ does not contain η or λ anymore; they only appear in the ratio $r = \lambda/\eta$. Indeed, that could already have been concluded from (1) with $\lambda/\eta = r$.
Knowledge of the nth moment of the Gamma distribution immediately yields that

$$E[\mathbf{V}_e^n] = \frac{n+a}{a}(\frac{r+1}{\mu})^n\frac{\Gamma(n + a\frac{r}{r+1})}{\Gamma(a\frac{r}{r+1})}, \quad n = 1, 2, \ldots . \tag{12}$$

In particular,

$$E[\mathbf{V}_e] = \frac{(1+a)r}{\mu}. \tag{13}$$

Apparently, the mean steady-state cash level grows linearly with the proportionality parameter a, the jump ratio $r = \lambda/\eta$, and the mean deposit size $1/\mu$.

(ii) If $R(x) = \sum_{n=-K}^{L} r_n x^n$, with $K, L \geq 0$ (and the r_n chosen such that $R(x) = \frac{\lambda(x)}{\eta(x)} > 0$ for all x), then the exponent in (9) can be evaluated by a partial fraction expansion. In particular, for $R(x) = r_{-1}/x$ we have:

$$f(x) = C(x^{a+1} + \frac{a}{\mu}x^a)\frac{1}{\eta(x)}(\frac{r_{-1}}{r_{-1}+x})^{a+1-\mu r_{-1}}\mathrm{e}^{-\mu x}, \quad x > 0. \tag{14}$$

If $K = L = 1$, so $R(x) = \frac{r_{-1}}{x} + r_0 + r_1 x$, the integral in (9) becomes

$$\begin{aligned} &\int^x \frac{1}{r_1}\frac{\mu y + a}{y^2 + \frac{r_0+1}{r_1}y + \frac{r_{-1}}{r_1}}\mathrm{d}y \\ = &\ \frac{1}{r_1}\int^x (\frac{A_1}{y-y_1} + \frac{A_2}{y-y_2})\mathrm{d}y, \end{aligned}$$

with

$$y_{1,2} = -\frac{r_0+1}{2r_{-1}} \pm \frac{1}{2}\sqrt{\{(\frac{r_0+1}{r_1})^2 - 4\frac{r_{-1}}{r_1}\}},$$

$$A_1 = \frac{\mu y_1 + a}{y_1 - y_2}, \quad A_2 = \frac{\mu y_2 + a}{y_2 - y_1}.$$

Hence, from (9),

$$f(x) = \frac{C}{r_1}(x^{a+1} + \frac{a}{\mu}x^a)\frac{1}{\eta(x)}(x-y_1)^{-1-A_1/r_1}(x-y_2)^{-1-A_2/r_1}, \quad x>0. \tag{15}$$

Note that for $r_1 > 0$ we do not get an exponential term, unlike the special cases $R(x) = r$ and $R(x) = r_{-1}/x$.

4 Case II: general jumps upward

In this section we allow the generic upward jump $\mathbf{G}$ (a deposit) to have a general distribution $G(\cdot)$ with LST $\gamma(s)$. In Subsection 4.1 we consider the case $\lambda(x) \equiv \lambda$, $\eta(x) \equiv \eta$, $h(x) = x^a$, and in Subsection 4.2 the case $R(x) = \frac{\lambda(x)}{\eta(x)} = \frac{r_{-1}}{x}$, $h(x) = x$.

4.1 Case II.a

When $\lambda(x) \equiv \lambda$, $\eta(x) \equiv \eta$, $h(x) = x^a$, the level-crossing equation (1) becomes

$$\int_{w=0}^{x} \lambda P(\mathbf{G} > x-w) f(w) \mathrm{d}w = \eta \int_{w=x}^{\infty} (\frac{x}{w})^a f(w) \mathrm{d}w. \tag{16}$$

Taking Laplace transforms, and introducing $\phi(s) := E[\mathrm{e}^{-s\mathbf{V}_e}] = \int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x$, we obtain

$$\lambda \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=0}^{x} P(\mathbf{G} > x-w) f(w)\mathrm{d}w\mathrm{d}x = \eta \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=x}^{\infty} (\frac{x}{w})^a f(w)\mathrm{d}w\mathrm{d}x. \tag{17}$$

The lefthand side of (17) equals

$$\lambda \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=0}^{x} P(\mathbf{G} > x-w) f(w)\mathrm{d}w\mathrm{d}x = \lambda \frac{1-\gamma(s)}{s}\phi(s). \tag{18}$$

In handling the righthand side of (17), we use partial integration to write

$$\int_{x=0}^{w} \mathrm{e}^{-sx} x^a \mathrm{d}x = -\frac{1}{s} w^a \mathrm{e}^{-sw} + \frac{a}{s}\int_{x=0}^{w} \mathrm{e}^{-sx} x^{a-1}\mathrm{d}x. \tag{19}$$

The last term of that expression should be integrated with respect to s to, once more, get a power x^a:

$$\begin{aligned} \int_{x=0}^{w} \mathrm{e}^{-sx} x^{a-1}\mathrm{d}x &= -\int_{x=0}^{w} x^{a-1} \int_{v=0}^{s} x \mathrm{e}^{-vx}\mathrm{d}v\mathrm{d}x + \int_{x=0}^{w} x^{a-1}\mathrm{d}x \\ &= -\int_{x=0}^{w} x^a \int_{v=0}^{s} \mathrm{e}^{-vx}\mathrm{d}v\mathrm{d}x + \frac{1}{a}w^a. \end{aligned} \tag{20}$$

The Laplace transform of the righthand side of (17) now becomes (using (17) itself and (18)):

$$
\begin{aligned}
& \eta \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=x}^{\infty} (\frac{x}{w})^a f(w) \mathrm{d}w \mathrm{d}x \\
= & -\frac{\eta}{s}\phi(s) - \frac{\eta a}{s} \int_{v=0}^{s} \frac{\lambda}{\eta} \frac{1-\gamma(v)}{v} \phi(v) \mathrm{d}v + \frac{\eta}{s} \int_{w=0}^{\infty} f(w) \mathrm{d}w. \qquad (21)
\end{aligned}
$$

We thus end up with the following equation:

$$
(\eta + \lambda(1-\gamma(s)))\phi(s) = \eta - a\lambda \int_0^s \frac{1-\gamma(v)}{v} \phi(v) \mathrm{d}v, \qquad (22)
$$

so after differentiation we obtain

$$
(\eta + \lambda(1-\gamma(s)))\phi'(s) = -\phi(s) \frac{d}{ds}(\eta + \lambda(1-\gamma(s))) - a\lambda \frac{1-\gamma(s)}{s} \phi(s). \qquad (23)
$$

Hence

$$
\frac{\phi'(s)}{\phi(s)} = -\frac{\frac{d}{ds}(\eta + \lambda(1-\gamma(s))}{\eta + \lambda(1-\gamma(s))} - \frac{a\lambda \frac{1-\gamma(s)}{s}}{\eta + \lambda(1-\gamma(s))}. \qquad (24)
$$

We have thus proven the following theorem.

Theorem 4.1. *The steady-state cash level LST for Case II.a is given by*

$$
E[\mathrm{e}^{-s\mathbf{V}_e}] = \phi(s) = \frac{\eta}{\eta + \lambda(1-\gamma(s))} \exp\Big(-a\lambda \int_0^s \frac{1-\gamma(u)}{u[\eta + \lambda(1-\gamma(u))]} \mathrm{d}u\Big). \qquad (25)
$$

Differentiation of (25) and subsequent substitution of $s = 0$ gives

$$
E[\mathbf{V}_e] = (1+a)\frac{\lambda E[\mathbf{G}]}{\eta}. \qquad (26)
$$

We end this subsection with several remarks.
Remark 1. It is interesting to see that a in (25) only appears as a factor in the exponent. Further observe that (26) generalizes (13) to the case of generally distributed upward jumps, for the case that $\lambda(x) \equiv \lambda$, $\eta(x) \equiv \eta$.

Remark 2. When $\mathbf{G} \sim \exp(\mu)$, (25) reduces to

$$
\begin{aligned}
\phi(s) &= \frac{\eta(\mu+s)}{\eta\mu + (\lambda+\eta)s} \exp\Big(-a \int_0^s \frac{\lambda}{\eta\mu + (\lambda+\eta)u} \mathrm{d}u\Big) \\
&= (\frac{\eta}{\lambda+\eta} + \frac{\lambda}{\lambda+\eta} \frac{\eta\mu}{\eta\mu + (\lambda+\eta)s})(\frac{\eta\mu}{\eta\mu + (\lambda+\eta)s})^{a\frac{\lambda}{\lambda+\eta}} \\
&= \frac{\eta}{\lambda+\eta}(\frac{\eta\mu}{\eta\mu + (\lambda+\eta)s})^{a\frac{\lambda}{\lambda+\eta}} + \frac{\lambda}{\lambda+\eta}(\frac{\eta\mu}{\eta\mu + (\lambda+\eta)s})^{a\frac{\lambda}{\lambda+\eta}+1}. \qquad (27)
\end{aligned}
$$

With $r = \lambda/\eta$, this becomes

$$
\phi(s) = \frac{1}{r+1}(\frac{\mu}{\mu + (r+1)s})^{a\frac{r}{r+1}} + \frac{r}{r+1}(\frac{\mu}{\mu + (r+1)s})^{a\frac{r}{r+1}+1}. \qquad (28)
$$

This case of exponential upward jumps was already discussed at the end of Section 3, where we already saw that $\mathbf{V}_e$ is distributed as a weighted sum of a Gamma$(a\frac{r}{r+1}, \frac{\mu}{r+1})$ and a Gamma$(a\frac{r}{r+1}+1, \frac{\mu}{r+1})$ distributed random variable. This is in agreement with (28).

Remark 3. We can interpret both factors of the cash level LST $\phi(s)$ in (25). First observe that $k(s) := \frac{\eta}{\eta+\lambda(1-\gamma(s))}$ is the LST of the total increment $\mathbf{K}$ during an $\exp(\eta)$ time interval of a compound Poisson process with jump rate λ and jump size LST $\gamma(s)$. Also observe that the integrand in the exponent of (25) can be rewritten as follows:

$$a\frac{\lambda\frac{1-\gamma(u)}{u}}{\eta+\lambda(1-\gamma(u))} = \frac{a\lambda E\mathbf{G}}{\eta}\frac{1-\frac{\eta}{\eta+\lambda(1-\gamma(u))}}{\frac{\lambda E[\mathbf{G}]}{\eta}u}. \tag{29}$$

Now recognize this as $\frac{a\lambda E[\mathbf{G}]}{\eta}$ times the LST of the *residual* of $\mathbf{K}$.
For a further interpretation of the second factor of (25), consider a so-called *shot noise queueing model*. This is an $M/G/1$ queue with the special feature that the service speed is proportional to the workload. Let us assume that the arrival rate of customers in that queue is λ, that their service time LST is $\beta(s)$ and that the service speed is αx if the workload equals x. The steady-state workload LST $E[\mathrm{e}^{-s\hat{\mathbf{V}}_e}]$ in such a shot noise queueing model is given by (cf. Bekker et al. [5]):

$$E[\mathrm{e}^{-s\hat{\mathbf{V}}_e}] = \exp\Big(-\frac{\lambda}{\alpha}\int_0^s \frac{1-\beta(u)}{u}\mathrm{d}u\Big). \tag{30}$$

In the case of (25) we should apparently take $\alpha = \eta$ and $\beta(s) = k(s)$; in other words, the upward jumps are distributed as $\mathbf{K}$. The decomposition $\phi(s) = k(s)\mathrm{e}^{-\int_0^s a\frac{\lambda\frac{1-\gamma(u)}{u}}{\eta+\lambda(1-\gamma(u))}\mathrm{d}u}$ in (25) immediately yields that the steady-state cash level $\mathbf{V}_e$ in our model can be written as

$$\mathbf{V}_e \stackrel{d}{=} \mathbf{K} + \hat{\mathbf{V}}_e, \tag{31}$$

$\mathbf{K}$ and $\hat{\mathbf{V}}_e$ being independent. This decomposition also quickly allows us to get moments (this is actually how (26) was found), and to obtain *tail asymptotics*. In particular, let us assume that the deposit size $\mathbf{G}$ is regularly varying of index $-\nu$, i.e.,

$$P(\mathbf{G} > x) \sim x^{-\nu}L(x), \quad x \to \infty, \tag{32}$$

with $L(x)$ a slowly varying function at infinity, so $\lim_{x\to\infty}\frac{L(gx)}{L(x)} = 1$ for any $g > 0$. The Tauberian Theorem 8.1.6 of [6] relates the tail behavior of a regularly varing random variable to the behavior of its LST near zero. It states that (32) with $1 < \nu < 2$ is equivalent with the following relation:

$$\gamma(s) - 1 + sE[\mathbf{G}] \sim -\Gamma(1-\nu)s^{\nu}L(\frac{1}{s}), \quad s \downarrow 0. \tag{33}$$

Now consider both terms of (25). Using that

$$\frac{\eta}{\eta+\lambda(1-\gamma(s))} - 1 + \frac{\lambda}{\eta}E\mathbf{G}s \sim -\frac{\lambda}{\eta}\Gamma(1-\nu)s^{\nu}L(\frac{1}{s}), \quad s \downarrow 0, \tag{34}$$

and, after some calculations,

$$\exp\Big(-a\lambda\int_0^s \frac{1-\gamma(u)}{u[\eta+\lambda(1-\gamma(u))]}\mathrm{d}u\Big) - 1 + \frac{a\lambda}{\eta}E\mathbf{G}s \sim -\frac{a\lambda}{\eta}\frac{\Gamma(1-\nu)}{\nu}s^{\nu}L(\frac{1}{s}), \quad s \downarrow 0, \tag{35}$$

it follows that (see also (26))

$$E[\mathrm{e}^{-s\mathbf{V}_e}] - 1 + E\mathbf{V}_e s \sim -\frac{\lambda}{\eta}(1+\frac{a}{\nu})\Gamma(1-\nu)s^{\nu}L(\frac{1}{s}), \quad s \downarrow 0. \tag{36}$$

Another application of Theorem 8.1.6 of [6] now implies that

$$P(\mathbf{V}_e > x) \sim \frac{\lambda}{\eta}(1+\frac{a}{\nu})x^{-\nu}L(x), \quad x \to \infty. \tag{37}$$

We conclude that, if $\mathbf{G}$ is regularly varying of index $-\nu \in (-2,-1)$, then the same holds for the cash level $\mathbf{V}_e$. More generally, a heavy tail of order $x^{-\nu}$ of the deposits results in an equally heavy tail of the cash level.

Remark 4. The proportionality property of the jumps down is indeed closely related to having a gradual decrease according to shot noise (even with $h(x) = x^a$). Consider a shot noise process that decreases at rate αx if the level is x, and that has Poisson arrivals at rate ζ. If, just after the nth arrival, the workload $\mathbf{X}_n = x$, then the probability that it decreases to a level $\mathbf{Y}_{n+1}$ below w in the arrival interval $\mathbf{T}_{n+1}$ between arrivals n and $n+1$ equals, for $w \le x$:

$$P(\mathbf{Y}_{n+1} < w | \mathbf{X}_n = x) = P(x\mathrm{e}^{-\alpha \mathbf{T}_{n+1}} < w) = P(\mathbf{T}_{n+1} > -\frac{1}{\alpha}\ln(\frac{w}{x})) = \mathrm{e}^{\frac{\zeta}{\alpha}\ln(\frac{w}{x}))} = (\frac{w}{x})^{\frac{\zeta}{\alpha}}. \tag{38}$$

This explains that $h(w) = w^a$ indeed is directly linked to shot noise.
To clarify the relation between the $M/G/1$ shot noise queue and the model of the present subsection, observe that the LST in (25) can be interpreted as the LST of the workload in the shot noise queue *immediately after an arrival.* Indeed, by PASTA, $E[\mathrm{e}^{-s\hat{\mathbf{V}}_e}]$ is the workload LST *just before an arrival.* To translate our model to the $M/G/1$ shot noise queue, we should compress the upward parts in between two consecutive downward jumps in our model to jumps upward distributed as $\mathbf{K}$, and we should replace the proportional jumps down by a shot noise decreasing path, as described in this remark. Finally, by applying PASTA to our cash management model, we see that $\mathbf{V}_e$ is also distributed as the capital just before a downward jump (crisis). Considering our model at such epochs corresponds to considering the shot noise queue just after jumps.

Remark 5. It is not hard to verify that the compound Poisson input of this subsection, with jump rate λ and jump size LST $\gamma(s)$, can be generalized to the case of a Lévy input process $\{\mathbf{X}(t), t \ge 0\}$ that is a subordinator, i.e., a non-decreasing Lévy process. In such a case, $E[\mathrm{e}^{-s\mathbf{X}(t)}] = \mathrm{e}^{-\tau(s)t}$. In our case of compound Poisson input, $\tau(s) = \lambda(1-\gamma(s))$; in the formulas in this subsection, we would simply have to replace $\lambda(1-\gamma(s))$ by $\tau(s)$, and $\lambda E[\mathbf{G}]$ by $\tau'(0)$.

Remark 6. One could extend the model of Section 4 by allowing not only level-proportional jumps downward, but also jumps downward whose size does not depend on the current cash level. Such jumps could represent withdrawals by lenders. Below we roughly sketch how this case could be analyzed. If we assume that such jumps downward occur according to a Poisson process with rate ψ and that they are independent, exponentially distributed with rate ω, then (17) becomes

$$\lambda \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=0}^{x} P(\mathbf{G} > x-w) f(w) \mathrm{d}w \mathrm{d}x + F(0)\lambda \int_0^{\infty} \mathrm{e}^{-sx}(1-G(x))\mathrm{d}x$$
$$= \eta \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=x}^{\infty} (\frac{x}{w})^a f(w)\mathrm{d}w\mathrm{d}x + \psi \int_{x=0}^{\infty} \mathrm{e}^{-sx} \int_{w=x}^{\infty} \mathrm{e}^{-\omega(w-x)} f(w)\mathrm{d}w\mathrm{d}x. \quad (39)$$

Here $F(0)$ is the probability that the process is at level 0. In the case of non-proportional downward jumps, that probability no longer is zero. Using the same steps as in (19) and (20), it follows that

$$\begin{aligned} & \lambda \frac{1-\gamma(s)}{s}[\phi(s) + F(0)] = \eta(1-\phi(s) - F(0)) \\ - & \frac{a}{s}\int_0^s [\lambda \frac{1-\gamma(v)}{v}(\phi(v)+F(0)) - \frac{\psi}{\omega - v}[\phi(v) - \phi(\omega)]]\mathrm{d}v \\ + & \frac{\psi}{\omega - s}[\phi(s) - \phi(\omega)]. \end{aligned} \quad (40)$$

Introducing $E[\mathrm{e}^{-s\mathbf{V}_e}] = \phi^*(s) = \phi(s) + F(0)$, we can rewrite this into

$$\begin{aligned} & \lambda \frac{1-\gamma(s)}{s}\phi^*(s) = \eta(1-\phi^*(s)) \\ - & \frac{a}{s}\int_0^s [\lambda \frac{1-\gamma(v)}{v}\phi^*(v) - \frac{\psi}{\omega - v}[\phi^*(v) - \phi^*(\omega)]]\mathrm{d}v \\ + & \frac{\psi}{\omega - s}[\phi^*(s) - \phi^*(\omega)]. \end{aligned} \quad (41)$$

Via differentiation we obtain a first-order linear differential equation in $\phi^*(s)$, which is similar to (23) but has an inhomogeneous part (involving the unknown $\phi^*(\omega)$). Its solution is reasonably straightforward; see, e.g., Case II.b below for another example in which a first-order linear nonhomogeneous differential equation with an unknown constant is treated.

4.2 Case II.b

In this subsection we assume that $R(x) = \frac{r_{-1}}{x}$, as in special case (ii) at the end of Section 3. We furthermore take $h(x) \equiv x$, and as before we introduce $z(x) = \frac{\eta(x)}{x} f(x)$. The level crossing equation (1) now translates into

$$\int_{w=0}^{x} wR(w)P(\mathbf{G} > x-w)z(w)\mathrm{d}w = x\int_{w=x}^{\infty} z(w)\mathrm{d}w. \quad (42)$$

Denote the Laplace transform of $z(\cdot)$ by $\zeta(\cdot)$. Taking transforms in (42) gives:

$$\begin{aligned} & r_{-1}\frac{1-\gamma(s)}{s}\zeta(s) = -\frac{d}{ds}\int_{w=0}^{\infty} z(w)\int_{x=0}^{w} \mathrm{e}^{-sx}\mathrm{d}x\mathrm{d}w \\ = & -\frac{d}{ds}\frac{\int_0^{\infty} z(w)\mathrm{d}w - \zeta(s)}{s} \\ = & \frac{\zeta'(s)}{s} + \frac{\int_0^{\infty} z(w)\mathrm{d}w - \zeta(s)}{s^2} = \frac{\zeta'(s)}{s} + \frac{\zeta(0) - \zeta(s)}{s^2}. \end{aligned} \quad (43)$$

We end up with the differential equation

$$\zeta'(s) = [\frac{1}{s} + r_{-1}(1-\gamma(s))]\zeta(s) - \frac{\zeta(0)}{s}. \tag{44}$$

The solution of this first-order linear nonhomogeneous differential equation is routinely found, using the method of variation of constants:

$$\zeta(s) = s \ \exp\Big(\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v\Big)\Big[C^* - \zeta(0)\int^s \frac{\exp\big(-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v\big)}{y^2}\mathrm{d}y\Big]. \tag{45}$$

It should be noticed that we have not yet specified the lower integration bound of the integral inside the square brackets. The reason is that the behavior of the expression on the righthand side of (45) when $s \downarrow 0$ is somewhat delicate. A careful study of this behavior will now allow us to determine the still unknown constant C^* – and also the remaining unknown $\zeta(0)$. We first observe that the term in front of the square brackets, on the righthand side of (45), behaves as s for $s \downarrow 0$. Hence the term in square brackets should go to infinity when $s \downarrow 0$. We now show that

$$\begin{aligned} & \lim_{s\downarrow 0} \ s \ \exp\Big(\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v\Big)\zeta(0)\int_s^\infty \frac{\exp\big(-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v\big)}{y^2}\mathrm{d}y \\ = \ & \lim_{s\downarrow 0}\frac{\int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y}{\frac{1}{s}} = 1. \end{aligned} \tag{46}$$

The last equality follows by using l'Hôpital's rule. This determines the choice of C^*; the term in square brackets in (45) should be $\zeta(0)\int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y$, and hence we can rewrite (45) into

$$\begin{aligned} \zeta(s) &= \zeta(0) \ s \ \exp\Big(\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v\Big)\int_s^\infty \frac{\exp\big(-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v\big)}{y^2}\mathrm{d}y \\ &= \zeta(0) \ s \ \int_s^\infty \frac{\exp\big(\int_y^s r_{-1}(1-\gamma(v))\mathrm{d}v\big)}{y^2}\mathrm{d}y. \end{aligned} \tag{47}$$

It remains to determine $\zeta(0)$. This constant is computed by using the normalizing condition $\int_0^\infty f(x)\mathrm{d}x = \int_0^\infty \frac{x}{\eta(x)}z(x)\mathrm{d}x = 1$. To obtain explicit results, we now assume that $\eta(x) \equiv \eta$ (so that $\lambda(x) = \frac{r_{-1}\eta}{x}$; we could also, e.g., have taken $\eta(x) \equiv \eta x$, which would immediately have given that $\zeta(0) = \eta$). It follows that $\int_0^\infty xz(x)\mathrm{d}x = \eta$, and hence $\zeta'(0) = -\eta$. Differentiating (47) yields

$$\begin{aligned} \zeta'(0) = \lim_{s\downarrow 0}\Big(-\frac{\zeta(0)}{s} + \zeta(0)\int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y \\ \times \ [\mathrm{e}^{\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v} + sr_{-1}(1-\gamma(s))\mathrm{e}^{\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v}]\Big). \end{aligned} \tag{48}$$

Using that $\frac{1}{s} = \int_s^\infty \frac{1}{y^2}\mathrm{d}y$, we can rewrite the above formula as

$$\begin{aligned} \zeta'(0) &= \zeta(0)\lim_{s\downarrow 0}\int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v} - 1}{y^2}\mathrm{d}y \\ &+ \ \zeta(0)\lim_{s\downarrow 0}\int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y\Big[\mathrm{e}^{\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v} - 1 + sr_{-1}(1-\gamma(s))\mathrm{e}^{\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v}\Big]. \end{aligned} \tag{49}$$

The term between square brackets in (49) is readily seen to be $\mathrm{O}(s^2)$, while the integral in front of it behaves like $1/s$ for $s \downarrow 0$ according to (46). Hence,

$$\zeta'(0) = \zeta(0)\lim_{s\downarrow 0} \int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v} - 1}{y^2}\mathrm{d}y. \tag{50}$$

While both parts of the outer integral behave like $1/s$, the integral is clearly negative and finite; notice that the integrand is approximately $-\frac{1}{2}r_{-1}E[\mathbf{G}]$ for very small y. Using the above-mentioned fact that $\zeta'(0) = -\eta$ we finally obtain $\zeta(0)$:

$$\zeta(0) = \frac{\eta}{\int_0^\infty \frac{1-\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y}. \tag{51}$$

Combining (47) and (51) we have determined $\zeta(s)$:

$$\zeta(s) = \int_0^\infty \mathrm{e}^{-sx} z(x)\mathrm{d}x = \frac{\eta s \mathrm{e}^{\int_0^s r_{-1}(1-\gamma(v))\mathrm{d}v} \int_s^\infty \frac{\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y}{\int_0^\infty \frac{1-\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v}}{y^2}\mathrm{d}y}. \tag{52}$$

Theorem 4.2. *The steady-state cash level LST for Case II.b, with $\eta(x) = \eta$, is given by*

$$E[\mathrm{e}^{-s\mathbf{V}_e}] = \frac{\zeta(0)}{\eta s} - \frac{1}{\eta}[\frac{1}{s} + r_{-1}(1-\gamma(s))]\zeta(s), \tag{53}$$

with $\zeta(s)$ given by (52) and $\zeta(0)$ by (51).

Proof. Observe that $E[\mathrm{e}^{-s\mathbf{V}_e}] = \int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{-sx}\frac{xz(x)}{\eta}\mathrm{d}x = -\frac{1}{\eta}\zeta'(s)$. Now use Expression (44) for $\zeta'(s)$. □

Remark 7. Remembering that $z(x) = \frac{\eta}{x}f(x)$, it is seen that

$$E[\mathbf{V}_e] = \int_0^\infty xf(x)\mathrm{d}x = \eta\zeta''(0). \tag{54}$$

Remark 8. In (14) $f(x)$ is given for the case in which $h(x) = x^a$, $\lambda(x)/\eta(x) = r_{-1}/x$ and $G(x) = 1 - \mathrm{e}^{-\mu x}$. Taking $a = 1$ in (14) and remembering that $z(x) = \frac{\eta(x)}{x}f(x)$, we have

$$z(x) = C(x + \frac{1}{\mu})(x + r_{-1})^{\mu r_{-1}-2}\mathrm{e}^{-\mu x}, \tag{55}$$

where C is not the same constant as the normalizing constant in (14). Let us now check that, indeed, its Laplace transform,

$$\int_0^\infty \mathrm{e}^{-sx} z(x)\mathrm{d}x = C\int_0^\infty (x + \frac{1}{\mu})(x + r_{-1})^{\mu r_{-1}-2}\mathrm{e}^{-(s+\mu)x}\mathrm{d}x, \tag{56}$$

agrees with $\zeta(s)$ as given in (52). When $G(x) = 1 - \mathrm{e}^{-\mu x}$, so $\gamma(s) = \frac{\mu}{\mu+s}$, we have $\mathrm{e}^{-\int_0^y r_{-1}(1-\gamma(v))\mathrm{d}v} = (\frac{\mu+y}{\mu})^{r_{-1}\mu}\mathrm{e}^{-r_{-1}y}$. Hence we can rewrite the transform in (56), taking $y = s + u$, into

$$\zeta(s) = \zeta(0)s\int_{u=0}^\infty \frac{(\mu+u+s)^{\mu r_{-1}}}{(\mu+s)^{\mu r_{-1}}}\frac{1}{(u+s)^2}\mathrm{e}^{-r_{-1}u}\mathrm{d}u. \tag{57}$$

Partial integration yields

$$\zeta(s) = \zeta(0) - \zeta(0)s \int_{u=0}^{\infty} r_{-1} \frac{(\mu+u+s)^{\mu r_{-1}-1}}{(\mu+s)^{\mu r_{-1}}} \mathrm{e}^{-r_{-1}u} \mathrm{d}u. \tag{58}$$

Taking, on the other hand, $(s+\mu)x = r_{-1}u$ in (56) changes that Laplace transform after some straightforward calculations into

$$\begin{aligned} \zeta(s) \quad &= \quad \frac{C^{**}}{(\mu+s)^{\mu r_{-1}}} [\int_0^{\infty} (\mu+s+u)^{\mu r_{-1}-2} (\mu+s)(1-\mu r_{-1}) \mathrm{e}^{-r_{-1}u} \mathrm{d}u \\ &+ \quad \int_0^{\infty} (\mu+s+u)^{\mu r_{-1}-1} \mu r_{-1} \mathrm{e}^{-r_{-1}u} \mathrm{d}u], \end{aligned} \tag{59}$$

with C^{**} some constant. Via a partial integration of the second integral and subsequently taking $s = 0$, it can be readily verified that $C^{**} = \zeta(0)$. We want to show that the expressions for $\zeta(s)$ in (58) and (59) coincide. By performing one partial integration with respect to the integral in (58), viz.,

$$\int_{u=0}^{\infty} r_{-1}(\mu+u+s)^{\mu r_{-1}-1} \mathrm{e}^{-r_{-1}u} \mathrm{d}u = (\mu+s)^{\mu r_{-1}-1} - (1-\mu r_{-1}) \int_0^{\infty} (\mu+s+u)^{\mu r_{-1}-2} \mathrm{e}^{-r_{-1}u} \mathrm{d}u, \tag{60}$$

we conclude that, indeed, (52) and (45) are in agreement.

5 A second P2P model: up and down periods, i.i.d. jumps up, shot noise down

In this section we shall study a slightly different cash management model. We assume that there is a background process for the cash level process; that background process alternates between *up* and *down* periods. During up periods, the cash level grows according to some compound Poisson process, and stays constant otherwise. During down periods, the cash level process grows according to another compound Poisson process, but in between upward jumps it decreases at a rate that is proportional to its level. The down periods represent recessions.

We would like to point out that the above-described model bears a strong resemblance to a polling model that was recently studied in [9]. A polling model is a queueing model in which a single server cyclically visits a number of queues, serving the customers of a visited queue for a certain time period. If one were to focus on one particular queue Q, then its workload increases during all the periods in which it is not visited (this corresponds to the up periods in the model of the present section). During visit periods of Q, the workload also increases because of customer arrivals, but in addition it decreases because the server is serving the queue. Both in [9] (during visit periods) and in the present section (during down periods), the process level decreases at a speed that is proportional to that level. Contrary to [9], the input processes during up and down periods may be different in the present model, and the up periods may have a general distribution.

The model is described in Subsection 5.1, and the steady-state analysis of the cash level is presented in Subsection 5.2.

5.1 Model description

We assume that the cash level process $\{\mathbf{V}(t), t \geq 0\}$ alternately goes through up periods and down periods. The up period lengths $\mathbf{U}_1, \mathbf{U}_2, ...$ are i.i.d. with distribution function $U(\cdot)$, density $\upsilon(\cdot)$ and LST $\phi_U(s)$, while the down period lengths $\mathbf{D}_1, \mathbf{D}_2, ...$ are i.i.d. with distribution function $D(\cdot)$, density $\delta(\cdot)$ and LST $\phi_D(s)$. All up periods are also independent of all down periods. During up periods, $\mathbf{V}(t)$ is increasing according to a compound Poisson process (we could generalize this to a Lévy subordinator) with jump rate λ_U and i.i.d. jumps, generically denoted by $\mathbf{G}_U$, with jump size LST $G_U^*(s)$. During down periods, $\mathbf{V}(t)$ is decreasing according to a shot noise process; when $\mathbf{V}(t) = x$, the process decreases at rate rx, $x > 0$ (notice that the process never can reach zero). But in addition, during down periods we allow increments according to a compound Poisson process with jump rate λ_D and i.i.d. jumps, generically denoted by $\mathbf{G}_D$, with jump size LST $G_D^*(s)$. The two compound Poisson processes are also assumed to be independent of everything else.

5.2 Steady-state analysis

In this subsection we study the steady-state distribution of the process $\{\mathbf{V}(t), t \geq 0\}$ – which always exists if $r > 0$ and the mean up periods and upward jumps are finite. Let $\mathbf{V}$ denote a random variable with that steady-state distribution, and let $\mathbf{V}_U$ and $\mathbf{V}_D$ denote the steady-state capital level at the end of an up, respectively down, period. Our first observation is that

$$E[\mathrm{e}^{-s\mathbf{V}_U}] = \phi_U(\lambda_U(1 - G_U^*(s)))E[\mathrm{e}^{-s\mathbf{V}_D}], \quad \mathrm{Re}\ s \geq 0. \tag{61}$$

Our second observation (see, e.g., [9]) is that, during a down period that starts at time 0, $\mathbf{V}(t)$ evolves as follows:

$$\mathbf{V}(t) = \mathbf{V}(0)\mathrm{e}^{-rt} + \sum_{i=1}^{A(t)} \mathbf{G}_{D,i}\mathrm{e}^{-r(t-t_i)}, \quad t \geq 0; \tag{62}$$

here $A(t)$ denotes the number of Poisson arrivals in $[0, t]$, and the sizes of successive upward jumps are denoted by $\mathbf{G}_{D,1}, \mathbf{G}_{D,2}, \ldots$. It readily follows from this relation (see also [9]) that

$$E[\mathrm{e}^{-s\mathbf{V}(t)}] = \exp(-s\mathbf{V}(0)\mathrm{e}^{-rt})\exp(-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rt}}^{s} \frac{1 - G_D^*(v)}{v}\mathrm{d}v), \quad \mathrm{Re}\ s \geq 0. \tag{63}$$

Expressing the LST of $\mathbf{V}_D$ into the preceding $\mathbf{V}_U$, it follows from (63) that

$$E[\mathrm{e}^{-s\mathbf{V}_D}|\mathbf{V}_U = x] = \int_{t=0}^{\infty} \mathrm{e}^{-s\mathrm{e}^{-rt}x}\mathrm{e}^{-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rt}}^{s}\frac{1-G_D^*(v)}{v}\mathrm{d}v}\mathrm{d}D(t), \tag{64}$$

and hence

$$E[\mathrm{e}^{-s\mathbf{V}_D}] = \int_{t=0}^{\infty} E[\mathrm{e}^{-s\mathrm{e}^{-rt}\mathbf{V}_U}]\mathrm{e}^{-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rt}}^{s}\frac{1-G_D^*(v)}{v}\mathrm{d}v}\mathrm{d}D(t). \tag{65}$$

Combining (61) and (65) we obtain a functional equation for the LST of $\mathbf{V}_U$:

$$E[\mathrm{e}^{-s\mathbf{V}_U}] = \phi_U(\lambda_U(1 - G_U^*(s)))\int_{t=0}^{\infty} E[\mathrm{e}^{-s\mathrm{e}^{-rt}\mathbf{V}_U}]\mathrm{e}^{-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rt}}^{s}\frac{1-G_D^*(v)}{v}\mathrm{d}v}\mathrm{d}D(t). \tag{66}$$

The transformation $s\mathrm{e}^{-rt} = y$ reduces this equation to

$$E[\mathrm{e}^{-s\mathbf{V}_U}] = \phi_U(\lambda_U(1 - G_U^*(s))) \int_{y=0}^{s} E[\mathrm{e}^{-y\mathbf{V}_U}]\mathrm{e}^{-\frac{\lambda_D}{r}\int_y^s \frac{1-G_D^*(v)}{v}\mathrm{d}v} \frac{1}{ry}\delta(-\frac{1}{r}\ln\frac{y}{s})\mathrm{d}y. \quad (67)$$

As mentioned in the beginning of this section, the model of this section is closely related to polling models studied in [9]. In Section 6 of [9], the case of constant down periods is treated, and in Section 7 of [9] the case of exponentially distributed down periods, with also exponential up periods, is studied. In both cases, the compound Poisson processes during up and down periods are the same. Below we briefly indicate how (67) can be solved in these two cases (but allowing different compound Poisson processes, and generally distributed up periods).

Remark 9. Differentiating (66) with respect to s we obtain $E[\mathbf{V}_U]$:

$$E[\mathbf{V}_U] = \lambda_U E[\mathbf{G}_U]E[\mathbf{U}] + E[\mathbf{V}_U]\int_{t=0}^{\infty} \mathrm{e}^{-rt}\mathrm{d}D(t) + \frac{\lambda_D}{r}\int_{t=0}^{\infty}(1 - \mathrm{e}^{-rt})E[\mathbf{G}_D]\mathrm{d}D(t),$$

so

$$E[\mathbf{V}_U] = \frac{\lambda_U E[\mathbf{G}_U]E[\mathbf{U}]}{1 - E[\mathrm{e}^{-r\mathbf{D}}]} + \frac{\lambda_D E[\mathbf{G}_D]}{r}. \quad (68)$$

In combination with (61) this yields

$$E[\mathbf{V}_D] = \frac{E[\mathrm{e}^{-r\mathbf{D}}]}{1 - E[\mathrm{e}^{-r\mathbf{D}}]}\lambda_U E[\mathbf{G}_U]E[\mathbf{U}] + \frac{\lambda_D E[\mathbf{G}_D]}{r}. \quad (69)$$

Remark 10. Once we have the LST of $\mathbf{V}_U$, the LST of $\mathbf{V}_D$ follows from (61); it is subsequently not hard to obtain the capital level LST's at arbitrary epochs in down and up periods via a stochastic mean value theorem; and finally one averages over the two periods to obtain the LST of $\mathbf{V}$. Below we discuss this for the case of exponential down periods.

Example 1: The case of exponential down periods.
For $\exp(\xi_D)$ down periods, the density $\delta(-\frac{1}{r}\ln\frac{y}{s}) = \xi_D(\frac{y}{s})^{\xi_D/r}$, and (67) simplifies considerably:

$$E[\mathrm{e}^{-s\mathbf{V}_U}] = \phi_U(\lambda_U(1 - G_U^*(s))) \int_{y=0}^{s} E[\mathrm{e}^{-y\mathbf{V}_U}]\mathrm{e}^{-\frac{\lambda_D}{r}\int_y^s \frac{1-G_D^*(v)}{v}\mathrm{d}v} \frac{\xi_D}{ry}(\frac{y}{s})^{\xi_D/r}\mathrm{d}y. \quad (70)$$

Differentiation w.r.t. s leads, after some calculations, to the following first-order linear differential equation in $F_U(s) := E[\mathrm{e}^{-s\mathbf{V}_U}]$:

$$\begin{aligned} F_U'(s) &= \frac{\frac{d}{ds}\phi_U(\lambda_U(1 - G_U^*(s)))}{\phi_U(\lambda_U(1 - G_U^*(s)))}F_U(s) \\ &+ \frac{\xi_D}{r}\frac{\phi_U(\lambda_U(1 - G_U^*(s)))}{s}F_U(s) - \frac{\lambda_D}{r}\frac{1 - G_D^*(s)}{s}F_U(s) - \frac{\xi_D}{r}\frac{1}{s}F_U(s), \quad (71) \end{aligned}$$

and hence

$$F_U(s) = E[\mathrm{e}^{-s\mathbf{V}_U}] = \phi_U(\lambda_U(1 - G_U^*(s)))\mathrm{e}^{-\frac{\xi_D}{r}\int_0^s \frac{1-\phi_U(\lambda_U(1-G_U^*(v)))}{v}\mathrm{d}v}\mathrm{e}^{-\frac{\lambda_D}{r}\int_0^s \frac{1-G_D^*(v)}{v}\mathrm{d}v}, \quad (72)$$

and, using (61),

$$E[\mathrm{e}^{-s\mathbf{V}_D}] = \mathrm{e}^{-\frac{\xi_D}{r}\int_0^s \frac{1-\phi_U(\lambda_U(1-G_U^*(v)))}{v}\mathrm{d}v}\mathrm{e}^{-\frac{\lambda_D}{r}\int_0^s \frac{1-G_D^*(v)}{v}\mathrm{d}v}. \tag{73}$$

Notice that both exponential terms have the form of the LST of the workload in a shot noise model; see also Remark 3. The last exponential is the workload LST in a shot noise model that exactly corresponds to the down periods in our model. The first exponential is the workload LST in a shot noise model with arrival rate ξ_D of shots, and with shot (jump) sizes corresponding to the total amount of work/capital arriving during an up period. Also notice that the first two terms of (72) correspond to the cash level LST in Theorem 4.1, in case $\phi_U(s) = \frac{\eta}{\eta+s}$.
We close this example by determining the LST of the steady-state workload $\mathbf{V}_e$ at an arbitrary epoch; cf. Remark 10. In steady state, we can restrict ourselves to considering an arbitrary sequence of one down period followed by one up period. Denoting the steady-state amount of work in an arbitrary down (respectively up) period by $\mathbf{V}_e^D$ (respectively $\mathbf{V}_e^U$), we can write:

$$E[\mathrm{e}^{-s\mathbf{V}_e}] = \frac{E\mathbf{D}}{E\mathbf{U}+E\mathbf{D}}E[\mathrm{e}^{-s\mathbf{V}_e^D}] + \frac{E\mathbf{U}}{E\mathbf{U}+E\mathbf{D}}E[\mathrm{e}^{-s\mathbf{V}_e^U}]. \tag{74}$$

It immediately follows by PASTA that $\mathbf{V}_e^D$ has the same distribution as $\mathbf{V}_D$. During the subsequent up period, the workload grows from $\mathbf{V}_D$ according to a compound Poisson process. At an arbitrary time epoch during this up period, the workload $\mathbf{V}_e^U$ equals the sum of $\mathbf{V}_D$ and, independently, the compound Poisson increment during the past part of the $\mathbf{U}$ period. Hence, observing that the length of that past part has density $P(\mathbf{U}>t)/E\mathbf{U}$:

$$E[\mathrm{e}^{-s\mathbf{V}_e^U}] = E[\mathrm{e}^{-s\mathbf{V}_D}]\int_{t=0}^{\infty}\mathrm{e}^{-\lambda_U(1-G_U^*(s))t}\frac{P(\mathbf{U}>t)}{E\mathbf{U}}\mathrm{d}t = E[\mathrm{e}^{-s\mathbf{V}_D}]\frac{1-\phi_U(\lambda_U(1-G_U^*(s)))}{E\mathbf{U}\lambda_U(1-G_U^*(s))}. \tag{75}$$

Combining (74) and (75) we obtain the LST of the steady-state workload, for the case of exponential down periods:

$$E[\mathrm{e}^{-s\mathbf{V}_e}] = E[\mathrm{e}^{-s\mathbf{V}_D}[\frac{E\mathbf{D}}{E\mathbf{U}+E\mathbf{D}} + \frac{E\mathbf{U}}{E\mathbf{U}+E\mathbf{D}}\frac{1-\phi_U(\lambda_U(1-G_U^*(s)))}{E\mathbf{U}\lambda_U(1-G_U^*(s))}]. \tag{76}$$

Example 2: The case of constant down periods.
For constant down periods of length T, (66) reduces to

$$F_U(s) = \phi_U(\lambda_U(1-G_U^*(s)))F_U(s\mathrm{e}^{-rT})\mathrm{e}^{-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rT}}^{s}\frac{1-G_D^*(v)}{v}\mathrm{d}v}. \tag{77}$$

This equation can easily be solved by iteration, resulting in an infinite product. Observing that $F_U(0)=1$, we get:

$$F_U(s) = \mathrm{e}^{-\frac{\lambda_D}{r}\int_0^s\frac{1-G_D^*(v)}{v}\mathrm{d}v}\prod_{j=0}^{\infty}\phi_U(\lambda_U(1-G_U^*(s\mathrm{e}^{-jrT}))). \tag{78}$$

It is readily verified that the infinite product converges, which is equivalent with convergence of $\sum_{j=0}^{\infty}[1-\phi_U(\lambda_U(1-G_U^*(s\mathrm{e}^{-jrT})))]$. The latter sum exhibits geometric convergence, as is seen by twice using $1-\mathrm{e}^{-x} \leq x$ for $x>0$:

$$\begin{aligned}
|1-\phi_U(\lambda_U(1-G_U^*(s\mathrm{e}^{-jrT})))| &= |\int_0^\infty (1-\mathrm{e}^{-\lambda_U(1-G_U^*(s\mathrm{e}^{-jrT}))t})\mathrm{d}U(t)| \\
&\leq \lambda_U E[\mathbf{U}]|1-G_U^*(s\mathrm{e}^{-jrT})| \\
&\leq \lambda_U E[\mathbf{U}] \int_0^\infty |1-\mathrm{e}^{-s\mathrm{e}^{-jrT}x}|\mathrm{d}G_U(x) \\
&\leq \lambda_U E[\mathbf{U}]E[\mathbf{G}_U]s\mathrm{e}^{-jrT}. \qquad (79)
\end{aligned}$$

Finally, we briefly consider the case in which down periods are equal to the constant T_1 with probability p_1 and equal to the constant T_2 with probability p_2, with $0 < p_1 < 1$ and $p_2 = 1-p_1$. Our approach seems of independent interest, and can be extended in a rather straightforward way to handle the case of $M \geq 3$ different constant down periods. Equation (66) reduces to

$$F_U(s) = \phi_U(\lambda_U(1-G_U^*(s))) \sum_{i=1}^{2} p_i F_U(s\mathrm{e}^{-rT_i})\mathrm{e}^{-\frac{\lambda_D}{r}\int_{s\mathrm{e}^{-rT_i}}^{s} \frac{1-G_D^*(v)}{v}\mathrm{d}v}. \qquad (80)$$

This generalizes (77), that considered the workload at the end of an arbitrary down period of constant length T. Just like for the latter equation, we attempt an iteration. This approach bears some similarity to the approach of Adan et al. [1] of the following recursion, with $\psi(s)$ the unknown LST: $\psi(s) = p_0 + \sum_{i=1}^{2} H_i(s)\psi(s+\theta_i)$. To keep the overview in our somewhat complicated iteration, we introduce the following shorthand notation.

$$e_i := \mathrm{e}^{-rT_i}, \quad i = 1, 2, \qquad (81)$$

$$k(s) := \phi_U(\lambda_U(1-G_U^*(s))), \quad l(b,s) := \mathrm{e}^{-\frac{\lambda_D}{r}\int_b^s \frac{1-G_D^*(v)}{v}\mathrm{d}v}. \qquad (82)$$

Equation(80) can now be rewritten as

$$F_U(s) = k(s)\sum_{i=1}^{2} p_i l(e_i s, s) F_U(e_i s). \qquad (83)$$

After $n-1$ iterations, one gets

$$F_U(s) = \sum_{k=0}^{n} p_1^k p_2^{n-k} l(e_1^k e_2^{n-k}s, s) K_{k,n-k}(s) F_U(e_1^k e_2^{n-k}s), \qquad (84)$$

where $K_{k,n-k}(s)$ are recursively defined as follows (with $K_{-1,\cdot}(s) = K_{\cdot,-1}(s) \equiv 0$):

$$\begin{aligned}
&K_{1,0}(s) := k(s), \quad K_{0,1}(s) := k(s), \\
&K_{k+1,n-k}(s) = K_{k,n-k}(s)k(e_1^k e_2^{n-k}s) + K_{k+1,n-1-k}(s)k(e_1^{k+1}e_2^{n-1-k}s), \\
&K_{k,n+1-k}(s) = K_{k-1,n+1-k}(s)k(e_1^{k-1}e_2^{n+1-k}s) + K_{k,n-k}(s)k(e_1^k e_2^{n-k}s). \qquad (85)
\end{aligned}$$

One can verify that $K_{k,n-k}(s)$ is a sum of $\binom{n}{k}$ terms. All these terms correspond to having k periods of length T_1 and $n-k$ periods of length T_2 in the last n down periods. There are $\binom{n}{k}$ ways to order those n periods.

We now claim that one can approximate $F_U(s)$ quite accurately by the following expression:

$$F_U(s) \approx l(0,s) \sum_{k=0}^{n} p_1^k p_2^{n-k} K_{k,n-k}(s), \tag{86}$$

for n sufficiently large (but actually quite small). Indeed, comparing (84) and (86), the error thus made equals

$$\begin{aligned} & \sum_{k=0}^{n} p_1^k p_2^{n-k} l(e_1^k e_2^{n-k} s, s) K_{k,n-k}(s) (F_U(e_1^k e_2^{n-k} s) - 1) \\ + & \sum_{k=0}^{n} p_1^k p_2^{n-k} (l(e_1^k e_2^{n-k} s, s) - l(0,s)) K_{k,n-k}(s). \end{aligned} \tag{87}$$

Introducing

$$e_* = \max(e_1, e_2),$$

and observing that $\frac{1-G_D^*(v)}{E[\mathbf{G}_D]v}$ is the LST of the residual of a jump during a down period, and hence bounded by one, one has the following bounds for terms appearing in the righthand side of (87):

$$|F_U(e_1^k e_2^{n-k} s) - 1| \leq \int_0^\infty |1 - \mathrm{e}^{-e_*^n st}| \mathrm{d}P(\mathbf{V}_U < t) \leq E[\mathbf{V}_U]|s|e_*^n; \tag{88}$$

$$\begin{aligned} & |l(e_1^k e_2^{n-k} s, s) - l(0,s)| \leq |1 - l(0, e_1^k e_2^{n-k} s)| \\ = & |1 - \mathrm{e}^{-\frac{\lambda_D}{r} \int_0^{e_1^k e_2^{n-k} s} \frac{1-G_d^*(v)}{v} \mathrm{d}v}| \leq \frac{\lambda_D}{r} E[\mathbf{G}_D]|s|e_*^n. \end{aligned} \tag{89}$$

Note that $0 < e_* < 1$ and that

$$\sum_{k=0}^{n} p_1^k p_2^{n-k} K_{k,n-k}(s) \leq 1,$$

because $K_{k,n-k}(s)$ contains $\binom{n}{k}$ products of $k(\cdot)$ terms while $k(\cdot)$ is the LST of a nonnegative random variable. Hence we have shown that $F_U(s)$ converges geometrically fast to the expression in the righthand side of (86).

Acknowledgment
The research of Onno Boxma is supported by the NWO Gravitation Programme NETWORKS (Grant number 024.002.003). The research of David Perry is partly supported by a grant of the Israel Science Foundation (Grant number 3274/19).

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