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Spatially inhomogeneous populations with seed-banks: II. Clustering regime

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Abstract

We consider a spatial version of the classical Moran model with seed-bank where the constituent populations have finite sizes. Individuals live in colonies labelled by $\mathbb{Z}^d$, $d \geq 1$, playing the role of a geographic space, and change type via *resampling* as long as they are *active*. Each colony contains a seed-bank into which individuals can enter to become *dormant*, suspending their resampling until they exit the seed-bank and become active again. Individuals resample not only from their own colony, but also from other colonies according to a symmetric random walk transition kernel. The latter is referred to as *migration*. The sizes of the active and the dormant populations depend on the colony and remain constant throughout the evolution.

It was shown in [1] that the spatial system is well-defined, has a unique equilibrium that depends on the initial density of types, and exhibits a dichotomy between *clustering* (mono-type equilibrium) and *coexistence* (multi-type equilibrium). This dichotomy is determined by a clustering criterion that is given in terms of the dual of the system, which consists of a system of *interacting* coalescing random walks. In this paper we provide an alternative clustering criterion, given in terms of an auxiliary dual that is simpler than the original dual, and identify the range of parameters for which the criterion is met, which we refer to as the *clustering regime.* It turns out that if the sizes of the active populations are non-clumping, i.e., do not take arbitrarily large values in finite regions of the geographic space, and the relative strengths of the seed-banks in the different colonies are bounded, then clustering prevails if and only if the symmetrised migration kernel is recurrent.

The spatial system is hard to analyse because of the interaction in the original dual and the inhomogeneity of the colony sizes. By comparing the auxiliary dual with a *non-interacting* two-particle system, we are able to control the correlations that are caused by the interactions. The work in [1] and the present paper is part of a broader attempt to include dormancy into interacting particle systems.

Keywords: Moran model, resampling, migration, seed-bank, inhomogeneity, duality, coexistence versus clustering.

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1 Introduction

1.1 Background and targets

In [2] and [3], the discrete-time Fisher-Wright model with *seed-bank* was introduced and analysed. Individuals live in a colony, are subject to *resampling* (i.e., they randomly adopt each others type), and move in and out of the seed-bank, where they suspend their resampling. The seed-bank acts as a repository for the genetic information of the population. Individuals that reside inside the seed-bank are called *dormant*, those that reside outside are called *active*. Both the long-time scaling and the genealogy of the population were analysed in the limit as the size of the colony tends to infinity. The continuous-time version of the model, called Moran model, has the same behaviour. For a recent overview on seed-bank models in population genetics we refer the reader to [4].

In [1], a *spatial* version of the Moran model was considered, in which individuals live in multiple colonies, labelled by $\mathbb{Z}^d$, $d \geq 1$, playing the role of a *geographic space*. Each colony has its own seed-bank, and individuals resample not only from their own colony, but also from other colonies according to a random walk transition kernel, which is referred to as *migration*. The sizes of the active and the dormant population depend on the colony. It was shown that, under mild conditions on the sizes, the system is well-defined, has a unique equilibrium that depends on the initial density of types, and exhibits a dichotomy between *clustering* (mono-type equilibrium) and *coexistence* (multi-type equilibrium). This dichotomy is determined by a *clustering criterion* that is given in terms of the *dual* of the system, which consists of a system of *interacting* coalescing random walks in an inhomogeneous environment.

The goal of the present paper is to identify the *clustering regime*, i.e., the range of parameters for which the clustering criterion is met. More precisely, we show that if the sizes of the active populations are non-clumping, i.e., do not take arbitrarily large values in finite regions of the geographic space, and the relative strengths of the seed-banks in the different colonies are bounded, then the dichotomy between coexistence and clustering is the classical dichotomy between transience and recurrence of the symmetrised migration kernel, a property that is known to hold for colonies without seed-bank.

Three open problems for the future are:

(A) Identify the clustering regime when the relative strengths of the seed-banks in the different colonies are unbounded. In that setting the clustering regime will be different, because it will be driven by a delicate interplay between migration and seed-bank.

(B) In the coexistence regime, identify the *domain of attraction* of the equilibria.

(C) In the clustering regime, identify the *growth rate of the mono-type clusters*.

In [1] we only showed convergence to equilibrium starting from a family of initial states that are labelled by the initial density of types and that are products of binomial distributions tuned to the inhomogeneity of the relative strengths of the seed-banks.

In [5], [6], [7] a *homogeneous* spatial version of the Fisher-Wright model was considered, in the large-colony-size limit. For three different choices of seed-bank, it was shown that the system is well-defined, has a unique equilibrium that depends on the initial density of types, and exhibits a dichotomy between clustering and coexistence. A full description of the clustering regime was obtained. In addition, the finite-systems scheme was established (i.e., how a truncated version of the system behaves on a properly tuned time scale as the truncation level tends to infinity). Moreover, a multi-scale renormalisation analysis was carried out for the case where the colonies are labelled by the hierarchical group. The respective duals for these models are easier, because they are non-interacting and have no inhomogeneity in space. The dual of our model is much harder, which is why our results are much more modest.

1.2 Outline

The paper is organised as follows. In Section 2 we give a quick definition of the model and state our main theorems about the dichotomy of clustering versus coexistence by identifying the clustering regime for both. In Section 3 we recall the basic dual of our system introduced in [1], and define three auxiliary duals that serve as comparison objects. We relate the coalescence probabilities of the different duals, which leads to a necessary and sufficient criterion for clustering in our system. In Section 4 we prove our main theorems.

2 Main theorems

In Section 2.1 we give a quick definition of the multi-colony system. In Section 2.2 we state our results about the dichotomy of clustering versus coexistence, which requires additional conditions on the sizes of the active and the dormant population.

2.1 Quick definition of the system

Individuals live in colonies labelled by $\mathbb{Z}^d$, $d \geq 1$, which plays the role of a *geographic space*. (In what follows, the geographic space can be any countable Abelian group.) Each colony has an *active* population and a *dormant* population. For $i \in \mathbb{Z}^d$, we write $(N_i, M_i) \in \mathbb{N}^2$ to denote the *size* of the active, respectively, dormant population at the colony i. Each individual carries one of two *types*: $\heartsuit$ and $\spadesuit$. Individuals are subject to:

(1) Active individuals in any colony *resample* with active individuals in any colony.

(2) Active individuals in any colony *exchange* with dormant individuals in the same colony.

For (1) we assume that each active individual at colony i at rate $a(i,j)$ uniformly draws an active individual at colony j and *adopts its type*. For (2) we assume that each active individual at colony i at rate $\lambda \in (0,\infty)$ uniformly draws a dormant individual at colony i and the two individuals *trade places while keeping their type* (i.e., the active individual becomes dormant and the dormant individual becomes active). Although the exchange rate λ could be made to vary across colonies, for the sake of simplicity we choose it to be constant, and we let the *migration kernel* $a(\cdot,\cdot)$ be translation invariant and irreducible. Note that dormant individuals do *not* resample.

Assumption 2.1. [Homogeneous migration] The migration kernel $a(\cdot,\cdot)$ satisfies:

- $a(\cdot,\cdot)$ is irreducible in $\mathbb{Z}^d$.
- $a(i,j) = a(0, j-i)$ for all $i, j \in \Omega$.
- $c := \sum_{i \in \mathbb{Z}^d \setminus \{0\}} a(0,i) < \infty$ and $a(0,0) = \frac{1}{2}$. □

The second assumption ensures that the way genetic information moves between colonies is homogeneous in space. The third assumption ensures that the total rate of resampling is finite and that resampling is possible also at the same colony.

Furthermore, in order to avoid trivial statement we assume the following:

Assumption 2.2. [Non-trivial colony sizes] In each colony, both the active and the dormant population consist of at least two individuals, i.e.,

(2.1) $$N_i \geq 2,\ M_i \geq 2, \qquad i \in \mathbb{Z}^d.$$

For colony sizes where Assumption 2.2 fails, all the results stated below can be obtained with minor technical modifications.

At each colony i we register the pair $(X_i(t), Y_i(t))$, representing the number of active, respectively, dormant individuals of type $\heartsuit$ at time t at colony i. The resulting Markov process is denoted by

(2.2) $$Z := (Z(t))_{t \geq 0}, \qquad Z(t) = ((X_i(t), Y_i(t))_{i \in \mathbb{Z}^d},$$

and lives on the state space

(2.3) $$\mathcal{X} = \prod_{i \in \mathbb{Z}^d} [N_i] \times [M_i],$$

where $[n] := \{0, 1, \dots, n\}$, $n \in \mathbb{N}$. In [1], it was shown that under mild assumptions on the model parameters, the Markov process in (2.2) is well defined and has a *dual* $(Z^*(t))_{t\geq 0}$ where the process

$$(2.4)\quad Z^* := (Z^*(t))_{t\geq 0}, \quad Z^*(t) := (n_i(t), m_i(t))_{i\in\mathbb{Z}^d},$$

lives on the state space

$$(2.5)\quad \mathcal{X}^* := \Big\{(n_i, m_i)_{i\in\mathbb{Z}^d} \in \mathcal{X} \colon \sum_{i\in\mathbb{Z}^d}(n_i + m_i) < \infty\Big\}.$$

The dual process Z^* consists of finite collections of particles that switch between an *active* state and a *dormant* state, and perform *interacting coalescing* random walks while in the active state, with rates that are controlled by the model parameters. We will give a brief description of the dual process $(Z^*(t))_{t\geq 0}$ in Section 3. We recall the results in [1] on the well-posedness of the process Z and the duality relation between Z and Z^*.

Theorem 2.3. [Well-posedness and duality][1, Theorem 2.2 and Corollary 3.11] *Suppose that Assumption* 2.1 *is in force. Then the martingale problem associated with* (2.2) *is well-posed under either of the two following conditions:*

(a) $\lim_{\|i\|\to\infty} \|i\|^{-1} \log N_i = 0$ *and* $\sum_{i\in\mathbb{Z}^d} \mathrm{e}^{\delta\|i\|} a(0, i) < \infty$ *for some* $\delta > 0$.

(b) $\sup_{i\in\mathbb{Z}^d\setminus\{0\}} \|i\|^{-\gamma} N_i < \infty$ *and* $\sum_{i\in\mathbb{Z}^d} \|i\|^{d+\gamma+\delta} a(0, i) < \infty$ *for* $\gamma > 0$ *and some* $\delta > 0$.

Furthermore, the Markov process $(Z(t))_{t\geq 0}$ *has a factorial moment dual* $(Z^*(t))_{t\geq 0}$*, living on the state space* $\mathcal{X}^* \subset \mathcal{X}$ *and consisting of all configurations with finite mass. The duality relation is given by*

$$(2.6)\quad \mathbb{E}_\eta\left[\prod_{i\in\mathbb{Z}^d} \frac{\binom{X_i(t)}{n_i}}{\binom{N_i}{n_i}} \frac{\binom{Y_i(t)}{m_i}}{\binom{M_i}{m_i}} \mathbf{1}_{\{n_i\leq X_i(t), m_i\leq Y_i(t)\}}\right] = \mathbb{E}^\xi\left[\prod_{i\in\mathbb{Z}^d} \frac{\binom{X_i}{n_i(t)}}{\binom{N_i}{n_i(t)}} \frac{\binom{Y_i}{m_i(t)}}{\binom{M_i}{m_i(t)}} \mathbf{1}_{\{n_i(t)\leq X_i, m_i(t)\leq Y_i\}}\right],$$

where the expectations are taken with respect to the laws of Z *and* Z^* *started at initial configurations* $\eta := (X_i, Y_i)_{i\in\mathbb{Z}^d} \in \mathcal{X}$ *and* $\xi := (n_i, m_i)_{i\in\mathbb{Z}^d} \in \mathcal{X}^*$*, respectively.*

In view of the above, from here onwards, we implicitly assume that the model parameters $(N_i)_{i\in\mathbb{Z}^d}$ and $a(\cdot\,,\cdot)$ are such that one of the two conditions (a) and (b) is satisfied.

We write $\hat{a}(\cdot\,,\cdot)$ to denote the *symmetrised migration kernel* defined by

$$(2.7)\quad \hat{a}(i, j) := \tfrac{1}{2}[a(i, j) + a(j, i)], \qquad i, j \in \mathbb{Z}^d,$$

and write $a_n(\cdot\,,\cdot)$ to denote the n-step transition probability kernel of the embedded chain associated to the continuous-time random walk on $\mathbb{Z}^d$ with rates $a(\cdot\,,\cdot)$. Furthermore, we denote by $\hat{a}_t(\cdot\,,\cdot)$ (respectively, $a_t(\cdot\,,\cdot)$), the time-t transition probability kernel of the continuous-time random walk with migration rates $\hat{a}(\cdot\,,\cdot)$ (respectively, $a(\cdot\,,\cdot)$), and put

$$(2.8)\quad K_i := \frac{N_i}{M_i}, \qquad i \in \mathbb{Z}^d,$$

for the *ratios* of the sizes of the active and the dormant population in each colony. Note that K_i^{-1} quantifies the *relative strength* of the seed-bank at colony $i \in \mathbb{Z}^d$.

Let $\mathcal{P}$ be the set of probability distributions on $\mathcal{X}$ defined by

$$(2.9)\quad \mathcal{P} := \big\{\mathcal{P}_\theta \colon \theta \in [0, 1]\big\}, \qquad \mathcal{P}_\theta := \theta \prod_{i\in\mathbb{Z}^d} \delta_{(0,0)} + (1-\theta) \prod_{i\in\mathbb{Z}^d} \delta_{(N_i, M_i)}.$$

We say that (2.2) exhibits *clustering* if the limiting distribution of $Z(t)$ (given that it exists) falls in $\mathcal{P}$. Otherwise we say that it exhibits *coexistence*. In the next section we state the *clustering criterion* from [1], given in terms of the original dual process Z^*, and provide an alternative equivalent criterion in terms of a simpler two-particle process that is absorbing.

2.2 Clustering versus coexistence

In [1], it was shown that the system admits a mono-type equilibrium (clustering) if and only if the following criterion is met:

Theorem 2.4. [Clustering condition][1, Theorem 3.17] *The system clusters if and only if in the dual process Z^* two particles, starting from any locations in $\mathbb{Z}^d$ and any states (active or dormant), coalesce with probability* 1.

Before we state our alternative criterion for clustering, we introduce an auxiliary two-particle dual process. In Proposition 3.10 below we will show the well-posedness of this process. Recall that λ is the exchange rate between active and dormant individuals in each colony.

Definition 2.5. [Auxiliary two-particle system] The two-particle process $\hat{\xi} := (\hat{\xi}(t))_{t\geq 0}$ is a continuous-time Markov chain on the state space

$$\mathcal{S} := (G \times G) \cup \{\circledast\}, \quad G := \mathbb{Z}^d \times \{0,1\} \tag{2.10}$$

with transition rates

$$[(i,\alpha),(j,\beta)] \to$$

$$\begin{cases} \circledast, & \text{at rate } \frac{\alpha\beta}{N_i}\delta_{i,j}, \\ [(i,1-\alpha),(j,\beta)], & \text{at rate } \lambda[\alpha+(1-\alpha)K_i] - \frac{\lambda}{M_i}\delta_{i,j}(1-\delta_{\alpha,\beta}), \\ [(i,\alpha),(j,1-\beta)], & \text{at rate } \lambda[\beta+(1-\beta)K_j] - \frac{\lambda}{M_j}\delta_{i,j}(1-\delta_{\alpha,\beta}), \\ [(k,\alpha),(j,\beta)], & \text{at rate } \alpha\, a(i,k) \quad \text{for } k\neq i\in\mathbb{Z}^d, \\ [(i,\alpha),(k,\beta)], & \text{at rate } \beta\, a(j,k) \quad \text{for } k\neq j\in\mathbb{Z}^d, \end{cases} \tag{2.11}$$

where $[(i,\alpha),(j,\beta)] \in G\times G$ and $\delta_{\cdot,\cdot}$ denotes the Kronecker delta-function. □

Here, $\hat{\xi}(t) = [(i,\alpha),(j,\beta)]$ capture the location $(i,j\in\mathbb{Z}^d)$ and the state $(\alpha,\beta\in\{0,1\})$ of the two particles at time t, where 0 stands for dormant and 1 stands for active, respectively. Note that $\circledast$ is an absorbing state for the process $\hat{\xi}$, which is absorbed at a location-dependent rate only when the two particles are on top of each other and in the active state. This is different from what happens in the two-particle system obtained from the original dual. Furthermore, the process $\hat{\xi}$ is much simpler than the original two-particle system, because here the particles do not interact unless they are on top of each other with opposite states. We write $\hat{\mathbb{P}}^\eta$ to denote the law of the process $\hat{\xi}$ started from $\eta\in\mathcal{S}$, and $\hat{\mathbb{E}}^\eta$ to denote the expectation w.r.t. $\hat{\mathbb{P}}^\eta$.

Remark 2.6. Note that, by virtue of Assumption 2.2, all states in $\mathcal{S}$ are accessible by $\hat{\xi}$. □

Theorem 2.7. [Clustering criterion] *The system clusters if the process $\hat{\xi}$ starting from an arbitrary configuration in $G\times G$ is absorbed with probability* 1*. Furthermore, if the sizes of the active populations are non-clumping, i.e.,*

$$\inf_{i\in\mathbb{Z}^d} \sum_{\|j-i\|\leq R} \tfrac{1}{N_j} > 0 \textit{ for some } R<\infty, \tag{2.12}$$

then the converse is true as well.

Remark 2.8. Note that the condition in (2.12) is equivalent to requiring that, for some constant $C<\infty$ and all $i\in\mathbb{Z}^d$, there exists a j with $\|j-i\|\leq R$ such that $N_j\leq C$. This requirement can be further relaxed to

$$\inf_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \frac{1}{N_j} \sum_{n\in\mathbb{N}} m^{2n}\, a_n(i,j)^2 > 0, \tag{2.13}$$

where $m := \frac{c}{2(c+\lambda)+1}$. □

To verify when the above clustering criterion is satisfied, we need to impose the following regularity condition on the migration kernel.

Assumption 2.9. **[Slowly varying migration kernel]** $\lim_{t\to\infty} \frac{\hat{a}_{pt}(0,0)}{\hat{a}_t(0,0)} = 1$ for all $p \in (0,\infty)$, where $\hat{a}_t(\cdot,\cdot)$ is the time-t symmetrised migration kernel.

When the relative strengths of the seed-banks are uniformly bounded, clustering is equivalent to the symmetrised migration kernel being recurrent, a setting that is classical. The following theorem provides a slightly weaker result.

Theorem 2.10. **[Clustering regime]** *Suppose that Assumption 2.9 is in force. Assume that the active population sizes are non-clumping, i.e.,* (2.12) *is satisfied, and the relative strengths of the seed-banks are uniformly bounded, i.e.,*

$$(2.14)\quad \sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty.$$

If the system clusters, then it is necessary that the symmetrised kernel $\hat{a}(\cdot,\cdot)$ is recurrent. Furthermore, if the migration kernel $a(\cdot,\cdot)$ is symmetric, then the converse holds as well.

The condition in (2.14) allows us to compare the auxiliary dual process $\hat{\xi}$ with a non-interacting two-particle system. We will exploit this fact to prove Theorem 2.10. We expect the symmetry assumption for the converse to be redundant, but are unable to remove it for technical reasons. It was shown in [5] that the above dichotomy is true when the seed-banks are homogeneous (i.e., $(N_i,M_i) = (N,M)$ for all $i \in \mathbb{Z}^d$) and the large-colony-size limit is taken (i.e., $N,M \to \infty$ such that $N/M \to K \in (0,\infty)$). In that case, the dual process is an independent particle system with coalescence.

3 Dual processes: comparison between different systems

In Section 3.1 we recall the *basic dual* of our original system introduced in [1]. In Section 3.2 we introduce *three auxiliary duals*, which are simpler versions of the basic dual and serve as comparison objects. To build up the comparison we follow the approach in [8]. In Sections 3.3–3.4 we relate the coalescence probabilities of the auxiliary duals. In Section 3.5 this leads to a necessary and sufficient criterion for clustering in our original system.

3.1 Basic duals

Definition 3.1. **[Dual]** The dual process

$$(3.1)\quad Z^* = (Z^*(t))_{t\geq 0}, \qquad Z^*(t) = (n_i(t), m_i(t))_{i\in\mathbb{Z}^d},$$

is a continuous-time Markov chain with state space

$$(3.2)\quad \mathcal{X}^* = \Big\{(n_i,m_i)_{i\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (n_i+m_i) < \infty\Big\}$$

and with transition rates

(3.3)

$$(n_k,m_k)_{k\in\mathbb{Z}^d} \to \begin{cases} (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A}, & \text{at rate } \frac{2a(i,i)}{N_i}\binom{n_i}{2}\mathbf{1}_{\{n_i\geq 2\}} + \sum_{j\in\mathbb{Z}^d\setminus\{i\}} \frac{n_i a(i,j) n_j}{N_j} \text{ for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \vec{\delta}_{i,D}, & \text{at rate } \frac{\lambda n_i(M_i-m_i)}{M_i} \quad \text{for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d} + \vec{\delta}_{i,A} - \vec{\delta}_{i,D}, & \text{at rate } \frac{\lambda(N_i-n_i)m_i}{M_i} \quad \text{for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d} - \vec{\delta}_{i,A} + \delta_{j,A}, & \text{at rate } \frac{n_i a(i,j)(N_j-n_j)}{N_j} \quad \text{for } i\neq j\in\mathbb{Z}^d, \end{cases}$$

where for $i\in\mathbb{Z}^d$, $\vec{\delta}_{i,A} = (X_n,Y_n)_{n\in\mathbb{Z}^d}$ and $\vec{\delta}_{i,D} = (\hat{X}_n,\hat{Y}_n)_{n\in\mathbb{Z}^d}$ are the configurations defined as

$$(3.4)\quad (X_n,Y_n) := (\mathbf{1}_{\{n=i\}},0), \quad (\hat{X}_n,\hat{Y}_n) := (0,\mathbf{1}_{\{n=i\}}) \qquad \forall\, n\in\mathbb{Z}^d,$$

and for two configurations $\eta_1 = (\bar{X}_i, \bar{Y}_i)_{i\in\mathbb{Z}^d}$ and $\eta_2 = (\hat{X}_i, \hat{Y}_i)_{i\in\mathbb{Z}^d}$, $\eta_1 \pm \eta_2 := (X_i, Y_i)_{i\in\mathbb{Z}^d}$ is defined componentwise by

$$\begin{aligned} X_i &= (\bar{X}_i \pm \hat{X}_i)\mathbf{1}_{\{0\le \bar{X}_i \pm \hat{X}_i \le N_i\}} + N_i\,\mathbf{1}_{\{\bar{X}_i \pm \hat{X}_i > N_i\}}, \\ Y_i &= (\bar{Y}_i \pm \hat{Y}_i)\mathbf{1}_{\{0\le \bar{Y}_i \pm \hat{Y}_i \le M_i\}} + M_i\,\mathbf{1}_{\{\bar{Y}_i \pm \hat{Y}_i > M_i\}}. \end{aligned} \tag{3.5}$$

□

Here, $n_i(t)$ and $m_i(t)$ are the number of active and dormant particles at site $i \in \mathbb{Z}^d$ at time t. The first transition describes the coalescence of an active particle at site i with active particles at other sites. The second and third transition describe the switching between the active and the dormant state of the particles at site i. The fourth transition describes the migration of an active particle from site i to site j.

Since by Theorem 2.4 the dichotomy between clustering and coexistence is solely determined by the coalescence of two dual particles, we only need to analyse the dual process starting from two particles. For the sake of clarity, we give the definition of the two-particle dual process explicitly.

Definition 3.2. **[Two-particle dual]** The two-particle dual process

$$\tilde{Z} := (\tilde{Z}(t))_{t\ge 0}, \qquad \tilde{Z}(t) := (\tilde{n}_i(t), \tilde{m}_i(t))_{i\in\mathbb{Z}^d}, \tag{3.6}$$

is the continuous-time Markov chain with state space

$$\tilde{\mathcal{X}} := \Big\{(\tilde{n}_i, \tilde{m}_i)_{\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n}_i + \tilde{m}_i) \le 2\Big\} \tag{3.7}$$

and with transition rates as in (3.3). The support of the distribution of $\tilde{Z}(0)$ is contained in

$$\tilde{\mathcal{X}}_0 := \Big\{(\tilde{n}_i, \tilde{m}_i)_{\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n}_i + \tilde{m}_i) = 2\Big\}. \tag{3.8}$$

□

Let $\tilde{\mathcal{X}}_1$ be the set of configurations containing a single particle, i.e.,

$$\tilde{\mathcal{X}}_1 := \Big\{(\tilde{n}_i, \tilde{m}_i)_{\in\mathbb{Z}^d} \in \tilde{\mathcal{X}} \colon \sum_{i\in\mathbb{Z}^d} (\tilde{n}_i + \tilde{m}_i) = 1\Big\}, \tag{3.9}$$

and let $\tilde{\tau}$ be the first time at which coalescence has occurred, i.e.,

$$\tilde{\tau} = \inf\{t \ge 0 \colon (\tilde{n}_i(t), \tilde{m}_i(t))_{i\in\mathbb{Z}^d} \in \tilde{\mathcal{X}}_1\}. \tag{3.10}$$

Since we are only interested in the coalescence probability of two dual particles, this motivates us to lump all the configurations in $\tilde{\mathcal{X}}_1$ into a single state $\circledast$ and consider the resulting lumped process. Note that, on the event $\{\tilde{\tau} < s\}$, the process $(\tilde{Z}(t))_{t\ge s}$ a.s. stays in $\tilde{\mathcal{X}}_1$. Therefore the lumped process is a well-defined continuous-time Markov chain with state space $\tilde{\mathcal{X}}_0 \cup \{\circledast\}$, where $\circledast$ is an absorbing state.

With a little abuse of notation, from here onwards we will denote the lumped process by $(\tilde{Z}(t))_{t\ge 0}$. We give the formal description of this process in a definition.

Definition 3.3. **[Lumped two-particle dual]** The lumped two-particle dual process

$$\tilde{Z} := (\tilde{Z}(t))_{t\ge 0} \tag{3.11}$$

is the continuous-time Markov chain with state space

$$\tilde{\mathcal{X}} := \Big\{(n_i, m_i)_{\in\mathbb{Z}^d} \in \prod_{i\in\mathbb{Z}^d} [N_i]\times[M_i] \colon \sum_{i\in\mathbb{Z}^d} (n_i + m_i) = 2\Big\} \bigcup \Big\{\circledast\Big\} \tag{3.12}$$

and with transition rates

(3.13)

$$(n_k,m_k)_{k\in\mathbb{Z}^d}\to$$
$$\begin{cases}\circledast, & \text{at rate } \sum_{i\in\mathbb{Z}^d}\left[\frac{2a(0,0)}{N_i}\binom{n_i}{2}\mathbf{1}_{\{n_i\geq 2\}}+\sum_{j\in\mathbb{Z}^d\setminus\{i\}}\frac{n_i a(i,j)n_j}{N_j}\right] \text{ for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d}-\vec{\delta}_{i,A}+\vec{\delta}_{i,D}, & \text{at rate } \frac{\lambda n_i(M_i-m_i)}{M_i} \quad \text{for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d}+\vec{\delta}_{i,A}-\vec{\delta}_{i,D}, & \text{at rate } \frac{\lambda(N_i-n_i)m_i}{M_i} \quad \text{for } i\in\mathbb{Z}^d,\\ (n_k,m_k)_{k\in\mathbb{Z}^d}-\vec{\delta}_{i,A}+\vec{\delta}_{j,A}, & \text{at rate } \frac{n_i a(i,j)(N_j-n_j)}{N_j} \quad \text{for } i\neq j\in\mathbb{Z}^d,\end{cases}$$

where, for $i\in\mathbb{Z}^d$, $\vec{\delta}_{i,A}$ and $\vec{\delta}_{i,D}$ are as in (3.4). □

We write $\tilde{\mathbb{P}}^\eta$ to denote the law of the process $\tilde{Z}$ started from $\eta\in\mathcal{X}$.

Remark 3.4. Note that the coalescence time $\tilde{\tau}$ is now the absorption time of the process $\tilde{Z}$. □

3.2 Auxiliary duals

In addition to the auxiliary two-particle system defined in Definition 2.5, we introduce two more two-particle systems, called *interacting RW1* and *independent RW*, on the state space

(3.14) $\mathcal{S}:=(G\times G)\cup\{\circledast\},\quad G:=\mathbb{Z}^d\times\{0,1\}.$

These will be used as intermediate comparison objects.

Definition 3.5. [Interacting RW1] The interacting RW1 process

(3.15) $\xi:=(\xi(t))_{t\geq 0}$

is the continuous-time Markov chain on the state space $\mathcal{S}$ with transition rates

$$[(i,\alpha),(j,\beta)]\to$$
$$(3.16)\quad \begin{cases}\circledast, & \text{at rate } \alpha\beta(1-\delta_{i,j})\left[\frac{a(i,j)}{N_j}+\frac{a(j,i)}{N_i}\right]+\frac{\alpha\beta}{N_i}\delta_{i,j},\\ [(i,1-\alpha),(j,\beta)], & \text{at rate } \lambda[\alpha+(1-\alpha)K_i]-\frac{\lambda}{M_i}\delta_{i,j}(1-\delta_{\alpha,\beta}),\\ [(i,\alpha),(j,1-\beta)], & \text{at rate } \lambda[\beta+(1-\beta)K_j]-\frac{\lambda}{M_j}\delta_{i,j}(1-\delta_{\alpha,\beta}),\\ [(k,\alpha),(j,\beta)], & \text{at rate } \alpha\, a(i,k)[1-\frac{\beta}{N_j}\delta_{k,j}] \quad \text{for } k\neq i\in\mathbb{Z}^d,\\ [(i,\alpha),(k,\beta)], & \text{at rate } \beta\, a(j,k)[1-\frac{\alpha}{N_i}\delta_{k,i}] \quad \text{for } k\neq j\in\mathbb{Z}^d,\end{cases}$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta-function. □

We write $\mathbb{P}^\eta$ to denote the law of the process ξ started from $\eta\in\mathcal{S}$, and $\mathbb{E}^\eta$ to denote the expectation w.r.t. $\mathbb{P}^\eta$.

Remark 3.6. Note that, by virtue of Assumption (2.2), all states in $\mathcal{S}$ are accessible by ξ. □

Proposition 3.7. [Equivalence between $\tilde{Z}$ and ξ] *Let $\xi=(\xi(t))_{t\geq 0}$ be the process defined in Definition 3.5 with initial distribution μ. Let $\phi\colon\mathcal{S}\to\tilde{\mathcal{X}}$ be the map defined by*

$$(3.17)\quad \phi(\eta):=\begin{cases}(\alpha\delta_{k,i}+\beta\delta_{k,j},(1-\alpha)\delta_{k,i}+(1-\beta)\delta_{k,j})_{k\in\mathbb{Z}^d}, & \textit{if } \eta=[(i,\alpha),(j,\beta)]\neq\circledast,\\ \circledast, & \textit{otherwise.}\end{cases}$$

For $t\geq 0$, let $\tilde{Z}(t):=\phi(\xi(t))$. Then the process $(\tilde{Z}(t))_{t\geq 0}$ is the lumped dual process defined in Definition 3.3, and its initial distribution is the push-forward of μ under the map ϕ.

Proof. Using (2.2), we see that $\phi(\eta)\in\tilde{\mathcal{X}}$, and so $\tilde{Z}(t)\in\mathcal{X}$ for all $t\geq 0$, and ϕ is onto. For $\eta\in\mathcal{S}$, define

$$(3.18)\quad \bar{\eta}:=\begin{cases}[(j,\beta),(i,\alpha)], & \text{if } \eta=[(i,\alpha),(j,\beta)]\neq\circledast,\\ \circledast, & \text{otherwise.}\end{cases}$$

Note that $\phi^{-1}(\phi(\eta)) = \{\eta, \bar{\eta}\}$. Let $Q(\eta_1, \eta_2)$ denote the transition rate from η_1 to η_2 for the process ξ, where $\eta_1 \neq \eta_2 \in \mathcal{S}$. Furthermore, let $z_1 \neq z_2 \in \tilde{\mathcal{X}}$ be fixed and $\eta_1 \in \mathcal{S}$ be such that $\phi(\eta_1) = z_1$. Since $Q(\eta_1, \eta_2) = Q(\bar{\eta}_1, \bar{\eta}_2)$ for any $\eta_1 \neq \eta_2 \in \mathcal{S}$, we have

$$\sum_{\eta \in \phi^{-1}(z_2)} Q(\eta_1, \eta) = \sum_{\eta \in \phi^{-1}(z_2)} Q(\bar{\eta}_1, \eta). \tag{3.19}$$

Hence the Dynkin criterion for lumpablity is satisfied, and ϕ preserves the Markov property. So $\tilde{Z}$ is a Markov process on $\tilde{X}$. We can easily verify that the sum in (3.19) is indeed the transition rate from z_1 to z_2 defined in (3.13). Thus, $\tilde{Z}$ is the lumped dual process defined in Definition 3.3. Clearly, the distribution of $\tilde{Z}(0)$ is $\mu \circ \phi^{-1}$. □

Let τ be the absorption time for the process ξ, i.e.,

$$\tau := \inf\{t \geq 0\colon\ \xi(t) = \circledast\}. \tag{3.20}$$

Lemma 3.8. **[Equivalence of absorption]** *Let ξ be the process defined in Definition 3.5 started from $\eta \in \mathcal{S}$. Furthermore, let $\tilde{Z}$ be the process defined in Definition 3.3 started from $\phi(\eta)$, where ϕ is the map defined in Proposition 3.7. Then ξ is absorbed at $\circledast$ if and only if $\tilde{Z}$ is absorbed at $\circledast$.*

Proof. By Proposition 3.7, we have $(\phi(\xi(t)))_{t\geq 0} \overset{d}{=} \tilde{Z}$. Since $\phi(\eta) = \circledast$ if and only if $\eta = \circledast$, the result trivially follows. □

Definition 3.9. **[Independent RW]** The independent RW process

$$\xi^* := (\xi^*(t))_{t\geq 0} \tag{3.21}$$

is the continuous-time Markov chain on the state space $\mathcal{S}$ with transition rates

$$[(i,\alpha),(j,\beta)] \to \begin{cases} \circledast, & \text{at rate } \frac{\alpha\beta}{N_i}\delta_{i,j}, \\ [(i,1-\alpha),(j,\beta)], & \text{at rate } \lambda[\alpha + (1-\alpha)K_i], \\ [(i,\alpha),(j,1-\beta)], & \text{at rate } \lambda[\beta + (1-\beta)K_j], \\ [(k,\alpha),(j,\beta)], & \text{at rate } \alpha\, a(i,k) \quad \text{for } k \neq i \in \mathbb{Z}^d, \\ [(i,\alpha),(k,\beta)], & \text{at rate } \beta\, a(j,k) \quad \text{for } k \neq j \in \mathbb{Z}^d. \end{cases} \tag{3.22}$$

□

We write $\mathbb{P}^*_\eta$ to denote the law of the process ξ^* started from $\eta \in \mathcal{S}$, and $\mathbb{E}^*_\eta$ to denote expectation w.r.t. $\mathbb{P}^*_\eta$.

Proposition 3.10. **[Stability]** *All three processes $\xi, \hat{\xi}, \xi^*$ are non-explosive continuous-time Markov chains on the countable state space $\mathcal{S}$.*

Proof. We prove this claim by using the Foster-Lyapunov criterion (see [9]). Let $B_0 := \{\circledast\}$, and for $n \in \mathbb{N}$ define $B_n := \{[(i,\alpha),(j,\beta)] \in \mathcal{S}\colon\ \max\{\|i\|,\|j\|\} < n\} \cup B_0$. Define

$$V(\eta) := \begin{cases} \|i\| + \|j\|, & \text{if } \eta = [(i,\alpha),(j,\beta)], \\ 0, & \text{otherwise,} \end{cases} \qquad \eta \in \mathcal{S}. \tag{3.23}$$

Furthermore, let $Q, \hat{Q}, Q^*$ be the infinitesimal generators of the processes $\xi, \hat{\xi}, \xi^*$, respectively. Note that, for $\eta = [(i,\alpha),(j,\beta)] \in \mathcal{S}$,

$$\begin{aligned} QV(\eta) &= \alpha \sum_{k\neq i} a(i,k)(\|k\| - \|i\|) + \beta \sum_{k \neq j} a(j,k)(\|k\| - \|j\|) \\ &\qquad - 2\alpha\beta(1-\delta_{i,j})\left[\tfrac{a(i,j)}{N_j}\|i\| + \tfrac{a(j,i)}{N_i}\|j\|\right] - \tfrac{\alpha\beta}{N_i}\delta_{i,j} \\ &\leq (\alpha+\beta)\mu_1 + 2\alpha\beta(1-\delta_{i,j})\left[\tfrac{a(i,j)}{N_j}\|i\| + \tfrac{a(j,i)}{N_i}\|j\|\right] \\ &\leq 2V(\eta) + (\alpha+\beta)\mu_1 \\ &\leq 2V(\eta) + 2\mu_1 \quad (\text{since } \alpha + \beta \leq 2), \end{aligned} \tag{3.24}$$

where $\mu_1 := \sum_{i\in\mathbb{Z}^d/\{0\}} \|i\| a(0,i)$. Let $V' : \mathcal{S} \to [0,\infty)$ be the function defined by $\eta \mapsto V(\eta) + \mu_1$. Note that $B_n \uparrow \mathcal{S}$ as $n \to \infty$ and $\inf_{\eta\in B_n^c} V'(\eta) \geq n$. Thus, $\inf_{\eta\in B_n^c} V'(\eta) \uparrow \infty$ as $n \to \infty$ and, by (3.24), $QV'(\eta) \leq 2V'(\eta)$. Hence the Foster-Lyapunov criterion is satisfied by the generator Q, and so ξ is non-explosive. Similar arguments show that $\hat{\xi}$ and ξ^* are non-explosive as well. □

3.3 Comparison between interacting duals

Proposition 3.11. **[Stochastic domination]** *Let $f : \mathcal{S} \to \mathbb{R}$ be bounded and such that $f(\eta) \leq f(\circledast)$ for all $\eta \in \mathcal{S}$. Let $(\xi(t))_{t\geq 0}$ and $(\hat{\xi}(t))_{t\geq 0}$ be the interacting RW1 and the auxiliary two-particle system defined in Definition 3.5 and Definition 2.5, respectively. Then, for any $\eta \in \mathcal{S}$ and $t \geq 0$, $\mathbb{E}^\eta[f(\xi(t))] \geq \hat{\mathbb{E}}^\eta[f(\hat{\xi}(t))]$.*

Proof. Let Q and $\hat{Q}$ be the generators of the processes $\xi, \hat{\xi}$, respectively. Since ξ and $\hat{\xi}$ are non-explosive continuous-time Markov processs on a countable state space, Q and $\hat{Q}$ generate unique Markov semigroups $(S_t)_{t\geq 0}$ and $(\hat{S}_t)_{t\geq 0}$, respectively, given by

$$(3.25)\quad (S_t g)(\eta) = \mathbb{E}^\eta[g(\xi(t))], \quad (\hat{S}_t g)(\eta) = \hat{\mathbb{E}}^\eta[g(\hat{\xi}(t))], \qquad t \geq 0,$$

where $g : \mathcal{S} \to \mathbb{R}$ is bounded and $\eta \in \mathcal{S}$. Since f is bounded, we can apply the variation of constants formula for semigroups, to obtain

$$(3.26)\quad (S_t f)(\eta) - (\hat{S}_t f)(\eta) = \int_0^t (S_{t-s}(Q - \hat{Q})\hat{S}_s f)(\eta)\,\mathrm{d}s.$$

Now, if $g : \mathcal{S} \to \mathbb{R}$ is bounded and such that $\sup_{\eta\in\mathcal{S}} g(\eta) = g(\circledast)$, then

(3.27)

$$((Q-\hat{Q})g)(\eta) = \begin{cases} \alpha\beta(1-\delta_{i,j})\frac{a(i,j)}{N_j}\Big[g(\circledast) - g([(j,\alpha),(j,\beta)])\Big] & \\ \quad +\alpha\beta(1-\delta_{i,j})\frac{a(j,i)}{N_i}\Big[g(\circledast) - g([(i,\alpha),(i,\beta)])\Big], & \text{if } \eta = [(i,\alpha),(j,\beta)] \neq \circledast, \\ 0, & \text{otherwise}, \end{cases}$$
$$\geq 0.$$

Note that the semigroup $(\hat{S}_t)_{t\geq 0}$ also has the property $\sup_{\eta\in\mathcal{S}}(\hat{S}_s f)(\eta) = f(\circledast) = (\hat{S}_s f)(\circledast)$ for any $s \geq 0$, since $f \leq f(\circledast)$ and $\circledast$ is absorbing. Thus, combining the above with (3.27), we get that $(Q - \hat{Q})\hat{S}_s f$ is a non-negative function for any $s \geq 0$. Therefore the right-hand side of (3.26) is non-negative as well, which proves the desired result. □

Corollary 3.12. **[Stochastic ordering of absorption times]** *Let τ and $\hat{\tau}$ denote the absorption time of the processes ξ and $\hat{\xi}$, respectively. Then, for any $\eta \in \mathcal{S}$ and $t > 0$,*

$$(3.28)\quad \mathbb{P}^\eta(\tau \leq t) \geq \hat{\mathbb{P}}^\eta(\hat{\tau} \leq t).$$

Proof. This follows by applying Proposition 3.11 to the function $f = \mathbf{1}_{\{\circledast\}}$ and using that $\circledast$ is absorbing for both ξ and $\hat{\xi}$. □

Theorem 3.13. **[Comparison of absorption probabilities]** *Let $\nu : \mathcal{S} \to [0,1]$ and $\hat{\nu} : \mathcal{S} \to [0,1]$ be defined by*

$$(3.29)\quad \nu(\eta) := \mathbb{P}^\eta(\tau < \infty), \quad \hat{\nu}(\eta) := \hat{\mathbb{P}}^\eta(\hat{\tau} < \infty),$$

i.e., $\nu(\eta)$ and $\hat{\nu}(\eta)$ are the absorption probabilities of the processes ξ and $\hat{\xi}$, respectively, started from η. Assume that

$$(3.30)\quad \inf\{\hat{\nu}([(i,1),(i,1)]) : i \in \mathbb{Z}^d\} > 0.$$

Then $\hat{\nu}(\eta) = 1$ whenever $\nu(\eta) = 1$ for some $\eta \in \mathcal{S}$.

Proof. The proof is by contradiction. If $\eta = \circledast$, then the claim is trivial. So assume that $\hat{\nu}(\eta) < 1$ and $\nu(\eta) = 1$ for some $\eta \neq \circledast$. Note that, by the strong Markov property,

$$\inf_{y\in\mathcal{S}} \hat{\nu}(y) = 0. \tag{3.31}$$

Moreover, since by Remark 3.6 the process ξ started from η can visit any configuration $y \in \mathcal{S}$ in finite time with positive probability, we have

$$\nu(y) = 1 \qquad \forall\, y \in \mathcal{S}. \tag{3.32}$$

We will show that (3.31) and (3.32) are contradictory.

Let $B \subset \mathcal{S}$ be defined as

$$B := \{[(i,1),(i,1)] \colon\ i \in \mathbb{Z}^d\} \cup \{\circledast\}. \tag{3.33}$$

Let $T_B, \hat{T}_B$ denote the first hitting time of the set B for the processes ξ and $\hat{\xi}$, respectively. Furthermore, let $g \colon \mathcal{S} \to [0,1]$ and $\hat{g} \colon \mathcal{S} \to [0,1]$ be defined as

$$g(y) := \mathbb{P}^y(T_B < \infty), \quad \hat{g}(y) := \hat{\mathbb{P}}^y(\hat{T}_B < \infty), \quad y \in \mathcal{S}. \tag{3.34}$$

We first show that

$$\hat{g}(y) \geq g(y) \text{ for any } y \in \mathcal{S}. \tag{3.35}$$

To that end, let Q and $\hat{Q}$ be the generators of the processes ξ and $\hat{\xi}$, respectively. Applying $Q - \hat{Q}$ to the function $\hat{g}$, we get from (3.27) that

$$\begin{aligned}&(Q\hat{g})(y) - (\hat{Q}\hat{g})(y)\\ &= \begin{cases} \alpha\beta(1-\delta_{i,j})\frac{a(i,j)}{N_j}\Big[\hat{g}(\circledast) - \hat{g}([(j,\alpha),(j,\beta)])\Big] & \\ \quad +\alpha\beta(1-\delta_{i,j})\frac{a(j,i)}{N_i}\Big[\hat{g}(\circledast) - \hat{g}([(i,\alpha),(i,\beta)])\Big], & \text{if } y = [(i,\alpha),(j,\beta)] \neq \circledast,\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{3.36}$$

By a first-step analysis of $\hat{\xi}$, we have $(\hat{Q}\hat{g})(y) = 0$ for any $y \notin B$ and $\hat{g} \equiv 1$ on B. Thus, the right-hand side of (3.36) is always 0, and so $(Q\hat{g})(y) = (\hat{Q}\hat{g})(y) = 0$ for any $y \notin B$. Let $y \in \mathcal{S}$ be fixed and let the process ξ be started from y. Since $\hat{g}$ is bounded and ξ is non-explosive, the process $(M_t)_{t\geq 0}$ defined by $M_t := \hat{g}(\xi(t)) - \int_0^t (Q\hat{g})(\xi(s))\,\mathrm{d}s$ is a martingale under the law $\mathbb{P}^y$ w.r.t. the natural filtration associated to the process ξ. Hence the stopped process $(M_{t\wedge T_B})_{t\geq 0}$ is also a martingale. Note that, since $Q\hat{g} = 0$ outside B, we have $\int_0^{t\wedge T_B} (Q\hat{g})(\xi(s))\,\mathrm{d}s = 0$ for any $t \geq 0$. Hence $M_{t\wedge T_B} = \hat{g}(\xi(t\wedge T_B))$ for any $t \geq 0$. By the martingale property, for any $t > 0$,

$$\hat{g}(y) = \mathbb{E}^y[\hat{g}(\xi(0))] = \mathbb{E}^y[\hat{g}(\xi(t\wedge T_B))] \geq \mathbb{E}^y[\hat{g}(\xi(T_B))\mathbf{1}_{T_B<t}] = \mathbb{P}^y(T_B < t). \tag{3.37}$$

Letting $t \to \infty$, we get $\hat{g}(y) \geq \mathbb{P}^y(T_B < \infty) = g(y)$, which proves (3.35).

Now, since $T_B \leq \tau$ a.s., we have $g(y) \geq \nu(y)$ for any $y \in \mathcal{S}$, and combined with (3.32) this implies that $g \equiv 1$ on $\mathcal{S}$. So, using (3.35), we have

$$\hat{g}(y) = \hat{\mathbb{P}}^y(\hat{T}_B < \infty) = 1 \text{ for all } y \in \mathcal{S}, \tag{3.38}$$

i.e., the process $\hat{\xi}$ started from any configuration $y \in \mathcal{S}$ enters B with probability 1. Let $\hat{T}$ be the hitting time of the set $\hat{B} := B\backslash\{\circledast\}$ for the process $\hat{\xi}$, and let

$$\epsilon := \inf\{\hat{\nu}(y) \colon\ y \in \hat{B}\}. \tag{3.39}$$

By (3.30), we have $\epsilon > 0$. Note that $\hat{T} \leq \hat{\tau}$ a.s. for the process $\hat{\xi}$, since two particles coalesce only when they are on top of each other and are both active, and so $\hat{T}_B = \hat{T} \wedge \hat{\tau} = \hat{T}$ a.s. Therefore, by

(3.38), $\hat{\mathbb{P}}^y(\hat{T} < \infty) = 1$ for any $y \in \mathcal{S}$. Therefore, for $y \in \mathcal{S}$,

$$\begin{aligned}
\hat{\nu}(y) &= \hat{\mathbb{P}}^y(\hat{\tau} < \infty) = \hat{\mathbb{P}}^y(\hat{T} \le \hat{\tau} < \infty) = \sum_{x \in \hat{B}} \hat{\mathbb{P}}^y(\hat{\xi}(\hat{T}) = x, \hat{T} < \infty, \hat{\tau} < \infty) \\
&= \sum_{x \in \hat{B}} \hat{\mathbb{P}}^y(\hat{\tau} < \infty \,|\, \hat{\xi}(\hat{T}) = x, \hat{T} < \infty)\, \hat{\mathbb{P}}^y(\hat{\xi}(\hat{T}) = x, \hat{T} < \infty) \\
&= \sum_{x \in \hat{B}} \hat{\mathbb{P}}^x(\hat{\tau} < \infty)\, \hat{\mathbb{P}}^y(\hat{\xi}(\hat{T}) = x, \hat{T} < \infty) \\
&= \sum_{x \in \hat{B}} \hat{\nu}(x)\, \hat{\mathbb{P}}^y(\hat{\xi}(\hat{T}) = x, \hat{T} < \infty) \ge \epsilon\, \hat{\mathbb{P}}^y(\hat{T} < \infty) \ge \epsilon,
\end{aligned} \tag{3.40}$$

which contradicts (3.31). □

Corollary 3.14. [Equivalence of absorption probabilities] *For any $\eta \in \mathcal{S}$, $\nu(\eta) = 1$ if $\hat{\nu}(\eta) = 1$. Furthermore, if (3.30) holds, then the converse is true as well.*

Proof. The claim follows from Corollary 3.12 and Theorem 3.13. □

3.4 Comparison with non-interacting dual

Theorem 3.15. [Comparison of absorption probabilities] *Let $\nu^*\colon \mathcal{S} \to [0,1]$ and $\hat{\nu}\colon \mathcal{S} \to [0,1]$ be defined by*

$$\nu^*(\eta) := \mathbb{P}^*_\eta(\tau^* < \infty), \quad \hat{\nu}(\eta) := \hat{\mathbb{P}}^\eta(\hat{\tau} < \infty). \tag{3.41}$$

Assume that

$$\inf\{\nu^*([(i,1),(i,1)])\colon\ i \in \mathbb{Z}^d\} > 0. \tag{3.42}$$

Then $\nu^*(\eta) = 1$ whenever $\hat{\nu}(\eta) = 1$ for some $\eta \in \mathcal{S}$.

Proof. The proof follows a similar argument as in the proof of Theorem 3.13. Suppose that $\hat{\nu}(\eta) = 1$ and $\nu^*(\eta) < 1$. By the strong Markov property,

$$\inf_{y \in \mathcal{S}} \nu^*(y) = 0. \tag{3.43}$$

Since, by Remark 2.6, the process $\hat{\xi}$ started from η can visit any configuration $y \in \mathcal{S}$ in finite time with positive probability, we have

$$\hat{\nu}(y) = 1 \qquad \forall\, y \in \mathcal{S}. \tag{3.44}$$

We will show that (3.43) and (3.44) are contradictory.

Let $B \subset \mathcal{S}$ be defined as

$$B := \Big\{[(i,\alpha),(i,\beta)] \in \mathcal{S}\colon\ \alpha \ne \beta, \nu^*([(i,1),(i,1)]) < \nu^*([(i,1),(i,0)])\Big\} \cup \{\circledast\}. \tag{3.45}$$

By symmetry and a first-step analysis, we have

$$\nu^*([(i,1),(i,0)]) = \nu^*([(i,0),(i,1)]) = \nu^*([(i,0),(i,0)]) \qquad \forall\, i \in \mathbb{Z}^d. \tag{3.46}$$

Let $\hat{T}_B$ denote the first hitting time of the set B for the process $\hat{\xi}$, and let

$$\epsilon := \inf\{\nu^*(y)\colon\ y \in B\}. \tag{3.47}$$

By (3.42) and (3.46), $\epsilon > 0$. Note that if $\hat{Q}$ and Q^* are the generators of the processes $\hat{\xi}$ and ξ^*, respectively, then

$$\begin{aligned}
&((\hat{Q} - Q^*)\nu^*)(x) \\
&= \begin{cases} \frac{\lambda}{M_i} \delta_{i,j}(1 - \delta_{\alpha,\beta})[\nu^*([(i,1),(i,0)]) - \nu^*([(i,1),(i,1)])], & x = [(i,\alpha),(j,\beta)] \ne \circledast, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{3.48}$$

where (3.46) is used. Moreover, the right-hand side of the above equation is negative whenever $x \notin B$. Since $Q^*\nu^* \equiv 0$, we have

$$(3.49)\quad (\hat{Q}\nu^*)(x) \le 0, \qquad x \notin B.$$

Let $y \in \mathcal{S}$ be fixed arbitrarily, and let the process $\hat{\xi}$ be started from y. Since ν^* is bounded and $\hat{\xi}$ is non-explosive, the process $(M_t)_{t\ge 0}$ with $M_t := \nu^*(\hat{\xi}(t)) - \int_0^t (\hat{Q}\nu^*)(\hat{\xi}(s))\,\mathrm{d}s$ is a martingale under the law $\hat{\mathbb{P}}^y$ w.r.t. the natural filtration associated to the process $\hat{\xi}$. Hence the stopped process $(M_{t\wedge \hat{T}_B})_{t\ge 0}$ is also a martingale. By (3.49), we have $\int_0^{t\wedge \hat{T}_B} (\hat{Q}\nu^*)(\hat{\xi}(s))\,\mathrm{d}s \le 0$ a.s. for any $t \ge 0$. Hence $M_{t\wedge \hat{T}_B} \ge \nu^*(\hat{\xi}(t\wedge \hat{T}_B))$ for any $t \ge 0$. By the martingale property, for any $t > 0$,

$$(3.50)\quad \begin{aligned} \nu^*(y) &= \hat{\mathbb{E}}^y[\nu^*(\hat{\xi}(0))] = \hat{\mathbb{E}}^y[M_{t\wedge \hat{T}_B}] \ge \hat{\mathbb{E}}^y[\nu^*(\hat{\xi}(t\wedge \hat{T}_B))] \\ &\ge \hat{\mathbb{E}}^y[\nu^*(\hat{\xi}(\hat{T}_B))\mathbf{1}_{\hat{T}_B<t}] \ge \epsilon\, \hat{\mathbb{P}}^y(\hat{T}_B < t) \ge \epsilon\, \hat{\mathbb{P}}^y(\hat{\tau} < t), \end{aligned}$$

where in the last inequality we use that $\hat{T}_B \le \hat{\tau}$ a.s. Letting $t \to \infty$, we find with the help of (3.44) that $\nu^*(y) \ge \epsilon\, \hat{\mathbb{P}}^y(\hat{\tau} < \infty) = \epsilon\, \hat{\nu}(y) = \epsilon$, which contradicts (3.43). □

Theorem 3.16. **[Comparison of absorption probabilities]** *Let ν^*, ν, $\hat{\nu}$ be the absorption probability of ξ^*, $\hat{\xi}$, ξ, respectively, i.e.,*

$$(3.51)\quad \nu^*(\eta) := \mathbb{P}^*_\eta(\tau^* < \infty), \quad \hat{\nu}(\eta) := \hat{\mathbb{P}}^\eta(\hat{\tau} < \infty), \quad \nu(\eta) := \mathbb{P}^\eta(\tau < \infty).$$

Assume that

$$(3.52)\quad \inf\{\hat{\nu}([(i,1),(i,0)])\colon\ i \in \mathbb{Z}^d\} > 0.$$

Then $\hat{\nu}(\eta) = 1$ whenever $\nu^*(\eta) = 1$ for some $\eta \in \mathcal{S}$, and hence $\nu(\eta) = 1$ as well.

Proof. By Corollary 3.12, it suffices to prove that $\hat{\nu}(\eta) = 1$. Suppose that this fails. Then, by the strong Markov property,

$$(3.53)\quad \inf_{y\in\mathcal{S}} \hat{\nu}(y) = 0.$$

Moreover, since the process ξ^* started from η can visit any configuration $y \in \mathcal{S}$ in finite time with positive probability, we have

$$(3.54)\quad \nu^*(y) = 1 \qquad \forall\, y \in \mathcal{S}.$$

We will show that (3.53) and (3.54) are contradictory.

Let $B \subset \mathcal{S}$ be defined as

$$(3.55)\quad B := \Big\{[(i,\alpha),(i,\beta)] \in \mathcal{S}\colon\ \alpha \ne \beta, \hat{\nu}([(i,1),(i,1)]) \ge \hat{\nu}([(i,1),(i,0)])\Big\} \cup \{\circledast\}.$$

By symmetry and a first-step analysis, we have

$$(3.56)\quad \hat{\nu}([(i,1),(i,0)]) = \hat{\nu}([(i,0),(i,1)]) = \hat{\nu}([(i,0),(i,0)]) \qquad \forall\, i \in \mathbb{Z}^d.$$

Let T^*_B denote the first hitting time of the set B for the process ξ^*, and let

$$(3.57)\quad \epsilon := \inf\{\hat{\nu}(y)\colon\ y \in B\}.$$

By (3.52) and (3.56), we have $\epsilon > 0$. Note that if $\hat{Q}$ and Q^* are the generators of the processes $\hat{\xi}$ and ξ^*, respectively, then

$$(3.58)\quad \begin{aligned} &((Q^* - \hat{Q})\hat{\nu})(x) \\ &= \begin{cases} \frac{\lambda}{M_i}\delta_{i,j}(1-\delta_{\alpha,\beta})[\hat{\nu}([(i,1),(i,1)]) - \hat{\nu}([(i,1),(i,0)])], & x = [(i,\alpha),(j,\beta)] \ne \circledast, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

where we use (3.56). Moreover, the right-hand side side of the above equation is negative whenever $x \notin B$. Since $\hat{Q}\hat{\nu} \equiv 0$, we have

$$(3.59)\quad (Q^*\hat{\nu})(x) \le 0, \qquad x \notin B.$$

Let $y \in \mathcal{S}$ be fixed arbitrarily, and let the process ξ^* be started from y. Since $\hat{\nu}$ is bounded and ξ^* is non-explosive, the process $(M_t)_{t\ge 0}$ with $M_t := \hat{\nu}(\xi^*(t)) - \int_0^t (Q^*\hat{\nu})(\xi^*(s))\,\mathrm{d}s$ is a martingale under the law $\mathbb{P}^*_y$ w.r.t. the natural filtration associated to the process ξ^*. Hence the stopped process $(M_{t\wedge T^*_B})_{t\ge 0}$ is also a martingale. By (3.59), we have $\int_0^{t\wedge T^*_B} (Q^*\hat{\nu})(\xi^*(s))\,\mathrm{d}s \le 0$ a.s. for any $t \ge 0$. Hence $M_{t\wedge T^*_B} \ge \hat{\nu}(\xi^*(t\wedge T^*_B))$ for any $t \ge 0$. By the martingale property, for any $t > 0$,

$$(3.60)\quad \begin{aligned} \hat{\nu}(y) &= \mathbb{E}^*_y[\hat{\nu}(\xi^*(0))] = \mathbb{E}^*_y[M_{t\wedge T^*_B}] \ge \mathbb{E}^*_y[\hat{\nu}(\xi^*(t\wedge T^*_B))] \\ &\ge \mathbb{E}^*_y[\hat{\nu}(\xi^*(T^*_B))\mathbf{1}_{T^*_B<t}] \ge \epsilon\,\mathbb{P}^*_y(T^*_B < t) \ge \epsilon\,\mathbb{P}^*_y(\tau^* < t), \end{aligned}$$

where in the last inequality we use that $T^*_B \le \tau^*$ a.s. Letting $t \to \infty$, we find via (3.54) that $\hat{\nu}(y) \ge \epsilon\,\mathbb{P}^*_y(\tau^* < \infty) = \epsilon\,\nu^*(y) = \epsilon$, which contradicts (3.53). □

Remark 3.17. Theorem 3.16 tells us that coalescence of independent particles is sufficient for coalescence of interacting particles. The condition is stronger, because it requires control on the growth of both N_i and M_i. □

3.5 Conclusion

Theorem 3.18. [Equivalence of absorption probabilities] *Let ν^*, ν and $\hat{\nu}$ be the functions defined by*

$$(3.61)\quad \nu^*(\eta) := \mathbb{P}^*_\eta(\tau^* < \infty), \quad \hat{\nu}(\eta) := \hat{\mathbb{P}}^\eta(\hat{\tau} < \infty), \quad \nu(\eta) := \mathbb{P}^\eta(\tau < \infty).$$

If

(a) $\inf\{\hat{\nu}([(i,1),(i,1)])\colon\ i \in \mathbb{Z}^d\} > 0$,

(b) $\inf\{\nu^*([(i,1),(i,1)])\colon\ i \in \mathbb{Z}^d\} > 0$,

then $\nu^*(\eta) = 1$ whenever $\nu(\eta) = 1$ for some $\eta \in \mathcal{S}$. If $\inf\{\hat{\nu}([(i,1),(i,0)])\colon\ i \in \mathbb{Z}^d\} > 0$, then the converse is true as well.

Proof. The forward direction follows by combining Theorem 3.13 and Theorem 3.15. The reverse direction is a direct consequence of Theorem 3.16 and Corollary 3.12. □

Remark 3.19. Theorem 3.18 tells us that if the interacting particle system coalesces with probability 1, then it is necessary that two independent particles coalesce with probability 1. The two conditions are trivially satisfied when $\sup_{i\in\mathbb{Z}^d} N_i < \infty$. If, furthermore, $\sup_{i\in\mathbb{Z}^d} M_i < \infty$, then the third condition is satisfied as well. □

We conclude this section by providing conditions on the sizes of the active and the dormant populations that are weaker than the ones mentioned in Remark 3.19, and under which the assumptions in Theorem 3.18 are satisfied.

Theorem 3.20. [Lower bound on absorption probabilities] *Let $\hat{\nu}$ and ν^* be the functions defined by*

$$(3.62)\quad \hat{\nu}(\eta) := \hat{\mathbb{P}}^\eta(\hat{\tau} < \infty), \quad \nu^*(\eta) := \mathbb{P}^*_\eta(\tau^* < \infty).$$

If the sizes of the active populations $(N_i)_{i\in\mathbb{Z}^d}$ are non-clumping, i.e.,

$$(3.63)\quad \inf_{i\in\mathbb{Z}^d} \sum_{\|j-i\|\le R} \tfrac{1}{N_j} > 0 \text{ for some } R < \infty,$$

then

(a) $\inf\{\hat{\nu}([(i,1),(i,1)])\colon\ i \in \mathbb{Z}^d\} > 0$.

(b) $\inf\{\nu^*([(i,1),(i,1)])\colon\ i\in\mathbb{Z}^d\}>0$.

Furthermore, if the relative strengths of the seed-banks are bounded, i.e.,

$$\sup_{i\in\mathbb{Z}^d}\frac{M_i}{N_i}<\infty, \tag{3.64}$$

then

(i) $\inf\{\hat{\nu}([(i,1),(i,0)])\colon\ i\in\mathbb{Z}^d\}>0$.

(ii) $\inf\{\nu^*([(i,1),(i,0)])\colon\ i\in\mathbb{Z}^d\}>0$.

Before we give the proof of Theorem 3.20 we derive a series representation of the absorption probabilities ν^* and $\hat{\nu}$ of the respective processes ξ^* and $\hat{\xi}$.

Lemma 3.21. [Series representation] *Let ν^* and $\hat{\nu}$ be the functions defined by*

$$\nu^*(\eta):=\mathbb{P}^*_\eta(\tau^*<\infty),\quad \hat{\nu}(\eta):=\hat{\mathbb{P}}^\eta(\hat{\tau}<\infty). \tag{3.65}$$

For $i\in\mathbb{Z}^d$, let R^*_i (respectively, $\hat{R}_i$) be the total number of visits to the state $[(i,1),(i,1)]\in\mathcal{S}$ made by the jump chain associated to the process ξ^* (respectively, $\hat{\xi}$). Then, for $\eta\in\mathcal{S}\backslash\{\circledast\}$,

(a) $\nu^*(\eta)=\sum_{i\in\mathbb{Z}^d}\frac{1}{2(c+\lambda)N_i+1}\,\mathbb{E}^*_\eta[R^*_i]$.

(b) $\hat{\nu}(\eta)=\sum_{i\in\mathbb{Z}^d}\frac{1}{2(c+\lambda)N_i+1}\,\hat{\mathbb{E}}^\eta[\hat{R}_i]$,

where c is the total migration rate defined in Assumption 2.1, and expectations are taken w.r.t. the respective laws of the jump chains associated to the processes ξ^* and $\hat{\xi}$.

Proof. We only prove part (a), because the proof of part (b) is the same. Let $\eta\in\mathcal{S}\backslash\{\circledast\}$ be fixed, and let $X^*:=(X^*_n)_{n\in\mathbb{N}_0}$ be the embedded jump chain associated to the process ξ^* started at state η. Since X^* is absorbed to $\circledast$ if and only if ξ^* is absorbed, it suffices to analyse X^*. Let $T:=\inf\{n\in\mathbb{N}_0\colon\ X^*_n=\circledast\}$ be the absorption time of X^*. Note that, because the absorbing state $\circledast$ can be reached in one step from the states $\{[(i,1),(i,1)]\colon\ i\in\mathbb{Z}^d\}\subset\mathcal{S}$, for all $n\in\mathbb{N}$ we have

$$\begin{aligned}\mathbb{P}^*_\eta(T=n)&=\sum_{i\in\mathbb{Z}^d}\mathbb{P}^*_\eta(X^*_{n-1}=[(i,1),(i,1)],T=n)\\&=\sum_{i\in\mathbb{Z}^d}\mathbb{P}^*_\eta(X^*_n=\circledast\,|\,X^*_{n-1}=[(i,1),(i,1)])\,\mathbb{P}^*_\eta(X^*_{n-1}=[(i,1),(i,1)])\\&=\sum_{i\in\mathbb{Z}^d}\tfrac{1}{2(c+\lambda)N_i+1}\,\mathbb{P}^*_\eta(X^*_{n-1}=[(i,1),(i,1)]),\end{aligned} \tag{3.66}$$

where in the last equality we use that, by the Markov property,

$$\mathbb{P}^*_\eta(X^*_n=\circledast\,|\,X^*_{n-1}=[(i,1),(i,1)])=\mathbb{P}^*_{[(i,1),(i,1)]}(X^*_1=\circledast)=\frac{1}{2(c+\lambda)N_i+1}. \tag{3.67}$$

Using that $\eta\neq\circledast$, we get

$$\begin{aligned}\nu^*(\eta)&=\mathbb{P}^*_\eta(T<\infty)=\sum_{n\in\mathbb{N}}\mathbb{P}^*_\eta(T=n)\\&=\sum_{n\in\mathbb{N}}\sum_{i\in\mathbb{Z}^d}\tfrac{1}{2(c+\lambda)N_i+1}\,\mathbb{P}^*_\eta(X^*_{n-1}=[(i,1),(i,1)])\\&=\sum_{i\in\mathbb{Z}^d}\tfrac{1}{2(c+\lambda)N_i+1}\sum_{n\in\mathbb{N}}\mathbb{P}^*_\eta(X^*_{n-1}=[(i,1),(i,1)])=\sum_{i\in\mathbb{Z}^d}\tfrac{1}{2(c+\lambda)N_i+1}\,\mathbb{E}^*_\eta[R^*_i],\end{aligned} \tag{3.68}$$

where in the fourth equality we interchange the two sums using Fubini's theorem, and in the last equality we use

$$\mathbb{E}^*_\eta[R^*_i]=\sum_{n\in\mathbb{N}_0}\mathbb{P}^*_\eta(X^*_n=[(i,1),(i,1)]),\qquad i\in\mathbb{Z}^d. \tag{3.69}$$

□

Proof of Theorem 3.20. We only prove parts (a) and (i), because the proof of parts (b) and (ii) is the same. Let $\hat{X} := (\hat{X}_n)_{n\in\mathbb{N}_0}$ be the embedded jump chain associated to the process $\hat{\xi}$. For $j \in \mathbb{Z}^d$, let $\hat{R}_j$ be the total number of visits made by $\hat{X}$ to the state $[(j,1),(j,1)]$. We first show that, for any $i,j \in \mathbb{Z}^d$,

$$\hat{\mathbb{E}}^{[(i,1),(i,1)]}[\hat{R}_j] \geq \sum_{n\in\mathbb{N}} m^{2n} a_n(i,j)^2, \tag{3.70}$$

where $m := \frac{c}{2(c+\lambda)+1}$. Note that, in the process $\hat{\xi}$, each of the two particles moves from i to j at rate $a(i,j)$ while in the active state, and becomes dormant at rate λ when the two particles are not on top of each other with one active and the other dormant. Thus, for $i,j,k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\hat{\mathbb{P}}^{[(k,1),(i,1)]}(\hat{X}_n = [(k,1),(j,1)]) \\
&\geq \sum_{l\neq i} \hat{\mathbb{P}}^{[(k,1),(i,1)]}(\hat{X}_1 = [(k,1),(l,1)])\, \hat{\mathbb{P}}^{[(k,1),(l,1)]}(\hat{X}_{n-1} = [(k,1),(j,1)]) \\
&= \sum_{l\neq i} \tfrac{c}{2(c+\lambda)+(1/N_i)\delta_{k,i}} \tfrac{a(i,l)}{c}\, \hat{\mathbb{P}}^{[(k,1),(l,1)]}(\hat{X}_{n-1} = [(k,1),(j,1)]) \\
&\geq m \sum_{l\neq i} a_1(i,l)\, \hat{\mathbb{P}}^{[(k,1),(l,1)]}(\hat{X}_{n-1} = [(k,1),(j,1)]),
\end{aligned} \tag{3.71}$$

where $a_1(\cdot\,,\cdot) := \frac{a(\cdot\,,\cdot)}{c}$ is the transition kernel of the embedded chain associated to the continuous-time random walk on $\mathbb{Z}^d$ with rates $a(\cdot\,,\cdot)$. Using the above recursively, we obtain that, for any $i,j,k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$,

$$\hat{\mathbb{P}}^{[(k,1),(i,1)]}(\hat{X}_n = [(k,1),(j,1)]) \geq m^n\, a_n(i,j). \tag{3.72}$$

Therefore, applying the above twice, for $i,j \in \mathbb{Z}^d$ we have

$$\begin{aligned}
\hat{\mathbb{P}}^{[(i,1),(i,1)]}(\hat{X}_{2n} = [(j,1),(j,1)]) &\geq \hat{\mathbb{P}}^{[(i,1),(i,1)]}(\hat{X}_n = [(i,1),(j,1)])\, \hat{\mathbb{P}}^{[(i,1),(j,1)]}(\hat{X}_n = [(j,1),(j,1)]) \\
&\geq m^n a_n(i,j)\, \hat{\mathbb{P}}^{[(j,1),(i,1)]}(\hat{X}_n = [(j,1),(j,1)]) \geq m^{2n} a_n(i,j)^2.
\end{aligned} \tag{3.73}$$

Hence, for $i,j \in \mathbb{Z}^d$,

$$\begin{aligned}
\hat{\mathbb{E}}^{[(i,1),(i,1)]}[\hat{R}_j] &= \sum_{n\in\mathbb{N}_0} \hat{\mathbb{P}}^{[(i,1),(i,1)]}(\hat{X}_n = [(j,1),(j,1)]) \\
&\geq \sum_{n\in\mathbb{N}_0} \hat{\mathbb{P}}^{[(i,1),(i,1)]}(\hat{X}_{2n} = [(j,1),(j,1)]) \geq \sum_{n\in\mathbb{N}} m^{2n} a_n(i,j)^2.
\end{aligned} \tag{3.74}$$

Finally, substituting the above into the series representation of $\hat{\nu}$ in part (b) of Lemma 3.21, we obtain that, for $i \in \mathbb{Z}^d$,

$$\begin{aligned}
\hat{\nu}([(i,1),(i,1)]) &= \sum_{j\in\mathbb{Z}^d} \tfrac{1}{2(c+\lambda)N_j+1} \hat{\mathbb{E}}^{[(i,1),(i,1)]}[\hat{R}_j] \\
&\geq \sum_{j\in\mathbb{Z}^d} \tfrac{1}{2(c+\lambda)N_j+1} \sum_{n\in\mathbb{N}} m^{2n} a_n(i,j)^2 \\
&\geq \tfrac{1}{2(c+\lambda)+1} \sum_{j\in B_R(i)} \tfrac{1}{N_j} \sum_{n\in\mathbb{N}} m^{2n} a_n(0,j-i)^2 \geq \epsilon_R \sum_{j\in B_R(i)} \tfrac{1}{N_j},
\end{aligned} \tag{3.75}$$

where

$$\epsilon_R := \min\left\{ \tfrac{1}{2(c+\lambda)+1} \sum_{n\in\mathbb{N}} m^{2n} a_n(0,l)^2 \colon\ l \in B_R(0) \right\} > 0. \tag{3.76}$$

Since, by assumption, $(N_i)_{i\in\mathbb{Z}^d}$ are non-clumping, the right-hand side of (3.75) is bounded away from zero irrespective of the choice $i \in \mathbb{Z}^d$, and so part (a) is proved.

To prove part (i), by doing a first-step analysis of the process $\hat{X}$ we get that, for $i \in \mathbb{Z}^d$,

$$\hat{\nu}([(i,1),(i,0)]) \geq \hat{\mathbb{P}}^{[(i,1),(i,0)]}(\hat{X}_1 = [(i,1),(i,1)])\,\hat{\nu}([(i,1),(i,1)]) = \tfrac{\lambda K_i}{c+\lambda+\lambda K_i}\hat{\nu}([(i,1),(i,1)]), \tag{3.77}$$

where $K_i = \frac{N_i}{M_i}$. Thus, if $(N_i)_{i\in\mathbb{Z}^d}$ are non-clumping and $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$, then

$$\hat{\nu}([(i,1),(i,0)]) \geq \frac{\lambda}{\lambda + (c+\lambda)(\sup_{i\in\mathbb{Z}^d} K_i^{-1})} \inf\{\hat{\nu}([(j,1),(j,1)])\colon\ j \in \mathbb{Z}^d\}, \tag{3.78}$$

which is bounded away from zero uniformly in $i \in \mathbb{Z}^d$, and so part (i) follows. □

4 Proofs: clustering criterion and clustering regime

In Section 4.1 we prove Theorem 2.7. In Section 4.2 we prove Theorem 2.10 with the help of the results proved in Section 3.

4.1 Clustering criterion

Proof of Theorem 2.7. By Theorem 2.4 and Lemma 3.8, the system clusters if and only if the two-particle process ξ defined in Definition 3.5 is absorbed to $\circledast$ with probability 1. Let $\hat{\xi}$ be the auxiliary two-particle process defined in Definition 2.5, and $\hat{\nu}(\eta)$ (respectively, $\nu(\eta)$) be the absorption probability of the process $\hat{\xi}$ (respectively, ξ) started from state $\eta \in G \times G$. The system Z clusters if and only if $\nu(\eta) = 1$ for any state $\eta \in G \times G$. By the forward direction of Corollary 3.14, we have that $\nu(\eta) = 1$ whenever $\hat{\nu}(\eta) = 1$, and hence the forward direction of Theorem 2.7 follows. To prove the converse we note that, under the non-clumping assumption of the active populations sizes $(N_i)_{i\in\mathbb{Z}^d}$ in (2.12), (3.30) in Corollary 3.14 holds by part (a) of Theorem 3.20, and hence $\hat{\nu}(\eta) = 1$ whenever $\nu(\eta) = 1$, so that the converse follows as well. □

4.2 Independent particle system and clustering regime

In order to prove Theorem 2.10, we need to take a closer look at the non-interacting two-particle process ξ^* introduced in Definition 3.9. In what follows we briefly describe the process ξ^* and derive conditions under which the process ξ^* is absorbed with probability 1.

We recall from Definition 3.9 that the process $\xi^* = (\xi^*(t))_{t\geq 0}$ is a continuous-time Markov process on the state space $\mathcal{S} = (G \times G) \cup \{\circledast\}$ with $G = \mathbb{Z}^d \times \{0,1\}$. Here, $\xi^*(t) = [(i,\alpha),(j,\beta)]$ captures the location ($i, j \in \mathbb{Z}^d$) and the state ($\alpha, \beta \in \{0,1\}$) of two independent particles at time t, where 0 stands for dormant state and 1 stands for active state, respectively. The evolution of the two independent particles is governed by the following transitions (see Fig. 1):

- **(Migration)** Each particle migrates from location i to j at rate $a(i,j)$ while being active.
- **(Active to Dormant)** An active particle becomes dormant (without changing location) at rate λ.
- **(Dormant to Active)** A dormant particle at location i becomes active (without changing location) at rate λK_i.
- **(Coalescence)** The two particles coalesce with each other, and are absorbed to the state $\circledast$, at rate $\frac{1}{N_i}$ when they are both at location i and both active.

The following lemma tells that if the mean wake-up time of a dormant particle is uniformly bounded over all the locations in $\mathbb{Z}^d$, then the accumulated activity time of a single particle increases linearly in time.

Lemma 4.1. [Linear activity time] *Let $S(t)$ be the total accumulated time spent in the active state during the time interval $[0,t]$ by a single particle that evolves according to the first three transitions described above. If $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$, then*

$$\liminf_{t\to\infty} \frac{S(t)}{t} > 0 \quad a.s. \tag{4.1}$$

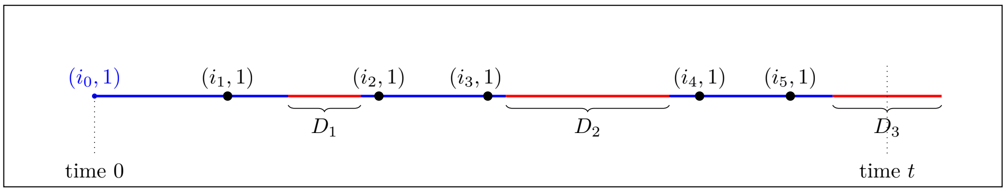

Figure 1: Evolution of a single particle started at location i_0 in the active state. Red and blue lines denote the dormant and the active phases of the particle. Each dot represents a migration step.

Proof. We prove the claim with the help of coupling in combination with a renewal argument. Let $(T_n)_{n\in\mathbb{N}}$ and $(D_n)_{n\in\mathbb{N}}$ be the successive time periods during which the particle is in the active and the dormant state, respectively (see Fig. 1). Note that $(T_n)_{n\in\mathbb{N}}$ are i.i.d. exponential random variables with mean $\frac{1}{\lambda}$. Let $K := \sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$ be fixed. Note that D_n is exponentially distributed with $\mathbb{E}[D_n] \le \frac{K}{\lambda}$, because the particle wakes up from the dormant state at rate $\lambda K_i \ge \frac{\lambda}{K}$ when it is at location i. Hence, using monotone coupling of exponential random variables, we can construct a sequence $(U_n)_{n\in\mathbb{N}}$ of i.i.d. exponential random variables on the same probability space with mean $\frac{K}{\lambda}$ such that $D_n \le U_n$ a.s. for all $n\in\mathbb{N}$. Consider the alternating renewal process $(R_t)_{t\ge 0}$ that takes value 0 (respectively, 1) during the time intervals $(T_n)_{n\in\mathbb{N}}$ (respectively, $(U_n)_{n\in\mathbb{N}}$), and let $D(t) := t - S(t)$ be the total accumulated time spent in the dormant state during the time interval $[0,t]$. Note that, because $D_n \le U_n$ a.s. for $n\in\mathbb{N}$, we have

$$(4.2)\quad D(t) \le \int_0^t \mathbf{1}_{\{R_s=1\}}\,\mathrm{d}s.$$

By applying the renewal reward theorem (see e.g. [10, Section 2b, Chapter VI] or [11, Theorem 1, Section 10.5]) to the process $(R_t)_{t\ge 0}$, we see that

$$(4.3)\quad \limsup_{t\to\infty} \frac{D(t)}{t} \le \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{\{R_s=1\}}\,\mathrm{d}s = \frac{\mathbb{E}[U_n]}{\mathbb{E}[T_n]+\mathbb{E}[U_n]} = \frac{\frac{K}{\lambda}}{\frac{1}{\lambda}+\frac{K}{\lambda}} = \frac{K}{1+K} \qquad \text{a.s.}$$

Hence

$$(4.4)\quad \liminf_{t\to\infty} \frac{S(t)}{t} = 1 - \limsup_{t\to\infty}\frac{D(t)}{t} \ge \frac{1}{1+K} > 0 \qquad \text{a.s.}$$

□

The following result provides a necessary and sufficient condition for the absorption of the process ξ^*.

Theorem 4.2. [Clustering regime] *Suppose that $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$ and Assumption 2.9 holds. If the process ξ^* is absorbed to $\circledast$ with probability 1, then it is necessary that the symmetrised kernel $\hat{a}(\cdot\,,\cdot)$ is recurrent, i.e.,*

$$(4.5)\quad \int_0^\infty \hat{a}_t(0,0)\,\mathrm{d}t = \infty.$$

Furthermore, if $(N_i)_{i\in\mathbb{Z}^d}$ satisfies the non-clumping condition in (2.12) *and $a(\cdot\,,\cdot)$ is symmetric, then* (4.5) *is also sufficient.*

Proof. Without loss of generality we may assume that the process starts at the state $\eta := [(0,1),(0,1)]$, i.e., both particles are initially at the origin $0\in\mathbb{Z}^d$ and in the active state. Since the process ξ^* has a positive rate of absorption only when the two independent particles are on top of each other and active, for the absorption probability to be equal to 1 it is necessary that, in the process where coalescence is switched off, the two independent particles meet infinitely often on the same location with probability 1. Let $S(t)$ and $S'(t)$ denote the total accumulated time spent in the active state by the two independent particles (where coalescence is switched off) during the time interval $[0,t]$. Since the two particles move according to $a(\cdot\,,\cdot)$ only when they are active, the total average time during which the two particles are on top of each other is given by

$$I := \int_0^\infty f(t)\,\mathrm{d}t, \tag{4.6}$$

where $f(t)$ is the probability that the two particles are on the same location at time t, which is given by

$$f(t) := \mathbb{E}^*_\eta\Big[\sum_{i\in\mathbb{Z}^d} a_{S(t)}(0,i)a_{S'(t)}(0,i)\Big]. \tag{4.7}$$

Thus, for the process ξ^* to be absorbed with probability 1, it is necessary that $I=\infty$.

Let us define

$$M(t) := S(t)\wedge S'(t), \quad L(t) := [S(t)\vee S'(t)] - [S(t)\wedge S'(t)] = |S(t)-S'(t)|. \tag{4.8}$$

Note that

$$\sum_{i\in\mathbb{Z}^d} a_{S(t)}(0,i)a_{S'(t)}(0,i) = \sum_{i\in\mathbb{Z}^d} \hat{a}_{2M(t)}(0,i)a_{L(t)}(i,0), \tag{4.9}$$

because the difference of two continuous-time random walks started at the origin that move independently in $\mathbb{Z}^d$ with rates $a(\cdot\,,\cdot)$ has distribution $\hat{a}_{2M(t)}(0,\cdot)$ at time $M(t)$ (because $a(\cdot\,,\cdot)$ is translation-invariant), and in order for the particle with the largest activity time to meet the other particle at the activity time $S(t)\vee S'(t) = M(t)+L(t)$, it must bridge the difference in the remaining time $L(t)$. We use the Fourier representation of the transition probability kernel $b(\cdot\,,\cdot)$, defined by

$$b(i,j) := \frac{a(i,j)}{c}\mathbf{1}_{i\neq j}, \qquad i,j\in\mathbb{Z}^d, \tag{4.10}$$

to further simplify the expression in (4.9). To this end, for $\theta\in\mathbb{T}^d := [-\pi,\pi]^d$, define

$$a(\theta) := \sum_{j\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\theta,j)}\, b(0,j), \quad \hat{a}(\theta) := \mathrm{Re}(a(\theta)), \quad \tilde{a}(\theta) := \mathrm{Im}(a(\theta)). \tag{4.11}$$

Then, for $j\in\mathbb{Z}^d$ and $t>0$,

$$\begin{aligned} \hat{a}_t(0,j) &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-\mathrm{i}(\theta,j)}\,\mathrm{e}^{-ct[1-\hat{a}(\theta)]}\,\mathrm{d}\theta,\\ a_t(0,j) &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-\mathrm{i}(\theta,j)}\,\mathrm{e}^{-ct[1-\hat{a}(\theta)-\mathrm{i}\tilde{a}(\theta)]}\,\mathrm{d}\theta. \end{aligned} \tag{4.12}$$

Using that $a(i,0)=a(0,-i)$, $i\in\mathbb{Z}^d$, and inserting the above into (4.9), we obtain

$$\begin{aligned} \sum_{i\in\mathbb{Z}^d} a_{S(t)}(0,i)a_{S'(t)}(0,i) &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-c[2M(t)+L(t)][1-\hat{a}(\theta)]}\cos(L(t)\tilde{a}(\theta))\,\mathrm{d}\theta\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-c[S(t)+S'(t)][1-\hat{a}(\theta)]}\cos(L(t)\tilde{a}(\theta))\,\mathrm{d}\theta\\ &\leq \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{e}^{-c[S(t)+S'(t)][1-\hat{a}(\theta)]}\,\mathrm{d}\theta\\ &= \hat{a}_{S(t)+S'(t)}(0,0), \end{aligned} \tag{4.13}$$

where we use that $\frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}(\theta-\theta',\,j)} = \delta(\theta-\theta')$, with $\delta(\cdot)$ the Dirac distribution (see e.g. [12, Chapter 7]). Finally, combining the above with (4.6)–(4.7), we see that

$$I \leq \int_0^\infty \mathbb{E}^*_\eta\Big[\hat{a}_{S(t)+S'(t)}(0,0)\Big]\,\mathrm{d}t \tag{4.14}$$

and therefore it is necessary that

$$(4.15)\quad \int_0^\infty \mathbb{E}_\eta^*\Big[\hat{a}_{S(t)+S'(t)}(0,0)\Big]\,\mathrm{d}t = \infty.$$

Since $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$, by Lemma 4.1 we know that $\liminf_{t\to\infty} \frac{S(t)+S'(t)}{t} \in [\delta, 2]$ a.s. for some $\delta \in (0,2]$. Also by Assumption 2.9, we have that

$$(4.16)\quad \lim_{t\to\infty} \frac{\hat{a}_{pt}(0,0)}{\hat{a}_t(0,0)} = 1,$$

where the convergence is uniform in $p \in [\delta, 2]$ (see e.g. [13, Theorem 1.2.1, Section 1.2]). Hence $\hat{a}_{S(t)+S'(t)}(0,0) \asymp \hat{a}_t(0,0)$ a.s. and thus, by (4.15), $\int_0^\infty \hat{a}_t(0,0)\,\mathrm{d}t = \infty$, which proves the forward direction.

To prove the converse, we first note that, because all the rates of absorption given by $(\frac{1}{N_i})_{i\in\mathbb{Z}^d}$ are such that (2.12) holds and $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$, whenever the two particles are on the same location, there is a positive probability of absorption that is uniformly bounded away from zero. Indeed, if $\nu^*(\eta)$ denote the absorption probability of ξ^* when started from state η, by Theorem 3.20 we have that

$$(4.17)\quad \begin{aligned} &\inf_{i\in\mathbb{Z}^d} \nu^*([(i,1),(i,1)]) > 0,\\ &\inf_{i\in\mathbb{Z}^d} \nu^*([(i,0),(i,1)]) = \inf_{i\in\mathbb{Z}^d} \nu^*([(i,0),(i,0)]) = \inf_{i\in\mathbb{Z}^d} \nu^*([(i,1),(i,0)]) > 0, \end{aligned}$$

where the last two equality follow from a first-jump analysis of the process ξ^* when started at the state $[(i,0),(i,0)],\ i \in \mathbb{Z}^d$. As a consequence, ξ^* is absorbed with probability 1 if and only if, in the corresponding process where coalescence is switched off, the two particles infinitely often meet each other with probability 1. In other words, $\nu^* \equiv 1$ if and only if $I = \infty$, where I is as in (4.6), the average accumulated time spent by the two particles at the same location. However, by the symmetry of the kernel $a(\cdot\,,\cdot)$, we have

$$(4.18)\quad I = \int_0^\infty \mathbb{E}_\eta^*\Big[a_{S(t)+S'(t)}(0,0)\Big]\,\mathrm{d}t = \int_0^\infty \mathbb{E}_\eta^*\Big[\hat{a}_{S(t)+S'(t)}(0,0)\Big]\,\mathrm{d}t$$

and thus, by virtue of Assumption 2.9 and Lemma 4.1, $I = \infty$ if and only if $\int_0^\infty a_t(0,0)\,\mathrm{d}t = \infty$, which due to the symmetry is equivalent to the symmetrised kernel being recurrent. This proves the backward direction. □

Now we are ready to prove Theorem 2.10 with the help of Theorem 4.2 and the results in Section 3.5.

Proof of Theorem 2.10. Let $\nu(\eta)$ denote the absorption probability of the process ξ (see Definition 3.5) started at state $\eta \in G \times G$. Recall from Theorem 2.4 and Lemma 3.8 that the system clusters if and only if $\nu \equiv 1$. By the irreducibility of the process ξ, we have $\nu \equiv 1$ if and only if $\nu([(0,0),(0,0)]) = 1$. Now, since $\sup_{i\in\mathbb{Z}^d} K_i^{-1} < \infty$ and (2.12) holds, we see that all the conditions of Theorem 3.18 are satisfied by virtue of Theorem 3.20, and hence $\nu([(0,0),(0,0)]) = 1$ if and only if $\nu^*([(0,0),(0,0)]) = 1$, where $\nu^*(\eta)$ denotes the absorption probability of the non-interacting two-particle process ξ^* (see Definition 3.9) started at state $\eta \in G \times G$. However, by the forward direction of Theorem 4.2, if $\nu^*([(0,0),(0,0)]) = 1$, then it is necessary that the symmetrised kernel $\hat{a}(\cdot\,,\cdot)$ is recurrent, and hence the forward direction is proved. Similarly, under the assumption of symmetry of the migration kernel, we can apply the converse direction of Theorem 3.18, to conclude that if the transition kernel $a(\cdot\,,\cdot)$ (which is the same as the symmetrised transition kernel) is recurrent, then $\nu^*([(0,0),(0,0)]) = 1$, and so the backward direction follows as well. □

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