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# Bounds on Dawson's integral occurring in the analysis of a line distribution network for electric vehicles

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**Abstract.**
The Dawson integral $F(y)$ arises in the analysis of a particular model (Distflow model) for charging electric vehicles (EVs) when one passes from the intrinsic discrete setting to an analytically more tractable continuous setting. The mathematical and computational properties of $F(y)$ have been developed in the context of the error function, with purely imaginary argument $iy$, $y \geq 0$, for which packages, such as Mathematica, exist. In this report, we focus on bounds on $F(y)$ that are sharp, both at $y = 0$ and $y = \infty$, a topic that has been hardly addressed in the existing literature. One of the bounds we show emerges naturally from the EV-application when one compares the Distflow model to a linearized version of it.

# 1 Introduction

We start this report by outlining how the Dawson integral arises in the analysis of a particular model (Distflow model) for charging electric vehicles (EVs) at $N$ (large) charging stations connected by a power line, with the requirement that the ratio between the voltages at the root and the last station at the power line should stay below a desired level. A full description of the problem at hand, comprising a comparison of the Distflow model and the linearized Distflow model, can be found in [1]. In [1], Subsections 2.3.1–2, the two models are introduced and discussed. Under the Distflow model, the normalized voltages $V_n$, $n = 0, 1, ..., N-1, N$ , with $V_N$ the voltage at the root station and $V_0$ the voltage at the last station of the power line, satisfy a recursion

$$V_0 = 1 \, , \quad V_1 = 1 + k_0 \; ; \qquad V_{n+1} - 2V_n + V_{n-1} = \frac{k_n}{V_n} \, , \quad n = 1, ..., N-1 \; , \quad (1)$$

see [1], (2.16–2.18). The $k_n$, comprising given charging rates $p_n$ and resistance/reactance values $r$ and $x$ as well as the arrival rate $\lambda$ at the stations, are normally small (of the order $a/N^2$ with $0 < a < 0.1$). The ratio $V_N/V_0 = V_N$ between the voltages at the root node ($N$) and the last node (0) should be below a level $1/(1-\Delta)$, where the tolerance $\Delta$ is small (of the order 0.1). Linearization of the Distflow model, as is done in [1], Subsection 2.3.2, yields the linearized Distflow model.

For analytically comparing the two models, it is assumed that all $k_n$ are equal to $a/N^2$, with $a \in (0, 0.1)$ independent of $n$ (for numerically comparing the two models, such an assumption does not need to be made). In [1], Section 5.3 and Appendix B, a major effort is made to establish a relationship between the sequence $V_n$, $n = 0, 1, ..., N$ , and the solution $f_0(t)$, $t \geq 0$, of the second-order boundary value problem

$$f_0''(t) = \frac{1}{f_0(t)} \, , \quad t \geq 0 \; ; \qquad f_0(0) = 1 \, , \quad f_0'(0) = 0 \; . \quad (2)$$

In particular, it is shown see [1], Section 5.4, that $V_N \to f_0(\sqrt{a})$ as $N \to \infty$ and $a$ is fixed.

The Dawson integral $F(y) = e^{-y^2} \, I(y)$, with

$$I(y) = \int_0^y e^{v^2} \, dv \; , \qquad y \geq 0 \; , \quad (3)$$

then arises as follows, see [1], Appendix C. We have

$$f_0(t) = \exp(U^2(t)) \; , \qquad t \geq 0 \; , \quad (4)$$

where $U(t)$ is defined implicitly by

$$\int_0^{U(t)} e^{v^2}\,dv = t/\sqrt{2}\ , \qquad t \geq 0\ . \tag{5}$$

That is, in the notation preferred in [1], Section 5.4,

$$U(t) = \text{inverfi}\Big(t\sqrt{\frac{2}{\pi}}\Big)\ , \qquad t \geq 0\ , \tag{6}$$

where inverfi is the inverse of the function

$$i\,\text{erf}(y/i) = \frac{2}{\sqrt{\pi}} \int_0^y e^{v^2}\,dv \quad \text{with} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds\ . \tag{7}$$

In the research effort, from which [1] and [2] resulted, properties and results about the sequence $V_n$, $n = 0, 1, \ldots$ , have been conjectured and proved by first establishing these for the analytically more tractable function $f_0(z)$. For instance, we have from the asymptotics

$$I(y) \sim \frac{e^{y^2}}{2y}\Big(1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \ldots\Big)\ , \qquad y \to \infty\ , \tag{8}$$

(asymptotic series) that

$$f_0(t) = t(2\ln t)^{1/2}\Big(1 + O\Big(\frac{\ln(\ln t)}{\ln t}\Big)\Big)\ , \qquad t \to \infty\ . \tag{9}$$

This leads to the conjecture that for fixed $k > 0$, and with $n$ in (1) allowed to tend to $\infty$,

$$V_n \sim n(2k\ln n)^{1/2}\ , \qquad n \to \infty\ . \tag{10}$$

The latter result will be proved in all detail in [2].

# 2 Basic properties and bounds of Dawson's integral

We shall now give basic properties and bounds for Dawson's integral $F(y) = e^{-y^2}\, I(y)$ and the intgral $I(y)$ in (3), as far as relevant for our purposes. We refer to [3], Ch. 7 for an extensive account, with references, of the properties of the error function and Dawson's integral. We have

A. $I(y) = \sum_{l=0}^{\infty} \frac{y^{2l+1}}{l!(2l+1)} = y(1 + \frac{1}{3}\, y^2 + \frac{1}{10}\, y^4 + \frac{1}{42}\, y^6 + ...)\,, \quad y \geq 0\,, \qquad (11)$

B. $$I(y) \sim \frac{e^{y^2}}{2y} \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot ... \cdot (2k-1)}{(2y^2)^k} = \frac{e^{y^2}}{2y} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(1/2)}\, y^{-2k}$$

$$= \frac{e^{y^2}}{2y} \left(1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \frac{15}{8y^6} + ...\right)\,, \qquad y \to \infty\,. \tag{12}$$

The series in (12) is an asymptotic series; see [4], (2.8) and further, for a Cauchy principal value integral for the remainder when finitely many terms of the series are included. When for a particular $y > 0$ the series is truncated at the integer $k$ nearest to $y^2 - 1/2$, the truncation error is of the order

$$\frac{\Gamma(k+3/2)}{\Gamma(1/2)}\, y^{-2k-2} \approx e^{-y^2} \sqrt{2} \tag{13}$$

(for $y = 8$ this is $\approx\ 2 \times 10^{-28}$ with $k = 64$).

C. In [3], §7.8 some bounds on $I(y)$ are given, viz.

$$I(y) < \frac{1}{3y}\,(2e^{y^2} + y^2 - 2)\,, \qquad y > 0\,, \tag{14}$$

$$I(y) < \frac{e^{y^2} - 1}{y}\,, \qquad y > 0\,, \tag{15}$$

The right-hand side of (14) has Taylor series

$$\frac{1}{3y}\,(2e^{y^2} + y^2 - 2) = y(1 + \tfrac{1}{3}\, y^2 + \tfrac{1}{9}\, y^4 + \tfrac{1}{36}\, y^6 + ...)\,, \qquad y \geq 0\,, \tag{16}$$

and the bound in (14) is therefore sharp, see (11), at $y = 0$; the bound in (14) is not sharp as $y \to \infty$, see (12). The right-hand side of (15) has Taylor series

$$\frac{e^{y^2} - 1}{y} = y(1 + \tfrac{1}{2}\, y^2 + \tfrac{1}{6}\, y^4 + ...)\,, \qquad y \geq 0\,, \tag{17}$$

and the bound in (15) is therefore sharp at $y = 0$ (though not as sharp as (16)). In [5], 2° on p. 180, there is given the bound

$$I(y) < \frac{\pi^2}{8y}\,(e^{y^2} - 1)\,, \qquad y > 0\,. \tag{18}$$

Since $\frac{\pi^2}{8} = 1.2337...$ , the bound 18) is not sharp at $y = 0$ nor at $y = \infty$. I haven't seen lower bounds on $I(y)$.

D. We now present some simple lower and upper bounds. We have

$$\frac{e^{y^2} - 1}{2y} \le I(y) \le \frac{e^{y^2} - 1}{y} \ , \qquad y \ge 0 \ . \tag{19}$$

Indeed,

$$\frac{e^{y^2} - 1}{y} = \sum_{l=0}^{\infty} \frac{y^{2l+1}}{(l+1)!} \ , \tag{20}$$

and

$$\frac{1}{2 \cdot (l+1)!} < \frac{1}{l!(2l+1)} \le \frac{1}{(l+1)!} \ , \qquad l = 0, 1, ... \ , \tag{21}$$

so (19) follows from (11). The lower bound in (19) is sharp at $y = \infty$ and not sharp at $y = 0$; the upper bound in (19) is sharp at $y = 0$ and not sharp at $y = \infty$. The lower bound in (19) can be sharpened to

$$I(y) > \frac{\sinh(y^2)}{y} = \frac{e^{y^2} - e^{-y^2}}{2y} \ , \qquad y > 0 \ . \tag{22}$$

Indeed, we have for $y > 0$

$$\begin{aligned} e^{-y^2}\, I(y) &= e^{-y^2} \int\limits_0^y e^{v^2}\, dv = \int\limits_0^y e^{(v+y)(v-y)}\, dv \\ &> \int\limits_0^y e^{2y(v-y)}\, dv = \frac{1}{2y}\,(1 - e^{-2y^2}) \ . \end{aligned} \tag{23}$$

The bound in (22) is sharp at both $y = 0$ and $y = \infty$. As a result of a numerical computation, it is found that the minimum of $\sinh(y^2)/(y\, I(y))$ over $y \ge 0$ equals 0.766769724... and is assumed at $y = 1.386079411...$ .

The lower bound in (22) arose in an early version of [1] (Nov. 2, 2021) where the Distflow and linearized Distflow models were compared. For this, it was required that

$$H(y) := \frac{1}{\sqrt{e^{2y^2} - 1}} \int\limits_0^y e^{v^2}\, dv = \frac{I(y)}{\sqrt{e^{2y^2} - 1}} \tag{24}$$

is strictly decreasing in $y > 0$. One computes

$$H'(y) = \frac{1}{\sqrt{e^{2y^2}-1}} \Big( -\frac{2y\, e^{2y^2}}{e^{2y^2}-1}\, I(y) + e^{y^2} \Big) ~, \tag{25}$$

and this negative for all $y > 0$ if and only if (22) holds.

# 3 More advanced bounds and approximations

We consider now the functions, with $\gamma \geq 0$,

$$B(y\,;\,\gamma) = \frac{e^{y^2}-1}{2y} ~/~ \Big( 1 - \frac{1-e^{-\gamma y^2}}{2\gamma y^2} \Big) ~, \qquad y \geq 0 ~, \tag{26}$$

as potential lower and upper bounds for $I(y)$ for all $y \geq 0$. The function $B(y\,;\,\gamma)$, case $\gamma = 1$, arises naturally when one wants to show that, see (19),

$$I(y) ~/~ \Big( \frac{e^{y^2}-1}{y} \Big) ~, \qquad y \geq 0 ~, \tag{27}$$

(strictly) decreases from 1 at $y = 0$ to 1/2 at $y = \infty$.

**Lemma 1** *We have for $y > 0$*

$$\Big( I(y) ~/~ \Big( \frac{e^{y^2}-1}{y} \Big) \Big)' < 0 \Leftrightarrow I(y) > B(y\,;\,1) = \frac{e^{y^2}-1}{2y} ~/~ \Big( 1 - \frac{1-e^{-y^2}}{2y^2} \Big) ~. \tag{28}$$

**Proof.** We have for $y > 0$

$$\Big( I(y) ~/~ \Big( \frac{e^{y^2}-1}{y} \Big) \Big)' = \frac{e^{y^2}-1-2y^2\, e^{y^2}}{(e^{y^2}-1)^2}\, I(y) + \frac{y\, e^{y^2}}{e^{y^2}-1} ~, \tag{29}$$

and this is negative if and only if

$$I(y) > \frac{y(e^{y^2}-1)\, e^{y^2}}{2y^2\, e^{y^2} - (e^{y^2}-1)} ~. \tag{30}$$

The right-hand side of (30) equals $B(y\,;\,1)$ and this completes the proof.

We next list a number of properties of the $B(y\,;\,\gamma)$.

**Proposition 1** *(a) We have, see (19),*

$$B(y\,;0)=\frac{e^{y^2}-1}{y}~,\quad y\geq 0~;\qquad \lim_{y\to\infty} B(y\,;\gamma)=\frac{e^{y^2}-1}{2y}~,\quad y>0~. \tag{31}$$

*(b) For $y\geq 0$ fixed, we have that $B(y\,;\gamma)$ is a decreasing function of $\gamma\geq 0$.*

*(c) We have for fixed $\gamma>0$ that*

$$B(y\,;\gamma)~/~\Big(\frac{e^{y^2}-1}{y}\Big) \tag{32}$$

*decreases in $y\geq 0$ from 1 at $y=0$ to $1/2$ at $y=\infty$.*

*(d) We have for $\gamma\geq 0$*

$$\begin{aligned} B(y\,;\gamma) &= y(1+\tfrac{1}{2}\,(1-\gamma)\,y^2+(\tfrac{5}{12}\,\gamma^2-\tfrac{1}{4}\,\gamma+\tfrac{1}{6})\,y^4 \\ &\quad +(-\tfrac{1}{3}\,\gamma^3+\tfrac{5}{24}\,\gamma^2-\tfrac{1}{12}\,\gamma+\tfrac{1}{24})\,\gamma^6+... \\ &= I(y)(1+O(y^2))~,\qquad y\downarrow 0~. \end{aligned} \tag{33}$$

*(e) We have (asymptotic equivalence)*

$$B(y\,;0)\sim\frac{e^{y^2}}{y}\Big(1+\frac{0}{y^2}+\frac{0}{y^4}+\frac{0}{y^6}+...\Big)~,\qquad y\to\infty~, \tag{34}$$

*and, for fixed $\gamma>0$,*

$$\begin{aligned} B(y\,;\gamma)\sim\frac{e^{y^2}}{2y}\,\frac{1}{1-(2\gamma y^2)^{-1}} &= \frac{e^{y^2}}{2y}\Big(1+\frac{1}{2\gamma y^2}+\frac{1}{4\gamma^2y^4}+\frac{1}{8\gamma^3y^6}+...\Big) \\ &= I(y)\Big(1+O\Big(\frac{1}{y^2}\Big)\Big)~,\qquad y\to\infty~. \end{aligned} \tag{35}$$

The proofs of these results are straightforward. It is used that the function $x\geq 0\mapsto(1-e^{-x})/x$ decreases from 1 at $x=0$ to 0 at $x=\infty$. Furthermore, it is used that $\exp(-y^2)$ and $\exp(-\gamma y^2)$, $\gamma>0$, are exponentially small as $y\to\infty$ and thus are asymptotically equivalent with $0/y^2+0/y^4+...$ .

**Theorem 1** *(a) $B(y\,;\gamma)\leq I(y)$ for all $y\geq 0\Leftrightarrow\gamma\geq 1$.*
*(b) $B(y\,;\gamma)\geq I(y)$ for all $y\geq 0\Leftrightarrow 0\leq\gamma\leq 1/3$.*

**Proof.** (a) It follows from (12) and (35) that $B(y\,;\gamma)\leq I(y)$ for all $y\geq 0$ implies that $1/2\gamma\leq 1/2$, i.e., $\gamma\geq 1$. Furthermore, from Proposition 1(b) we have that $B(y\,;\,\gamma) \leq B(y\,;\,1)$ for all $y \geq 0$ and all $\gamma \geq 1$. Hence, it is sufficient to show that $B(y\,;\,1) \leq I(y)$, $y \geq 0$. We have $I(0) = 0 = B(0\,;\,1)$, and so it is sufficient to show that for $y > 0$

$$I'(y) = e^{y^2} > \Big(\frac{y(e^{y^2}-1)}{2y^2-(1-e^{-y^2})}\Big)' = B'(y\,;\,1)\ . \tag{36}$$

We compute for $y > 0$

$$\begin{aligned}&\Big(\frac{y(e^{y^2}-1)}{2y^2-(1-e^{y^2})}\Big)' \\ &= \frac{-4x(e^x-1)+4x^2e^x-(e^x-1)(1-e^{-x})+2x(1-e^{-x})}{(2x-(1-e^{-x}))^2}\ ,\end{aligned} \tag{37}$$

where we have set $x = y^2 > 0$. Thus (36) is equivalent with

$$-4x(e^x-1)+4x^2e^x-(e^x-1)(1-e^{-x})+2x(1-e^{-x}) < e^x(2x-(1-e^{-x}))^2\ , \tag{38}$$

with $x = y^2 > 0$. The right-hand side of (38) equals

$$4x^2e^x - 4x(e^x-1) + e^x(1-e^{-x})^2\ , \tag{39}$$

and so, cancelling the $4x^2e^x - 4x(e^x-1)$ from both sides of (38) and dividing through by a factor $1-e^{-x}$, we have that (36) is equivalent with

$$-(e^x-1)+2x < e^x(1-e^{-x})\ , \qquad x = y^2 > 0\ , \tag{40}$$

i.e., with $(e^x-1)/x > 1$ for $x > 0$. This is obviously true.

(b) It follows from (11) and (33) that $B(y\,;\,\gamma) \geq I(y)$ for all $y \geq 0$ implies that $\frac{1}{2}(1-\gamma) \geq 1/3$, i.e., $\gamma \leq 1/3$. Furthermore, from Proposition 1 (b), we have that $B(y\,;\,\gamma) \geq B(y\,;\,1/3)$ for all $y \geq 0$ and all $\gamma$, $0 \leq \gamma \leq 1/3$. Hence, it is sufficient to show that $B(y\,;\,1/3) \geq I(y)$, $y \geq 0$. We have $I(0) = 0 = B(0\,;\,1/3)$, and so it is sufficient to show that for all $y > 0$

$$I'(y) = e^{y^2} < \Big(\frac{\gamma y(e^{y^2}-1)}{2\gamma y^2-(1-e^{-\gamma y^2})}\Big)' = B'(y\,;\,\gamma)\ , \qquad \gamma = 1/3\ . \tag{41}$$

We compute for $y > 0$

$$\begin{aligned}\Big(\frac{\gamma y(e^{y^2}-1)}{2\gamma y^2-(1-e^{-\gamma y^2})}\Big)' &= \gamma\ \big\{-2\gamma x(e^x-1)+4\gamma x^2e^x-(e^x-1)(1-e^{-\gamma x}) \\ &\qquad -2xe^x(1-e^{-\gamma x})+2\gamma x(e^x-1)\,e^{-\gamma x}\big\}\ / \\ &\qquad \big\{(2\gamma x-(1-e^{-\gamma x}))^2\big\}\ ,\end{aligned} \tag{42}$$

where we have set $x = y^2 > 0$. Thus (41) is equivalent with

$$\begin{aligned}\frac{1}{\gamma}\,(2\gamma x-(1-e^{-\gamma x}))^2\,e^x \quad &< \quad -2\gamma x(e^x-1)+4\gamma x^2e^x-(e^x-1)(1-e^{-\gamma x}) \\ &\qquad -2xe^x(1-e^{-\gamma x})+2\gamma x(e^x-1)\,e^{-\gamma x}\;, \qquad (43)\end{aligned}$$

with $x = y^2 > 0$. The left-hand side of (43) equals

$$4\gamma x^2e^x-4x(1-e^{-\gamma x})\,e^x+\frac{1}{\gamma}\,(1-e^{-\gamma x})^2\,e^x\;, \qquad (44)$$

while the right-hand side of (43) equals

$$\begin{aligned}4\gamma x^2e^x \quad &- \quad 2x(1-e^{-\gamma x})\,e^x-(e^x-1)(1-e^{-\gamma x}) \\ &- \quad 2\gamma x(e^x-1)+2\gamma x(e^x-1)\,e^{-\gamma x}\;. \qquad (45)\end{aligned}$$

Simplifying then in (43), we get that (41) is equivalent with

$$\begin{aligned}&-2x(1-e^{-\gamma x})\,e^x+\frac{1}{\gamma}\,(1-e^{-\gamma x})^2\,e^x \\ &< \; -2\gamma x(e^x-1)(1-e^{-\gamma x})-(e^x-1)(1-e^{-\gamma x})\,, \quad x=y^2>0\,. \,(46)\end{aligned}$$

Dividing through in (46) by $1 - e^{-\gamma x}$, we have that (41) is equivalent with

$$-2x\,e^x+\frac{1}{\gamma}\,(1-e^{-\gamma x})\,e^x<-2\gamma x(e^x-1)-(e^x-1)\;, \qquad (47)$$

i.e., with

$$-2(1-\gamma)\,x\,e^x+\left(\frac{1}{\gamma}+1\right)e^x-\frac{1}{\gamma}\,e^{(1-\gamma)x}<1+2\gamma x \qquad (48)$$

with $x = y^2 > 0$. We have equality in (48) for $x = 0$, and so, taking derivatives in (48), it is sufficient to show that

$$-2(1-\gamma)\,e^x-2(1-\gamma)\,x\,e^x+\left(\frac{1}{\gamma}+1\right)e^x-\frac{1-\gamma}{\gamma}\,e^{(1-\gamma)x}<2\gamma\,, \qquad x>0\,, \;(49)$$

i.e., that

$$\left(\frac{1}{\gamma}+2\gamma-1\right)e^x-2(1-\gamma)\,x\,e^x-\frac{1-\gamma}{\gamma}\,e^{(1-\gamma)x}<2\gamma\;, \qquad x>0\;. \qquad (50)$$

We have equality in (50) for $x = 0$, and so, taking derivatives in (50), it is sufficient to show that

$$\left(\frac{1}{\gamma} + 2\gamma - 1\right) e^x - 2(1-\gamma)\, e^x - 2(1-\gamma)\, x\, e^x - \frac{(1-\gamma)^2}{\gamma}\, e^{(1-\gamma)x} < 0 \ , \quad (51)$$

i.e., that

$$\left(\frac{1}{\gamma} + 4\gamma - 3\right) e^x - 2(1-\gamma)\, x\, e^x - \frac{(1-\gamma)^2}{\gamma}\, e^{(1-\gamma)x} < 0 \ , \qquad x > 0 \ . \quad (52)$$

The left-hand side of (52) equals $3\gamma - 1$ at $x = 0$ and thus vanishes at $x = 0$ since $\gamma = 1/3$, For $\gamma = 1/3$, the left-hand side of (52) becomes

$$\tfrac{4}{3}\,(e^x - x\, e^x - e^{\frac{2}{3}x}) \ , \quad (53)$$

and this is negative for $x > 0$ since $e^x(1-x) < 1 < e^{\frac{2}{3}x}$, $x > 0$. This completes the proof.

We have the Taylor developments, relevant for small $y > 0$,

$$I(y) = y(1 + \tfrac{1}{3}\, y^2 + \tfrac{1}{10}\, y^4 + \tfrac{1}{42}\, y^6 + ...) \ , \quad (54)$$

$$B(y\,;\,1) = y(1 + 0 \cdot y^2 + \tfrac{1}{3}\, y^4 - \tfrac{1}{12}\, y^6 + ...) \ , \quad (55)$$

$$B(y\,;\,1/3) = y(1 + \tfrac{1}{3}\, y^2 + \tfrac{7}{54}\, y^4 + \tfrac{2}{81}\, y^6 + ...) \ , \quad (56)$$

and the asymptotic expansions, relevant for $y \to \infty$,

$$I(y) \sim \frac{e^{y^2}}{2y}\left(1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \frac{15}{8y^6} + ...\right) \ , \quad (57)$$

$$B(y\,;\,1) \sim \frac{e^{y^2}}{2y}\left(1 + \frac{1}{2y^2} + \frac{1}{4y^4} + \frac{1}{8y^6} + ...\right) \ , \quad (58)$$

$$B(y\,;\,1/3) \sim \frac{e^{y^2}}{2y}\left(1 + \frac{3}{2y^2} + \frac{9}{4y^4} + \frac{27}{8y^6} + ...\right) \ . \quad (59)$$

Thus, in particular

$$\frac{B(y\,;\,1/3)}{I(y)} = 1 + \tfrac{8}{270}\, y^4 + O(y^6) \ , \qquad y \downarrow 0 \ , \quad (60)$$

and

$$\frac{B(y\,;\,1)}{I(y)} = 1 - \frac{1}{2y^4} + O\Big(\frac{1}{y^6}\Big) \ , \qquad y \to \infty \ . \quad (61)$$

Compare Proposition 1 (e).

By a numerical effort, it has been found that

$$\min_{y\geq 0} \frac{B(y\,;\,1)}{I(y)} = \frac{B(Y_1\,;\,1)}{I(Y_1)} = 0.852634652...\ , \tag{62}$$

$$\max_{y\geq 0} \frac{B(y\,;\,1/3)}{I(y)} = \frac{B(Y_{1/3}\,;\,1/3)}{I(Y_{1/3})} = 1.135207141...\ , \tag{63}$$

with $Y_1 = 1.180392274...$ and $Y_{1/3} = 2.324381951...$ .

In Figure 1, we present plots of

$$R(y\,;\,\gamma) = \frac{B(y\,;\,\gamma)}{I(y)}\ , \qquad y \in [0, 9]\ , \tag{64}$$

with $\gamma = 1/3$, 0.54 and 1, illustrating that $B(y\,;\,1/3)$ is an upper bound for $I(y)$, $y \geq 0$, and that $B(y\,;\,1)$ is a lower bound for $I(y)$, $y \geq 0$, with maximal absolute relative errors of less than 15%. The choice $\gamma = 0.54$ has

$$\max_{y\geq 0}\,(R(y\,;\,\gamma) - 1) \approx \max_{y\geq 0}\,(1 - R(y\,;\,\gamma)) \approx 0.044\ . \tag{65}$$

# 4 Bounding and approximating $f_0(t)$

We recall that

$$f_0(t) = \exp(U^2(t))\ , \quad y = U(t) = (\ln(f_0(t)))^{1/2}\ , \qquad t \geq 0\ , \tag{66}$$

where $U(t)$ is defined implicitly by

$$\int_0^{U(t)} e^{v^2}\, dv = \frac{t}{\sqrt{2}}\ , \qquad t \geq 0\ . \tag{67}$$

Suppose we have a bound (or approximation) $B(y)$ for

$$I(y) = \int_0^{y} e^{v^2}\, dv\ , \qquad y \geq 0\ . \tag{68}$$

Then we can find a bound (or approximation)

$$z_B(t) = \exp(y_B^2(t))\ , \qquad y_B(t) = (\ln(z_B(t)))^{1/2} \tag{69}$$

of $f_0(t)$ by letting $y = y_B(t)$ solve the equation

$$B(y) = \frac{t}{\sqrt{2}} \ . \tag{70}$$

With reference to (19) and to Theorem 1 in Section 3, we shall consider the lower bounds

$$\frac{e^{y^2}-1}{2y} = B(y\,;\,\infty) \quad \text{and} \quad \frac{e^{y^2}-1}{2y}\Big(1-\frac{1-e^{-y^2}}{2y^2}\Big)^{-1} = B(y\,;\,1)\ , \tag{71}$$

the upper bound

$$\frac{e^{y^2}-1}{2y}\Big(1-\frac{1-e^{-y^2/3}}{2y^2/3}\Big)^{-1} = B(y\,;\,1/3)\ , \tag{72}$$

and the approximation

$$\frac{e^{y^2}-1}{2y}\Big(1-\frac{1-e^{-\gamma y^2}}{2\gamma y^2}\Big)^{-1} = B(y\,;\,\gamma)\ , \qquad \gamma = 0.54\ , \tag{73}$$

as bound or approximation of $I(y)$.

Evidently, when $B(y)$ is a lower (upper) bound for $I(y)$, $y \geq 0$, we have from $I(U(t)) = t/\sqrt{2} = B(y_B(t))$ that $y_B(t) \geq\ (\leq)\ U(t)$.

First consider the lower bound $B(y\,;\,\infty)$ in (71). Then solving $y = y_\infty(t)$ from the equation

$$\frac{e^{y^2}-1}{2y} = \frac{t}{\sqrt{2}}\ , \tag{74}$$

yields for $z = z_\infty(t) = \exp(y_\infty^2(t))$ the fixed-point equation

$$z = 1 + t(2\ln z)^{1/2} =: F_\infty(z\,;\,t)\ . \tag{75}$$

With $t > 1$ being fixed, it is elementary to show that the mapping $z \geq 1 \mapsto F_\infty(z\,;\,t)$ has two fixed-points $z \in [1,\infty)$, viz. $z = 1$ and $z = z_\infty(t) > 1$. To compute the latter fixed-point, we iterate by successive substitution according to

$$z^{(0)} = t(2\ln t)^{1/2}\ ; \qquad z^{(j+1)} = 1 + t(2\ln z^{(j)})^{1/2}\ , \quad j = 0, 1, \dots\ , \tag{76}$$

where the initial value $z^{(0)}$ is suggested by the asymptotic result (9) for $f_0(t)$.

Next consider the general function $B(y\,;\,\gamma)$ from Section 3 with $1/3 \leq \gamma \leq 1$. Now solving $y = y_\gamma(t)$ from the equation

$$B(y\,;\,\gamma) = \frac{e^{y^2}-1}{2y}\Big(1-\frac{1-e^{-\gamma y^2}}{2\gamma y^2}\Big)^{-1} = \frac{t}{\sqrt{2}}\ , \tag{77}$$

yields for $z = z_\gamma(t) = \exp(y_\gamma^2(t))$ the fixed-point equation

$$z = 1 + t(2\ln z)^{1/2} - t\,\frac{1 - z^{-\gamma}}{\gamma(2\ln z)^{1/2}} =: F_\gamma(z\,;\,t)\ . \tag{78}$$

We have for $F_\gamma(z\,;\,t)$ with fixed $t > 1$ the following:

– $F_\gamma(1\,;\,t) = 1$,

– $F_\gamma(z\,;\,t) > z$ for $z$ in a right-neighbourhood of 1,

– $\dfrac{d}{dz}\,[F_\gamma(z\,;\,t)]$ is positive and decreasing in $z \geq 1$ when $0 < \gamma \leq 1$.

As to the latter property, one computes explicitly

$$\begin{aligned}\frac{1}{t}\,\frac{d}{dz}\,[F_\gamma(z\,;\,t)] &= z^{-1}(1 - z^{-\gamma})\Big((2\ln z)^{-1/2} + \frac{1}{\gamma}\,(2\ln z)^{-3/2}\Big) \\ &= \frac{e^{-y^2}}{y\sqrt{2}}\,(2\gamma y^2 + 1)\cdot\frac{1 - e^{-\gamma y^2}}{2\gamma y^2}\ ,\end{aligned} \tag{79}$$

where we have set $z = \exp(y^2)$ with $y \geq 0$. Now

$$\frac{d}{dy}\Big[\frac{e^{-y^2}}{y}\,(2\gamma y^2 + 1)\Big] = \frac{e^{-y^2}}{y^2}\,(2\gamma y^2 - 1 - 2y^2 - 4\gamma y^4) < 0 \tag{80}$$

for all $y > 0$ when $0 < \gamma \leq 1$, and this shows that the first factor on the second line of (79) decreases in $y > 0$ when $0 < \gamma \leq 1$. The second factor on the second line of (79) decreases in $y > 0$ for all $\gamma > 0$.

We conclude that the mapping $z \geq 1 \mapsto F_\gamma(z\,;\,t)$ has two fixed-points $z \in [1,\infty)$, viz. $z = 1$ and $z = z_\gamma(t) > 1$. The latter fixed-point can again be computed iteratively by successive substitution using (78) with $z^{(0)} = t(\ln t)^{1/2}$ as initial value.

We illustrate all this for the case that $t = 10$. We have

$$f_0(10) = 19.25011998\,,\quad U(10) = 1.719743380\,;\quad z^{(0)} = 21.45966026\ . \tag{81}$$

We find then

1. Lower bound $B(y\,;\,\infty)$ yields, using 12 iterations in (76) with initial value $z^{(0)}$,

$$z_\infty(10) = 26.61881448\,,\quad y_\infty(10) = (\ln z_\infty(10))^{1/2} = 1.811523745\ . \tag{82}$$

2. Lower bound $B(y\,;\,1)$ yields, using 12 iterations based on the fixed-point equation (78) with $\gamma = 1$ and with initial value $z^{(0)}$,

$$z_1(10) = 22.03097612\,, \quad y_1(10) = (\ln z_1(10))^{1/2} = 1.758536172\ . \tag{83}$$

3. Upper bound $B(y\,;\,1/3)$ yields using 15 iterations based on the fixed-point equation (78) with $\gamma = 1/3$ and with initial value $z^{(0)}$,

$$z_{1/3}(10) = 17.133554664\,, \quad y_{1/3}(10) = (\ln z_{1/3}(10))^{1/2} = 1.685537995\ . \tag{84}$$

4. Approximation $B(y\,;\,0.54)$ yields, using 14 iterations based on the fixed-point equation (78) with $\gamma = 0.54$ and with initial value $z^{(0)}$,

$$z_{0.54}(10) = 19.24791149\,, \quad y_{0.54} = (\ln z_{0.54}(10))^{1/2} = 1.719710022\ . \tag{85}$$

It is observed that in all cases (except, perhaps, in case 1), the $y$-values obtained are (quite) close to $U(10)$ in (81), already for the relatively small value 10 of $t$. This observation can be used to do a quality assessment of the estimates $z$ of $f_0(t)$, for values of $t = 10$ and larger. We recall that, given an approximation $B(y)$ of $I(y)$, we solve $y = y_B$ from

$$B(y_B) = \frac{t}{\sqrt{2}} = \int_0^{U(t)} e^{v^2}\,dv\ . \tag{86}$$

Now

$$B(y_B) = R_B(y_B)\int_0^{y_B} e^{v^2}\,dv\ . \tag{87}$$

where the $R_B$-function, compare Figure 1, is given by

$$R_B(y) = \frac{B(y)}{\int_0^y e^{v^2}\,dv}\,, \qquad y \ge 0\ . \tag{88}$$

In all considered cases, this $R_B$-function is a well-behaved smooth function. Next we have in terms of the Dawson function $F(y)$

$$I(y) = \int_0^y e^{v^2}\,dv = e^{y^2}\,F(y)\ , \qquad y \ge 0\ . \tag{89}$$

The function $F$ is, compared to $\exp(y^2)$, a mildly varying function in the sense that $F'(y)/F(y)$ is not large when $y$ is away from 0. Now, from (86, 87, 89)

$$e^{U^2(t)}\,F(U(t)) = R_B(y_B)\,e^{y_B^2}\,F(y_B)\ . \tag{90}$$

Hence, using

$$f_0(t) = \exp(U^2(t)) \ , \qquad z_B = \exp(y_B^2) \ , \tag{91}$$

we get

$$\frac{z_B}{f_0(t)} = R_B^{-1}(y_B)\,\frac{F(U(t))}{F(y_B)} \approx R_B^{-1}(y_B) \approx R_B^{-1}(U(t)) \ , \tag{92}$$

where the latter two near-equalities hold when $y_B$ is close to $U(t)$ and mild variation of $F$ and $R_B$.

We conclude that the relative error, made by approximating $f_0(t)$ by $z_B$, can be read off accurately from the $R$-plots in Figure 1. Thus we find

1. $\dfrac{z_\infty(10)}{f_0(10)} = 1.38 \approx 1.33 = R^{-1}(U(t)\,;\,\infty)$ ,

2. $\dfrac{z_1(10)}{f_0(10)} = 1.14 \approx 1.12 = R^{-1}(U(t)\,;\,1)$ ,

3. $\dfrac{z_{1/3}(10)}{f_0(10)} = 0.89 \approx 0.91 = R^{-1}(U(t)\,;\,1/3)$ ,

4. $\dfrac{z_{0.54}(10)}{f_0(10)} = 1.00 = R^{-1}(U(t)\,;\,0.54)$ .

by looking at the values of the $R$-functions in Figure 1 at $y = U(10) = 1.72$ (for the first case, a separate consideration is required).

The first identity in (92) can be explored further by elaborating the factor $F(U(t))/F(y_B)$, yielding improved estimates of $f_0(t)$.

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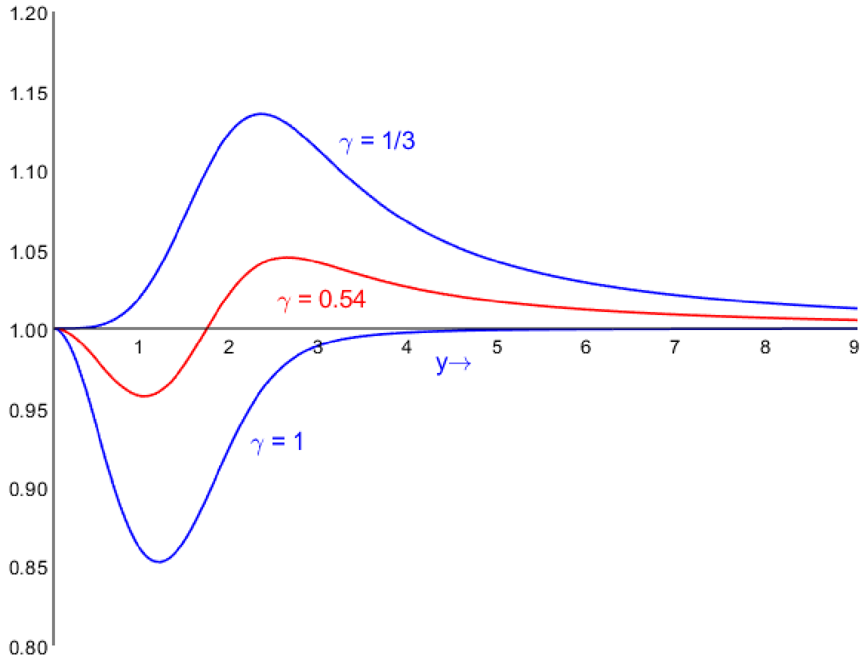


Figure 1: The ratio $R(y\,;\gamma) = B(y\,;\gamma)/I(y)$ in the range $0 \leq y \leq 9$ for $\gamma = 1/3$, 0.54 and 1.