Bounds on Dawson’s integral occurring in the analysis of a line distribution network for electric vehicles

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Abstract.
The Dawson integral $F(y)$ arises in the analysis of a particular model (Distflow model) for charging electric vehicles (EVs) when one passes from the intrinsic discrete setting to an analytically more tractable continuous setting. The mathematical and computational properties of $F(y)$ have been developed in the context of the error function, with purely imaginary argument $iy$, $y \geq 0$, for which packages, such as Mathematica, exist. In this report, we focus on bounds on $F(y)$ that are sharp, both at $y = 0$ and $y = \infty$, a topic that has been hardly addressed in the existing literature. One of the bounds we show emerges naturally from the EV-application when one compares the Distflow model to a linearized version of it.
1 Introduction

We start this report by outlining how the Dawson integral arises in the analysis of a particular model (Distflow model) for charging electric vehicles (EVs) at \( N \) (large) charging stations connected by a power line, with the requirement that the ratio between the voltages at the root and the last station at the power line should stay below a desired level. A full description of the problem at hand, comprising a comparison of the Distflow model and the linearized Distflow model, can be found in [1]. In [1], Subsections 2.3.1–2, the two models are introduced and discussed. Under the Distflow model, the normalized voltages \( V_n, n = 0, 1, \ldots, N - 1, N \), with \( V_N \) the voltage at the root station and \( V_0 \) the voltage at the last station of the power line, satisfy a recursion

\[
V_0 = 1, \quad V_1 = 1 + k_0; \quad V_{n+1} - 2V_n + V_{n-1} = \frac{k_n}{V_n}, \quad n = 1, \ldots, N - 1,
\]

see [1], (2.16–2.18). The \( k_n \), comprising given charging rates \( p_n \) and resistance/reactance values \( r \) and \( x \) as well as the arrival rate \( \lambda \) at the stations, are normally small (of the order \( a/N^2 \) with \( 0 < a < 0.1 \)). The ratio \( V_N/V_0 = V_N \) between the voltages at the root node (\( N \)) and the last node (0) should be below a level \( 1/(1 - \Delta) \), where the tolerance \( \Delta \) is small (of the order 0.1). Linearization of the Distflow model, as is done in [1], Subsection 2.3.2, yields the linearized Distflow model.

For analytically comparing the two models, it is assumed that all \( k_n \) are equal to \( a/N^2 \), with \( a \in (0, 0.1) \) independent of \( n \) (for numerically comparing the two models, such an assumption does not need to be made). In [1], Section 5.3 and Appendix B, a major effort is made to establish a relationship between the sequence \( V_n, n = 0, 1, \ldots, N \), and the solution \( f_0(t), t \geq 0 \), of the second-order boundary value problem

\[
f''_0(t) = \frac{1}{f_0(t)}, \quad t \geq 0; \quad f_0(0) = 1, \quad f'_0(0) = 0.
\]

In particular, it is shown see [1], Section 5.4, that \( V_N \to f_0(\sqrt{a}) \) as \( N \to \infty \) and \( a \) is fixed.

The Dawson integral \( F(y) = e^{-y^2} I(y) \), with

\[
I(y) = \int_0^y e^{v^2} dv, \quad y \geq 0,
\]

then arises as follows, see [1], Appendix C. We have

\[
f_0(t) = \exp(U^2(t)), \quad t \geq 0,
\]
where $U(t)$ is defined implicitly by
\[
\int_0^{U(t)} e^{v^2} dv = t/\sqrt{2}, \quad t \geq 0.
\] (5)

That is, in the notation preferred in [1], Section 5.4,
\[
U(t) = \text{inverfi}(t \sqrt{\frac{2}{\pi}}), \quad t \geq 0,
\] (6)

where inverfi is the inverse of the function
\[
i \text{erf}(y/i) = \frac{2}{\sqrt{\pi}} \int_0^y e^{v^2} dv \quad \text{with} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.
\] (7)

In the research effort, from which [1] and [2] resulted, properties and results about the sequence $V_n$, $n = 0, 1, \ldots$, have been conjectured and proved by first establishing these for the analytically more tractable function $f_0(z)$. For instance, we have from the asymptotics
\[
I(y) \sim \frac{e^{y^2}}{2y} \left(1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \ldots\right), \quad y \to \infty,
\] (asymptotic series) that
\[
f_0(t) = t (2 \ln t)^{1/2} \left(1 + O\left(\frac{\ln(\ln t)}{\ln t}\right)\right), \quad t \to \infty.
\] (9)

This leads to the conjecture that for fixed $k > 0$, and with $n$ in (1) allowed to tend to $\infty$,
\[
V_n \sim n (2k \ln n)^{1/2}, \quad n \to \infty.
\] (10)

The latter result will be proved in all detail in [2].

## 2 Basic properties and bounds of Dawson’s integral

We shall now give basic properties and bounds for Dawson’s integral $F(y) = e^{-y^2} I(y)$ and the integral $I(y)$ in (3), as far as relevant for our purposes. We refer to [3], Ch. 7 for an extensive account, with references, of the properties of the error function and Dawson’s integral. We have
A. \[ I(y) = \sum_{l=0}^{\infty} \frac{y^{2l+1}}{l!(2l+1)} = y(1 + \frac{1}{3} y^2 + \frac{1}{10} y^4 + \frac{1}{42} y^6 + \ldots), \quad y \geq 0 \] \quad (11)

B. \[ I(y) \sim \frac{e^{y^2}}{2y} \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{(2y^2)^k} = \frac{e^{y^2}}{2y} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(1/2)} y^{-2k} \]
\[ = \frac{e^{y^2}}{2y} \left(1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \frac{15}{8y^6} + \ldots\right), \quad y \to \infty. \] \quad (12)

The series in (12) is an asymptotic series; see [4], (2.8) and further, for a Cauchy principal value integral for the remainder when finitely many terms of the series are included. When for a particular \( y > 0 \) the series is truncated at the integer \( k \) nearest to \( y^2 - 1/2 \), the truncation error is of the order
\[ \frac{\Gamma(k+3/2)}{\Gamma(1/2)} y^{-2k-2} \approx e^{-y^2} \sqrt{2} \] \quad (13)

(for \( y = 8 \) this is \( \approx 2 \times 10^{-28} \) with \( k = 64 \)).

C. In [3], §7.8 some bounds on \( I(y) \) are given, viz.
\[ I(y) < \frac{1}{3y} (2e^{y^2} + y^2 - 2), \quad y > 0 \], \quad (14)
\[ I(y) < \frac{e^{y^2} - 1}{y}, \quad y > 0 \]. \quad (15)

The right-hand side of (14) has Taylor series
\[ \frac{1}{3y} (2e^{y^2} + y^2 - 2) = y(1 + \frac{1}{3} y^2 + \frac{1}{5} y^4 + \frac{1}{35} y^6 + \ldots), \quad y \geq 0 \] \quad (16)
and the bound in (14) is therefore sharp, see (11), at \( y = 0 \); the bound in (14) is not sharp as \( y \to \infty \), see (12). The right-hand side of (15) has Taylor series
\[ \frac{e^{y^2} - 1}{y} = y(1 + \frac{1}{2} y^2 + \frac{1}{6} y^4 + \ldots), \quad y \geq 0 \] \quad (17)
and the bound in (15) is therefore sharp at \( y = 0 \) (though not as sharp as (16)). In [5], \( 2^o \) on p. 180, there is given the bound
\[ I(y) < \frac{\pi^2}{8y} (e^{y^2} - 1), \quad y > 0 \]. \quad (18)
Since $\frac{\pi^2}{8} = 1.2337...$, the bound (18) is not sharp at $y = 0$ nor at $y = \infty$. I haven’t seen lower bounds on $I(y)$.

D. We now present some simple lower and upper bounds. We have

$$\frac{e^{y^2} - 1}{2y} \leq I(y) \leq \frac{e^{y^2} - 1}{y}, \quad y \geq 0.$$  \hfill (19)

Indeed,

$$\frac{e^{y^2} - 1}{y} = \sum_{l=0}^{\infty} \frac{y^{2l+1}}{(l+1)!},$$  \hfill (20)

and

$$\frac{1}{2 \cdot (l+1)!} < \frac{1}{l!(2l+1)} \leq \frac{1}{(l+1)!}, \quad l = 0, 1, \ldots,$$  \hfill (21)

so (19) follows from (11). The lower bound in (19) is sharp at $y = \infty$ and not sharp at $y = 0$; the upper bound in (19) is sharp at $y = 0$ and not sharp at $y = \infty$. The lower bound in (19) can be sharpened to

$$I(y) > \frac{\sinh(y^2)}{y} = \frac{e^{y^2} - e^{-y^2}}{2y}, \quad y > 0.$$  \hfill (22)

Indeed, we have for $y > 0$

$$e^{-y^2} I(y) = e^{-y^2} \int_0^y e^{v^2} dv = \int_0^y e^{(v+y)(v-y)} dv > \int_0^y e^{2y(v-y)} dv = \frac{1}{2y} (1 - e^{-2y^2}).$$  \hfill (23)

The bound in (22) is sharp at both $y = 0$ and $y = \infty$. As a result of a numerical computation, it is found that the minimum of $\sinh(y^2)/(y I(y))$ over $y \geq 0$ equals 0.766769724... and is assumed at $y = 1.386079411\ldots$.

The lower bound in (22) arose in an early version of [1] (Nov. 2, 2021) where the Distflow and linearized Distflow models were compared. For this, it was required that

$$H(y) := \frac{1}{\sqrt{e^{2y^2} - 1}} \int_0^y e^{v^2} dv = \frac{I(y)}{\sqrt{e^{2y^2} - 1}}$$  \hfill (24)
is strictly decreasing in \( y > 0 \). One computes

\[
H'(y) = \frac{1}{\sqrt{e^{2y^2} - 1}} \left( -\frac{2y e^{2y^2}}{e^{2y^2} - 1} I(y) + e^{y^2} \right),
\]

and this negative for all \( y > 0 \) if and only if (22) holds.

### 3 More advanced bounds and approximations

We consider now the functions, with \( \gamma \geq 0 \),

\[
B(y; \gamma) = \frac{e^{y^2} - 1}{2y} / \left(1 - \frac{1 - e^{-\gamma y^2}}{2\gamma y^2}\right), \quad y \geq 0,
\]

as potential lower and upper bounds for \( I(y) \) for all \( y \geq 0 \). The function \( B(y; \gamma) \), case \( \gamma = 1 \), arises naturally when one wants to show that, see (19),

\[
I(y) / \left(\frac{e^{y^2} - 1}{y}\right) , \quad y \geq 0,
\]

(strictly) decreases from 1 at \( y = 0 \) to 1/2 at \( y = \infty \).

**Lemma 1** We have for \( y > 0 \)

\[
\left( I(y) / \left(\frac{e^{y^2} - 1}{y}\right) \right)' < 0 \Leftrightarrow I(y) > B(y; 1) = \frac{e^{y^2} - 1}{2y} / \left(1 - \frac{1 - e^{-y^2}}{2y^2}\right).
\]

**Proof.** We have for \( y > 0 \)

\[
\left( I(y) / \left(\frac{e^{y^2} - 1}{y}\right) \right)' = \frac{e^{y^2} - 1 - 2y^2 e^{y^2}}{(e^{y^2} - 1)^2} I(y) + \frac{y e^{y^2}}{e^{y^2} - 1},
\]

and this is negative if and only if

\[
I(y) > \frac{y(e^{y^2} - 1) e^{y^2}}{2y^2 e^{y^2} - (e^{y^2} - 1)}. \quad (30)
\]

The right-hand side of (30) equals \( B(y; 1) \) and this completes the proof.

We next list a number of properties of the \( B(y; \gamma) \).
**Proposition 1**

(a) We have, see (19),

\[ B(y; 0) = \frac{e^{y^2} - 1}{y}, \quad y \geq 0; \quad \lim_{y \to \infty} B(y; \gamma) = \frac{e^{y^2} - 1}{2y}, \quad y > 0. \]  

(b) For \( y \geq 0 \) fixed, we have that \( B(y; \gamma) \) is a decreasing function of \( \gamma \geq 0 \).

(c) We have for fixed \( \gamma > 0 \) that

\[ \frac{B(y; \gamma)}{e^{y^2} - 1} \] decreases in \( y \geq 0 \) from 1 at \( y = 0 \) to 1/2 at \( y = \infty \).

(d) We have for \( \gamma \geq 0 \)

\[ B(y; \gamma) = y(1 + \frac{1}{2}(1 - \gamma) y^2 + (\frac{5}{12} \gamma^2 - \frac{1}{4} \gamma + \frac{1}{6} y^2) \quad \text{for all } y \geq 0 \implies \gamma \geq 1. \]

(e) We have (asymptotic equivalence)

\[ B(y; 0) \sim \frac{e^{y^2}}{y} \left( 1 + \frac{0}{y^2} + \frac{0}{y^4} + \frac{0}{y^6} + \ldots \right), \quad y \to \infty, \] and, for fixed \( \gamma > 0 \),

\[ B(y; \gamma) \sim \frac{e^{y^2}}{2y} \frac{1}{1 - (2\gamma/y^2)} = \frac{e^{y^2}}{2y} \left( 1 + \frac{1}{2\gamma y^2} + \frac{1}{4\gamma^2 y^4} + \frac{1}{8\gamma^3 y^6} + \ldots \right) \]

\[ = I(y) \left( 1 + \frac{1}{O(1)} \right), \quad y \to \infty. \]

The proofs of these results are straightforward. It is used that the function \( x \geq 0 \mapsto (1 - e^{-x})/x \) decreases from 1 at \( x = 0 \) to 0 at \( x = \infty \). Furthermore, it is used that \( \exp(-y^2) \) and \( \exp(-\gamma y^2), \gamma > 0, \) are exponentially small as \( y \to \infty \) and thus are asymptotically equivalent with \( 0/y^2 + 0/y^4 + \ldots \).

**Theorem 1**

(a) \( B(y; \gamma) \leq I(y) \) for all \( y \geq 0 \iff \gamma \geq 1 \).

(b) \( B(y; \gamma) \geq I(y) \) for all \( y \geq 0 \iff 0 \leq \gamma \leq 1/3 \).

**Proof.** (a) It follows from (12) and (35) that \( B(y; \gamma) \leq I(y) \) for all \( y \geq 0 \) implies that \( 1/2\gamma \leq 1/2, \) i.e., \( \gamma \geq 1 \). Furthermore, from Proposition 1(b)
we have that $B(y; \gamma) \leq B(y; 1)$ for all $y \geq 0$ and all $\gamma \geq 1$. Hence, it is sufficient to show that $B(y; 1) \leq I(y)$, $y \geq 0$. We have $I(0) = 0 = B(0; 1)$, and so it is sufficient to show that for $y > 0$

$$I'(y) = e^{y^2} > \left( \frac{y(e^{y^2} - 1)}{2y^2 - (1 - e^{-y^2})} \right)' = B'(y; 1). \quad (36)$$

We compute for $y > 0$

$$\left( \frac{y(e^{y^2} - 1)}{2y^2 - (1 - e^{-y^2})} \right)' = \frac{-4xe^x + 4x^2e^x - (e^x - 1)(1 - e^{-x}) + 2x(1 - e^{-x})}{(2x - (1 - e^{-x}))^2}, \quad (37)$$

where we have set $x = y^2 > 0$. Thus (36) is equivalent with

$$-4xe^x + 4x^2e^x - (e^x - 1)(1 - e^{-x}) + 2x(1 - e^{-x}) < e^x(2x - (1 - e^{-x}))^2, \quad (38)$$

with $x = y^2 > 0$. The right-hand side of (38) equals

$$4x^2e^x - 4xe^x - (e^x - 1) + e^x(1 - e^{-x})^2, \quad (39)$$

and so, cancelling the $4x^2e^x - 4xe^x - (e^x - 1)$ from both sides of (38) and dividing through by a factor $1 - e^{-x}$, we have that (36) is equivalent with

$$-(e^x - 1) + 2x < e^x(1 - e^{-x}), \quad x = y^2 > 0, \quad (40)$$

i.e., with $(e^x - 1)/x > 1$ for $x > 0$. This is obviously true.

(b) It follows from (11) and (33) that $B(y; \gamma) \geq I(y)$ for all $y \geq 0$ implies that $\frac{1}{2}(1 - \gamma) \geq 1/3$, i.e., $\gamma \leq 1/3$. Furthermore, from Proposition 1 (b), we have that $B(y; \gamma) \geq B(y; 1/3)$ for all $y \geq 0$ and all $\gamma, 0 \leq \gamma \leq 1/3$. Hence, it is sufficient to show that $B(y; 1/3) \geq I(y)$, $y \geq 0$. We have $I(0) = 0 = B(0; 1/3)$, and so it is sufficient to show that for all $y > 0$

$$I'(y) = e^{y^2} < \left( \frac{\gamma y(e^{y^2} - 1)}{2\gamma y^2 - (1 - e^{-\gamma y^2})} \right)' = B'(y; \gamma), \quad \gamma = 1/3 \quad (41)$$

We compute for $y > 0$

$$\left( \frac{\gamma y(e^{y^2} - 1)}{2\gamma y^2 - (1 - e^{-\gamma y^2})} \right)' = \gamma \left\{ -\gamma xe^x + 4\gamma x^2e^x - (e^x - 1)(1 - e^{-\gamma x}) - 2xe^x(1 - e^{-\gamma x}) + 2\gamma xe^{x - 1}e^{-\gamma x} \right\} / \left\{ (2\gamma x - (1 - e^{-\gamma x}))^2 \right\}, \quad (42)$$

8
where we have set \( x = y^2 > 0 \). Thus (41) is equivalent with

\[
\frac{1}{\gamma} \left( 2\gamma x - (1 - e^{-\gamma x})^2 e^x \right) < -2\gamma x(e^x - 1) + 4\gamma x^2 e^x - (e^x - 1)(1 - e^{-\gamma x}) \\
-2xe^x(1 - e^{-\gamma x}) + 2\gamma x(e^x - 1) e^{-\gamma x} ,
\]

with \( x = y^2 > 0 \). The left-hand side of (43) equals

\[
4\gamma x^2 e^x - 4x(1 - e^{-\gamma x}) e^x + \frac{1}{\gamma} (1 - e^{-\gamma x})^2 e^x ,
\]

while the right-hand side of (43) equals

\[
4\gamma x^2 e^x - 2x(1 - e^{-\gamma x}) e^x - (e^x - 1)(1 - e^{-\gamma x}) \\
- 2\gamma x(e^x - 1) + 2\gamma x(e^x - 1) e^{-\gamma x} .
\]

Simplifying then in (43), we get that (41) is equivalent with

\[
-2x(1 - e^{-\gamma x}) e^x + \frac{1}{\gamma} (1 - e^{-\gamma x})^2 e^x \\
< -2\gamma x(e^x - 1)(1 - e^{-\gamma x}) - (e^x - 1)(1 - e^{-\gamma x}) , \quad x = y^2 > 0 .
\]

Dividing through in (46) by \( 1 - e^{-\gamma x} \), we have that (41) is equivalent with

\[
-2x e^x + \frac{1}{\gamma} (1 - e^{-\gamma x}) e^x < -2\gamma x(e^x - 1) - (e^x - 1) ,
\]

i.e., with

\[
-2(1 - \gamma) x e^x + \left( \frac{1}{\gamma} + 1 \right) e^x - \frac{1}{\gamma} e^{(1-\gamma)x} < 1 + 2\gamma x
\]

with \( x = y^2 > 0 \). We have equality in (48) for \( x = 0 \), and so, taking derivatives in (48), it is sufficient to show that

\[
-2(1 - \gamma) e^x - 2(1 - \gamma) x e^x + \left( \frac{1}{\gamma} + 1 \right) e^x - \frac{1 - \gamma}{\gamma} e^{(1-\gamma)x} < 2\gamma , \quad x > 0 ,
\]

i.e., that

\[
\left( \frac{1}{\gamma} + 2\gamma - 1 \right) e^x - 2(1 - \gamma) x e^x - \frac{1 - \gamma}{\gamma} e^{(1-\gamma)x} < 2\gamma , \quad x > 0 .
\]
We have equality in (50) for \( x = 0 \), and so, taking derivatives in (50), it is sufficient to show that
\[
\left( \frac{1}{\gamma} + 2\gamma - 1 \right) e^x - 2(1 - \gamma) e^x - 2(1 - \gamma) x e^x - \frac{(1 - \gamma)^2}{\gamma} e^{(1-\gamma)x} < 0 \ ,
\]
(51)
i.e., that
\[
\left( \frac{1}{\gamma} + 4\gamma - 3 \right) e^x - 2(1 - \gamma) x e^x - \frac{(1 - \gamma)^2}{\gamma} e^{(1-\gamma)x} < 0 \ , \quad x > 0 \ .
\]
(52)
The left-hand side of (52) equals \( 3\gamma - 1 \) at \( x = 0 \) and thus vanishes at \( x = 0 \) since \( \gamma = 1/3 \), For \( \gamma = 1/3 \), the left-hand side of (52) becomes
\[
\frac{4}{3} \left( e^x - x e^x - e^{\frac{2}{3}x} \right) ,
\]
(53)
and this is negative for \( x > 0 \) since \( e^x(1 - x) < 1 < e^{\frac{2}{3}x} \), \( x > 0 \). This completes the proof.

We have the Taylor developments, relevant for small \( y > 0 \),
\[
I(y) = y(1 + \frac{1}{3} y^2 + \frac{1}{10} y^4 + \frac{1}{45} y^6 + \ldots) \ ,
\]
(54)
\[
B(y; 1) = y(1 + 0 \cdot y^2 + \frac{1}{3} y^4 + \frac{1}{12} y^6 + \ldots) \ ,
\]
(55)
\[
B(y; 1/3) = y(1 + \frac{1}{3} y^2 + \frac{7}{54} y^4 + \frac{2}{51} y^6 + \ldots) \ ,
\]
(56)
and the asymptotic expansions, relevant for \( y \to \infty \),
\[
I(y) \sim \frac{e^{y^2}}{2y} \left( 1 + \frac{1}{2 y^2} + \frac{3}{4 y^4} + \frac{15}{8 y^6} + \ldots \right) ,
\]
(57)
\[
B(y; 1) \sim \frac{e^{y^2}}{2y} \left( 1 + \frac{1}{2 y^2} + \frac{1}{4 y^4} + \frac{1}{8 y^6} + \ldots \right) ,
\]
(58)
\[
B(y; 1/3) \sim \frac{e^{y^2}}{2y} \left( 1 + \frac{3}{2 y^2} + \frac{9}{4 y^4} + \frac{27}{8 y^6} + \ldots \right) .
\]
(59)
Thus, in particular
\[
\frac{B(y; 1/3)}{I(y)} = 1 + \frac{8}{270} y^4 + O(y^6) , \quad y \downarrow 0 \ ,
\]
(60)
and
\[
\frac{B(y; 1)}{I(y)} = 1 - \frac{1}{2 y^2} + O\left( \frac{1}{y^6} \right) , \quad y \to \infty \ .
\]
(61)
Compare Proposition 1 (e).

By a numerical effort, it has been found that

$$\min_{y \geq 0} \frac{B(y; 1)}{I(y)} = \frac{B(Y_1; 1)}{I(Y_1)} = 0.852634652... \quad (62)$$

$$\max_{y \geq 0} \frac{B(y; 1/3)}{I(y)} = \frac{B(Y_{1/3}; 1/3)}{I(Y_{1/3})} = 1.135207141... \quad (63)$$

with $Y_1 = 1.18039274...$ and $Y_{1/3} = 2.324381951...$

In Figure 1, we present plots of

$$R(y; \gamma) = \frac{B(y; \gamma)}{I(y)} , \quad y \in [0, 9] \quad (64)$$

with $\gamma = 1/3, 0.54$ and 1, illustrating that $B(y; 1/3)$ is an upper bound for $I(y), y \geq 0$, and that $B(y; 1)$ is a lower bound for $I(y), y \geq 0$, with maximal absolute relative errors of less than 15%. The choice $\gamma = 0.54$ has

$$\max_{y \geq 0} (R(y; \gamma) - 1) \approx \max_{y \geq 0} (1 - R(y; \gamma)) \approx 0.044 \quad (65)$$

4 Bounding and approximating $f_0(t)$

We recall that

$$f_0(t) = \exp(U^2(t)) , \quad y = U(t) = (\ln(f_0(t)))^{1/2} , \quad t \geq 0 \quad (66)$$

where $U(t)$ is defined implicitly by

$$\int_0^{U(t)} e^{v^2} dv = \frac{t}{\sqrt{2}} , \quad t \geq 0 \quad (67)$$

Suppose we have a bound (or approximation) $B(y)$ for

$$I(y) = \int_0^y e^{v^2} dv , \quad y \geq 0 \quad (68)$$

Then we can find a bound (or approximation)

$$z_B(t) = \exp(y_B^2(t)) , \quad y_B(t) = (\ln(z_B(t)))^{1/2} \quad (69)$$
of \( f_0(t) \) by letting \( y = y_B(t) \) solve the equation

\[
B(y) = \frac{t}{\sqrt{2}} .
\]

(70)

With reference to (19) and to Theorem 1 in Section 3, we shall consider the lower bounds

\[
\frac{e^{y^2} - 1}{2y} = B(y; \infty) \quad \text{and} \quad \frac{e^{y^2} - 1}{2y} \left(1 - \frac{1 - e^{-y^2}}{2y^2}\right)^{-1} = B(y; 1),
\]

(71)

the upper bound

\[
\frac{e^{y^2} - 1}{2y} \left(1 - \frac{1 - e^{-y^2/3}}{2y^2/3}\right)^{-1} = B(y; 1/3),
\]

(72)

and the approximation

\[
\frac{e^{y^2} - 1}{2y} \left(1 - \frac{1 - e^{-\gamma y^2}}{2\gamma y^2}\right)^{-1} = B(y; \gamma), \quad \gamma = 0.54 ,
\]

(73)

as bound or approximation of \( I(y) \).

Evidently, when \( B(y) \) is a lower (upper) bound for \( I(y) \), \( y \geq 0 \), we have from \( I(U(t)) = t/\sqrt{2} = B(y_B(t)) \) that \( y_B(t) \geq (\leq) U(t) \).

First consider the lower bound \( B(y; \infty) \) in (71). Then solving \( y = y_\infty(t) \) from the equation

\[
\frac{e^{y^2} - 1}{2y} = \frac{t}{\sqrt{2}},
\]

(74)

yields for \( z = z_\infty(t) = \exp(y_\infty^2(t)) \) the fixed-point equation

\[
z = 1 + t(2 \ln z)^{1/2} =: F_\infty(z; t).
\]

(75)

With \( t > 1 \) being fixed, it is elementary to show that the mapping \( z \geq 1 \mapsto F_\infty(z; t) \) has two fixed-points \( z \in [1, \infty) \), viz. \( z = 1 \) and \( z = z_\infty(t) > 1 \). To compute the latter fixed-point, we iterate by successive substitution according to

\[
z^{(0)} = t(2 \ln t)^{1/2} ; \quad z^{(j+1)} = 1 + t(2 \ln z^{(j)})^{1/2}, \quad j = 0, 1, \ldots ,
\]

(76)

where the initial value \( z^{(0)} \) is suggested by the asymptotic result (9) for \( f_0(t) \).

Next consider the general function \( B(y; \gamma) \) from Section 3 with \( 1/3 \leq \gamma \leq 1 \). Now solving \( y = y_\gamma(t) \) from the equation

\[
B(y; \gamma) = \frac{e^{y^2} - 1}{2y} \left(1 - \frac{1 - e^{-\gamma y^2}}{2\gamma y^2}\right)^{-1} = \frac{t}{\sqrt{2}},
\]

(77)
yields for $z = z_\gamma(t) = \exp(y_\gamma^2(t))$ the fixed-point equation

$$z = 1 + t(2\ln z)^{1/2} - t \frac{1 - z^{-\gamma}}{\gamma(2\ln z)^{1/2}} =: F_\gamma(z ; t) . \tag{78}$$

We have for $F_\gamma(z ; t)$ with fixed $t > 1$ the following:

- $F_\gamma(1 ; t) = 1$,
- $F_\gamma(z ; t) > z$ for $z$ in a right-neighbourhood of 1,
- $\frac{d}{dz}[F_\gamma(z ; t)]$ is positive and decreasing in $z \geq 1$ when $0 < \gamma \leq 1$.

As to the latter property, one computes explicitly

$$\frac{1}{t} \frac{d}{dz}[F_\gamma(z ; t)] = z^{-1}(1 - z^{-\gamma})\left((2\ln z)^{-1/2} + \frac{1}{\gamma} (2\ln z)^{-3/2}\right) = \frac{e^{-y^2}}{y \sqrt{2}} (2\gamma y^2 + 1) \cdot \frac{1 - e^{-y^2}}{2\gamma y^2} , \tag{79}$$

where we have set $z = \exp(y^2)$ with $y \geq 0$. Now

$$\frac{d}{dy} \left[ \frac{e^{-y^2}}{y} (2\gamma y^2 + 1) \right] = \frac{e^{-y^2}}{y^2} (2\gamma y^2 - 1 - 2y^2 - 4\gamma y^4) < 0 \tag{80}$$

for all $y > 0$ when $0 < \gamma \leq 1$, and this shows that the first factor on the second line of (79) decreases in $y > 0$ when $0 < \gamma \leq 1$. The second factor on the second line of (79) decreases in $y > 0$ for all $\gamma > 0$.

We conclude that the mapping $z \geq 1 \mapsto F_\gamma(z ; t)$ has two fixed-points $z \in [1, \infty)$, viz. $z = 1$ and $z = z_\gamma(t) > 1$. The latter fixed-point can again be computed iteratively by successive substitution using (78) with $z^{(0)} = t(\ln t)^{1/2}$ as initial value.

We illustrate all this for the case that $t = 10$. We have

$$f_0(10) = 19.25011998 , \quad U(10) = 1.719743380 ; \quad z^{(0)} = 21.45966026 . \tag{81}$$

We find then

1. Lower bound $B(y ; \infty)$ yields, using 12 iterations in (76) with initial value $z^{(0)}$,

$$z_\infty(10) = 26.61881448 , \quad y_\infty(10) = (\ln z_\infty(10))^{1/2} = 1.811523745 . \tag{82}$$
2. Lower bound $B(y; 1)$ yields, using 12 iterations based on the fixed-point equation (78) with $\gamma = 1$ and with initial value $z^{(0)}$,

$$z_1(10) = 22.03097612, \quad y_1(10) = (\ln z_1(10))^{1/2} = 1.758536172 . \quad (83)$$

3. Upper bound $B(y; 1/3)$ yields using 15 iterations based on the fixed-point equation (78) with $\gamma = 1/3$ and with initial value $z^{(0)}$,

$$z_{1/3}(10) = 17.133554664, \quad y_{1/3}(10) = (\ln z_{1/3}(10))^{1/2} = 1.685537995 . \quad (84)$$

4. Approximation $B(y; 0.54)$ yields, using 14 iterations based on the fixed-point equation (78) with $\gamma = 0.54$ and with initial value $z^{(0)}$,

$$z_{0.54}(10) = 19.24791149, \quad y_{0.54} = (\ln z_{0.54}(10))^{1/2} = 1.719710022 . \quad (85)$$

It is observed that in all cases (except, perhaps, in case 1), the $y$-values obtained are (quite) close to $U(10)$ in (81), already for the relatively small value 10 of $t$. This observation can be used to do a quality assessment of the estimates $z$ of $f_0(t)$, for values of $t = 10$ and larger. We recall that, given an approximation $B(y)$ of $I(y)$, we solve $y = y_B$ from

$$B(y_B) = \frac{t}{\sqrt{2}} = \int_0^{U(t)} e^{v^2} dv . \quad (86)$$

Now

$$B(y_B) = R_B(y_B) \int_0^{y_B} e^{v^2} dv . \quad (87)$$

where the $R_B$-function, compare Figure 1, is given by

$$R_B(y) = \frac{B(y)}{\int_0^{y} e^{v^2} dv}, \quad y \geq 0 . \quad (88)$$

In all considered cases, this $R_B$-function is a well-behaved smooth function. Next we have in terms of the Dawson function $F(y)$

$$I(y) = \int_0^{y} e^{v^2} dv = e^{y^2} F(y) , \quad y \geq 0 . \quad (89)$$

The function $F$ is, compared to $\exp(y^2)$, a mildly varying function in the sense that $F'(y)/F(y)$ is not large when $y$ is away from 0. Now, from (86, 87, 89)

$$e^{U^2(t)} F(U(t)) = R_B(y_B) e^{y_B^2} F(y_B) . \quad (90)$$
Hence, using
\[ f_0(t) = \exp(U^2(t)), \quad z_B = \exp(y_B^2), \] (91)
we get
\[ \frac{z_B}{f_0(t)} = R_B^{-1}(y_B) \frac{F(U(t))}{F(y_B)} \approx R_B^{-1}(y_B) \approx R_B^{-1}(U(t)) , \] (92)
where the latter two near-equalities hold when \( y_B \) is close to \( U(t) \) and mild variation of \( F \) and \( R_B \).

We conclude that the relative error, made by approximating \( f_0(t) \) by \( z_B \), can be read off accurately from the \( R \)-plots in Figure 1. Thus we find
1. \[ \frac{z_{\infty}(10)}{f_0(10)} = 1.38 \approx 1.33 = R^{-1}(U(t) ; \infty) , \]
2. \[ \frac{z_1(10)}{f_0(10)} =1.14 \approx 1.12 = R^{-1}(U(t) ; 1) , \]
3. \[ \frac{z_{1/3}(10)}{f_0(10)} = 0.89 \approx 0.91 = R^{-1}(U(t) ; 1/3) , \]
4. \[ \frac{z_{0.54}(10)}{f_0(10)} = 1.00 = R^{-1}(U(t) ; 0.54) . \]
by looking at the values of the \( R \)-functions in Figure 1 at \( y = U(10) = 1.72 \) (for the first case, a separate consideration is required).

The first identity in (92) can be explored further by elaborating the factor \( F(U(t))/F(y_B) \), yielding improved estimates of \( f_0(t) \).

References


Figure 1: The ratio $R(y; \gamma) = B(y; \gamma)/I(y)$ in the range $0 \leq y \leq 9$ for $\gamma = 1/3, 0.54$ and 1.