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Switching interacting particle systems: scaling limits, uphill diffusion and boundary layer

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Abstract

In this paper we consider three classes of interacting particle systems on $\mathbb{Z}$: independent random walks, the exclusion process, and the inclusion process. We allow particles to switch their jump rate between 1 and $\epsilon \in [0,1]$. The switch between the two jump rates happens at rate $\gamma \in (0,\infty)$. In the exclusion process, the interaction is such that each site can be occupied by at most one particle of each type. In the inclusion process, the interaction takes places between particles of the same type at different sites and between particles of different type at the same site.

We derive the macroscopic limit equations for the three systems, obtained after scaling space by N^{-1}, time by N^2, the switching rate by N^{-2}, and letting $N \to \infty$. The limit equations for the macroscopic densities associated to the fast and slow particles is the well-studied double diffusivity model. This system of reaction-diffusion equations was introduced to model polycrystal diffusion and dislocation pipe diffusion, with the goal to overcome the limitations imposed by Fick's law. In order to investigate the microscopic out-of-equilibrium properties, we analyse the system on two copies of $[N] = \{1,\dots,N\}$, adding boundary reservoirs at sites 1 and N in both layers. Inside $[N]$ particles move as before, but now particles are injected at site 1 and absorbed at site N at prescribed rates that depend on the layer. We compute the steady-state distribution and the steady-state current. It turns out that uphill diffusion is possible, i.e., the total flow can be opposite to the gradient imposed by the total injection rate and the total absorption rate. This phenomenon, which cannot occur in a single-layer system, is a violation of Fick's law made possible by the switching between the layers. We rescale the microscopic steady-state distribution and steady-state current and obtain the steady-state solution of a boundary-value problem for the double diffusivity model.

Keywords: Switching random walks, fast and slow particles, duality, scaling limits, uphill diffusion, Fick's law.

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Contents

1 Introduction

Section 1.1 provides the background and the motivation for the paper. Section 1.2 defines the model. Section 1.3 identifies the dual and the stationary measures. Section 1.4 gives a brief outline of the remainder of the paper.

1.1 Background and motivation

Interacting particle systems are used to model and analyse properties of *non-equilibrium systems*, such as macroscopic profiles, long-range correlations and macroscopic large deviations. Some models have additional structure, such as duality or integrability properties, which allow for a study of the fine details of non-equilibrium steady states, such as microscopic profiles and correlations. Examples include zero-range processes, exclusion processes, and models that fit into the algebraic approach to duality, such as inclusion processes and related diffusion processes, or models of heat conduction, such as the Kipnis-Marchioro-Presutti model [8, 18, 19, 27, 34]. Most of these models have indistinguishable particles that are preserved, and so the relevant macroscopic quantity is the *density* of particles.

Turning to more complex models of non-equilibrium, various exclusion processes with *multi-type particles* have been studied [21, 22, 36], as well as reaction-diffusion processes [6, 7, 16, 14, 15], where non-linear reaction-diffusion equations are obtained in the hydrodynamic limit, and large deviations around such equations have been analysed. In the present paper, we focus on a reaction-diffusion model that on the one hand is simple enough such that via duality a complete microscopic analysis of the non-equilibrium profiles can be carried out, but on the other hand exhibit interesting phenomena, such as *uphill diffusion* and *boundary-layer effects*. In our model we have two types of particles, type 0 and type 1, that jump at rate 1 and $\epsilon \in [0, 1]$, respectively. Particles of identical type are allowed to interact via exclusion or inclusion. There is no interaction between particles of different type that are at different sites. Each particle can change type at a rate that is adapted to the particle interaction (inclusion or exclusion), and is therefore interacting with particles of different type at the same site. An alternative and equivalent view is to consider two layers of particles, where the layer determines the jump rate and where on each layer the particles move according to exclusion or inclusion, and particles can change layer at a rate that is appropriately chosen in accordance with the interaction. In the limit as $\epsilon \downarrow 0$, particles are immobile on the top layer.

We show that the *hydrodynamic limit* of all three dynamics is a linear reaction-diffusion system known under the name of *double diffusivity model*, namely,

$$(1.1)\quad \begin{cases} \partial_t \rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t \rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1), \end{cases}$$

where ρ_i, $i \in \{0, 1\}$, are the macroscopic densities of the two types of particles, and $\Upsilon \in (0, \infty)$ is the scaled switching rate. The above system was introduced in [1] to model polycrystal diffusion (more generally, diffusion in inhomogeneous porous media) and dislocation pipe diffusion, with the goal to overcome the restrictions imposed by Fick's law. Non-Fick behaviour is immediate from the fact that the total density $\rho = \rho_0 + \rho_1$ does not satisfy the classical diffusion equation.

The double diffusivity model was studied extensively in the PDE literature [30, 31, 33], while its discrete counterpart was analysed in terms of a single random walk switching between two layers [32]. The same macroscopic model was studied independently in the mathematical finance literature in the context of switching diffusion processes [42]. Thus, we have a family of interacting particle systems whose macroscopic limit is relevant in several contexts. Another context our three dynamics fit into are models of interacting active random walks with an internal state that changes randomly (e.g. activity, internal source of energy) and that determines their diffusion rate and or drift [13, 25, 29, 35, 38, 40].

An additional motivation to study two-layer models comes from population genetics. Individuals live in colonies, carry different genetics types, and can be either active or dormant. While active, individuals resample by adopting the type of a randomly sampled individual in the same colony, and migrate between colonies by hopping around. Active individuals can become dormant, after which they suspend resampling and migration, until they become active again. Dormant individuals reside in what is called a *seed bank*. The overall effect of dormancy is that extinction of types is slowed down, and so genetic diversity is enhanced by the presence of the seed bank. A wealth of phenomena can occur, depending on the parameters that control the rates of resampling, migration, falling asleep and waking up [5, 28]. Dormancy not only affects the long-term behaviour of the population quantitatively. It may also lead to qualitatively different equilibria and time scales of convergence. For a panoramic view on the role of dormancy in the life sciences, we refer the reader to [37].

From the point of view of non-equilibrium systems driven by boundary reservoirs, switching interacting particle systems have not been studied. On the one hand, such systems have both reaction and diffusion and therefore exhibit a richer non-equilibrium behaviour. On the other hand, the macroscopic equations are linear and exactly solvable in one dimension, and so these systems are simple enough to make a detailed microscopic analysis possible. As explained above, the system can be viewed as an interacting particle system on two layers. Therefore duality properties are available, which allows for a detailed analysis of the system coupled to reservoirs, dual to an absorbing system. In one dimension the analysis of the microscopic density profile reduces to a computation of the absorption probabilities of a simple random walk on a two-layer system absorbed at the left and right boundaries, which can be computed analytically. From the analytic solution, we can identify both the density profile and the microscopic stationary current in the system. This leads to two interesting phenomena. The first phenomenon is *uphill diffusion* (see e.g. [2, 11, 17]), i.e., in a well-defined parameter regime the current can go against the particle density gradient: when the total density of particles at the left end is higher than at the right end, the current can still go from right to left. The second phenomenon is *boundary-layer behaviour*: in the limit as $\epsilon \downarrow 0$, in the macroscopic stationary profile the densities in the top and bottom layer are equal, which for unequal boundary conditions in the top and bottom layer results in a *discontinuity* in the stationary profile. Corresponding to this jump in the macroscopic system, in the microscopic system we see a boundary layer of size $\sqrt{\epsilon}\,\log(1/\epsilon)$ where the densities are unequal. The quantification of the *size* of this boundary layer is an interesting corollary of the exact microscopic analysis via duality.

1.2 Three models

For $\sigma \in \{-1, 0, 1\}$ we introduce an interacting particle system on $\mathbb{Z}$ where the particles randomly switch their jumping rate between two possible values, 1 and ϵ, with $\epsilon \in [0, 1]$. For $\sigma = -1$ the particles are subject to the exclusion interaction, for $\sigma = 0$ the particles are independent, while for $\sigma = 1$ the particles are subject to the inclusion interaction. Let

$$\eta_0(x) := \text{number of particles at site } x \text{ jumping at rate } 1,$$
$$\eta_1(x) := \text{number of particles at site } x \text{ jumping at rate } \epsilon.$$

The configuration of the system is

$$\eta := \{\eta(x)\}_{x\in\mathbb{Z}} \in \mathcal{X} = \begin{cases} \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, & \text{if } \sigma = -1 \\ \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}}, & \text{if } \sigma = 0, 1, \end{cases}$$

where

$$\eta(x) := (\eta_0(x), \eta_1(x)), \qquad x \in \mathbb{Z}.$$

We call $\eta_0 = \{\eta_0(x)\}_{x\in\mathbb{Z}}$ and $\eta_1 = \{\eta_1(x)\}_{x\in\mathbb{Z}}$ the configurations of *fast particles*, respectively, *slow particles*. When $\epsilon = 0$ we speak of *dormant particles* (see Fig. 3).

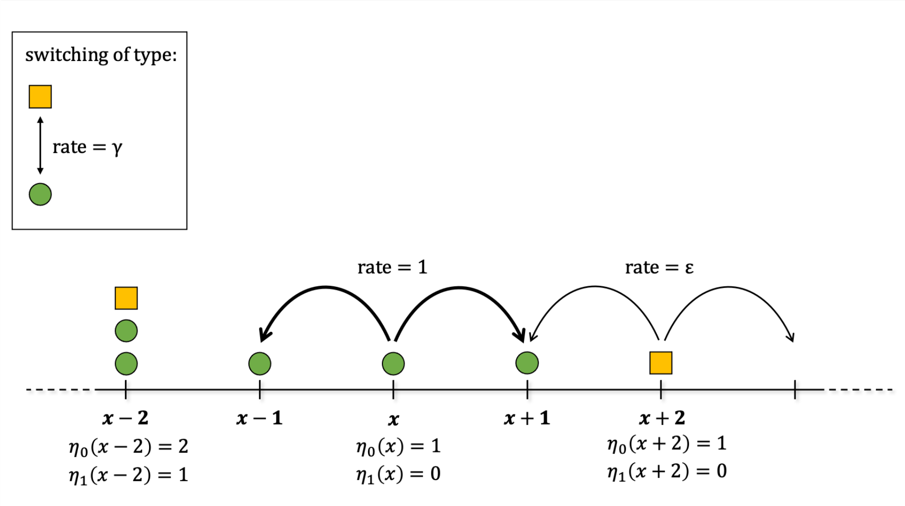

Figure 1: Schematic representation of switching independent random walks ($\sigma = 0$).

Definition 1.1. [Switching interacting particle systems] For $\epsilon \in [0, 1]$ and $\gamma \in (0, \infty)$, let $L_{\epsilon,\gamma}$ be the generator, acting on bounded cylindrical functions $f: \mathcal{X} \to \mathbb{R}$, given by

(1.2) $(L_{\epsilon,\gamma} f)(\eta) := L_0 f(\eta) + \epsilon L_1 f(\eta) + \gamma L_{0\updownarrow 1} f(\eta)$

with

$$\begin{aligned}(L_0 f)(\eta) = \sum_{|x-y|=1} \Big\{&\eta_0(x)(1+\sigma\eta_0(y))\,[f((\eta_0-\delta_x+\delta_y,\eta_1))-f(\eta)]\\ &+\eta_0(y)(1+\sigma\eta_0(x))\,[f((\eta_0+\delta_x-\delta_y,\eta_1))-f(\eta)]\Big\},\\ (L_1 f)(\eta) = \sum_{|x-y|=1} \Big\{&\eta_1(x)(1+\sigma\eta_1(y))\,[f((\eta_0,\eta_1-\delta_x+\delta_y))-f(\eta)]\\ &+\eta_1(y)(1+\sigma\eta_1(x))\,[f((\eta_0,\eta_1+\delta_x-\delta_y))-f(\eta)]\Big\},\\ (L_{0\updownarrow 1} f)(\eta) = \gamma\sum_{x\in\mathbb{Z}^d} \Big\{&\eta_0(x)(1+\sigma\eta_1(x))\,[f((\eta_0-\delta_x,\eta_1+\delta_x))-f(\eta)]\\ &+\eta_1(x)(1+\sigma\eta_0(x))\,[f((\eta_0+\delta_x,\eta_1-\delta_x))-f(\eta)]\Big\}.\end{aligned}$$

The Markov process $\{\eta(t) \colon t \geq 0\}$ on state space $\mathcal{X}$ with

$$\eta(t) := \{\eta(x,t)\}_{x\in\mathbb{Z}} = \{(\eta_0(x,t), \eta_1(x,t))\}_{x\in\mathbb{Z}}$$

with hopping rates $1, \epsilon$ and switching rate γ is called *switching exclusion process* for $\sigma = -1$, *switching random walks* for $\sigma = 0$ (see Fig. 1), and *switching inclusion process* for $\sigma = 1$. ♠

1.3 Duality and stationary measures

The systems defined in (1.2) can be equivalently formulated as jump processes on the graph (see Fig. 2) with vertex set $\{(x, i) \in \mathbb{Z}^d \times I\}$, with $I = \{0, 1\}$ labelling the two layers, and edge set given by the nearest-neighbour relation

$$(x,i) \sim (y,j) \quad \text{when} \quad \begin{cases} |x-y| = 1 \text{ and } i = j, \\ x = y \text{ and } |i-j| = 1. \end{cases}$$

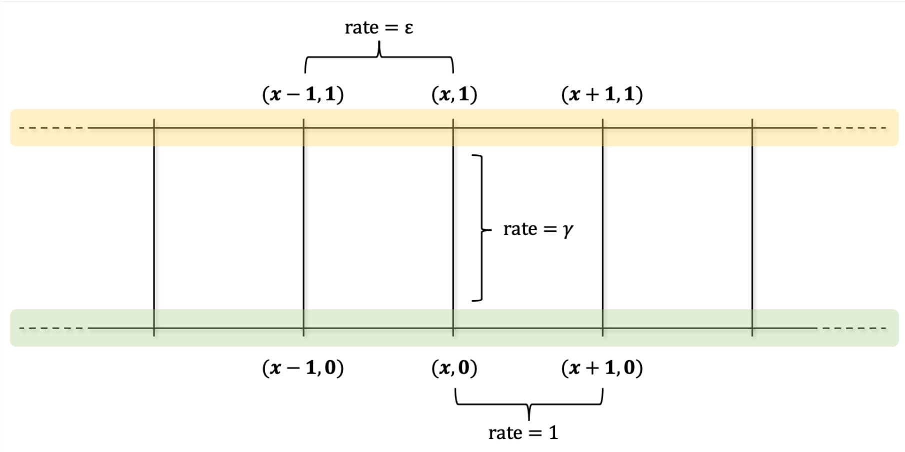

Figure 2: Schematic representation of the two-layer graph.

In this formulation the particle configuration is

$$\eta = (\eta_i(x))_{(x,i)\in\mathbb{Z}\times I}$$

and the generator is given by

$$(1.3)\quad \begin{aligned}(Lf)(\eta) := \sum_{i\in I}\sum_{|x-y|=1} & \epsilon^i \eta_i(x)(1+\sigma\eta_i(y))\,[f(\eta-\delta_{(x,i)}+\delta_{(y,i)})-f(\eta)]\\ & +\epsilon^i\eta_i(y)(1+\sigma\eta_i(x))\,[f(\eta-\delta_{(y,i)}+\delta_{(x,i)})-f(\eta)]\\ & +\sum_{i\in I}\sum_{x\in\mathbb{Z}}\eta_i(x)(1+\sigma\eta_{1-i})\,[f(\eta-\delta_{(x,i)}+\delta_{(x,1-i)})-f(\eta)].\end{aligned}$$

Thus, a single particle (when no other particles are present) is subject to two movements:

i) *Horizontal movement*: In layer $i = 0$ and $i = 1$ the particle performs a nearest-neighbour random walk on $\mathbb{Z}$ at rate 1, respectively, ϵ.

ii) *Vertical movement*: The particle switches layer at the same site at rate γ.

It is well known that for these systems there exists a one-parameter family of reversible product measures

$$\{\mu_\theta = \otimes_{(x,i)\in\mathbb{Z}\times I}\nu_{(x,i),\theta}\colon\ \theta\in\Theta\}$$

with $\Theta = [0,1]$ if $\sigma = -1$ and $\Theta = [0,\infty)$ if $\sigma \in \{0,1\}$, and with marginals given by

$$(1.4)\quad \nu_{(x,i),\theta} = \begin{cases}\text{Bernoulli}\,(\theta), & \sigma=-1,\\ \text{Poisson}\,(\theta), & \sigma = 0,\\ \text{Negative–Binomial}\,(1,\frac{\theta}{1+\theta}), & \sigma = 1.\end{cases}$$

Moreover, the usual *self-duality relation* holds with self-duality function $D\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ given by

$$(1.5)\quad D(\xi,\eta) := \prod_{(x,i)\in\mathbb{Z}^d\times I} d(\xi_i(x),\eta_i(x)),$$

with

$$(1.6)\quad d(k,n) := \frac{n!}{(n-k)!}\frac{1}{w(k)}\mathbf{1}_{\{k\le n\}}$$

and

$$(1.7)\quad w(k) := \begin{cases}\frac{\Gamma(1+k)}{\Gamma(1)}, & \sigma = 1,\\ 1, & \sigma=-1,0.\end{cases}$$

Remark 1.2. [Possible extensions] Note that we could allow for inhomogeneous rates and non-nearest neighbour jumps as well, and the same duality relation would still hold (see e.g. [23] for an inhomogeneous version of the exclusion process). More precisely, let $\{\omega_i(\{x,y\})_{x,y\in\mathbb{Z}}$ and $\{\alpha_i(x)\}_{x\in\mathbb{Z}}$ be collections of bounded weights for $i \in I$. Then the interacting particle systems with generator (1.3), and with modified transitions rates

$$\eta \longrightarrow (\eta_0 - \delta_x + \delta_y, \eta_1) \quad \text{at rate} \quad \omega_0(\{x,y\})\,\eta_0(x)\,(\alpha_0(y) + \sigma\eta_0(y)),$$
$$\eta \longrightarrow (\eta_0, \eta_1 - \delta_x + \delta_y) \quad \text{at rate} \quad \omega_1(\{x,y\})\,\eta_1(x)\,(\alpha_1(y) + \sigma\eta_1(y)),$$

are still self-dual with duality function as in (1.5), but with single-site duality functions given by $d_{(x,i)}(k,n) = \frac{n!}{(n-k)!}\frac{1}{w_{(x,i)}(k)}\,\mathbf{1}_{\{k\le n\}}$ with

$$w_{(x,i)}(k) = \begin{cases} \dfrac{\alpha_i(x)!}{(\alpha_i(x)-k)!}\,\mathbb{1}_{\{k\le\alpha_i(x)\}}, & \sigma = -1, \\ \alpha_i(x)^k, & \sigma = 0, \\ \dfrac{\Gamma(\alpha_i(x)+k)}{\Gamma(\alpha_i(x))}, & \sigma = 1. \end{cases}$$

In the present paper we prefer to stick to the homogeneous setting in order not to introduce extra notation. The extension would not lead to qualitatively different behaviour. ♠

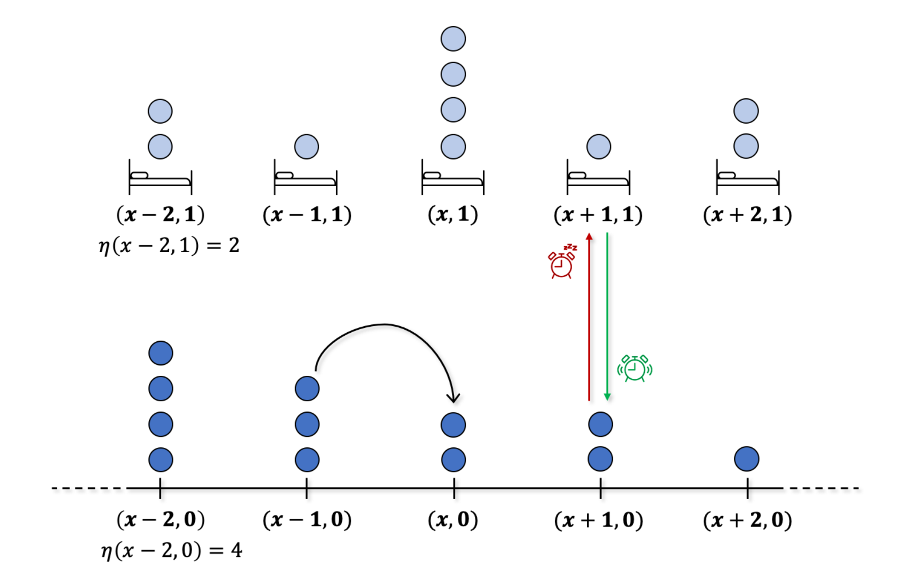

Figure 3: Schematic representation of the system of switching independent random walks with dormant particles. On the bottom layer, particles move as independent random walks with jump rate 1. On the top layer, particles do not move and sleep on top of each other.

1.4 Outline

Section 2 identifies and analyses the *hydrodynamic limit* of the system in Definition 1.1 after scaling space, time and switching rate diffusively. We thereby exhibit a class of interacting particle systems whose microscopic dynamics scales to a macroscopic dynamics called the double diffusivity model. Moreover, we provide a discussion on the solutions of this model, connecting mathematical literature applied to material science and to financial mathematics. Section 3 looks at what happens, both microscopically and macroscopically, when *boundary reservoirs* are added, resulting in a non-equilibrium flow. Here the possibility of *uphill diffusion* becomes manifest, which is absent in single-layer systems, i.e., the two-layers interact in a way that allows for a violation of Fick's law. We characterise the parameter regime for uphill diffusion. Moreover, we show that, in the limit $\epsilon \downarrow 0$, the macroscopic stationary profile of the type-1 particles adapts to the microscopic stationary profile of the type-0 particles, resulting in a *discontinuity* at the boundary in the case of unequal boundary conditions on the top layer and the bottom layer. Appendix A provides the inverse of a certain boundary-layer matrix. Appendix B lists three models for which a similar analysis can be carried through in principle. These include systems with infinitely many layers and with particles that interact via a more complicated exclusion rule.

2 The hydrodynamic limit

In this section we scale space, time and switching diffusively, so as to obtain a *hydrodynamic limit*. In Section 2.1 we scale space by $1/N$, time by N^2, the switching rate by $1/N^2$, introduce scaled microscopic empirical distributions, and let $N \to \infty$ to obtain a system of macroscopic equations. In Section 2.2 we recall some known results for this system, namely, there exists a unique solution that can be represented in terms of an underlying diffusion equation or, alternatively, via a Feynman-Kac formula involving the switching diffusion process.

2.1 From microscopic to macroscopic

Let $N \in \mathbb{N}$, and consider the scaled generator L_{ϵ,γ_N} (recall (1.2) with $\gamma_N = \Upsilon/N^2$ for some $\Upsilon \in (0,\infty)$, i.e., the reaction term is slowed down by a factor N^2 in anticipation of the diffusive scaling we are going to consider.

In order to study the collective behaviour of the particles after scaling of space and time, we introduce the following empirical density fields, which are Radon measure-valued processes that are right continuous with left limits:

$$\mathsf{X}_0^N(t) := \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_0(x,tN^2)\,\delta_{x/N}, \qquad \mathsf{X}_1^N(t) := \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_1(x,tN^2)\,\delta_{x/N}.$$

These are *microscopic* quantities. Given a test function $g \in C_c^\infty(\mathbb{R})$, we have

$$\langle \mathsf{X}_0^N(t), g\rangle = \frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_0(x,tN^2), \qquad \langle \mathsf{X}_1^N(t), g\rangle = \frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_1(x,tN^2).$$

The corresponding macroscopic quantities are

$$\rho_0(x,t) := \text{macroscopic density of fast particles,}$$
$$\rho_1(x,t) := \text{macroscopic density of slow particles.}$$

We put $\rho(x,t) := \rho_0(x,t) + \rho_1(x,t)$ for the total density.

In order to derive the hydrodynamic limit for the switching interacting particle systems, we need the following set of assumptions.

Assumption 2.1. [Compatible initial conditions] Let $\bar\rho_0\colon \mathbb{R} \to \mathbb{R}_+$ and $\bar\rho_1\colon \mathbb{R} \to \mathbb{R}_+$ be two given continuous and bounded functions, called initial macroscopic profiles. We say that a sequence $(\mu_N)_{N\in\mathbb{N}}$ of measures on $\mathcal{X}$ is a sequence of compatible initial conditions when

(i) For any $N \in \mathbb{N}$,

$$\mathbb{E}_{\mu_N}[\eta_0(x)] = \bar\rho_0(x/N), \qquad \mathbb{E}_{\mu_N}[\eta_1(x)] = \bar\rho_1(x/N).$$

(ii) There exists a constant $C < \infty$ such that

$$\text{(2.1)} \quad \sup_{(x,i)\in\mathbb{Z}\times I} \mathbb{E}_{\mu_N}[\eta(x,i)^2] \le C.$$

♠

Note that Assumption 2.1(ii) is the same as employed in [10, Theorem 1, assumption (b)]. As remarked there, the bound in (2.1) implies that

(2.2)

$$\mathbb{E}_{\mu_N}[D(\xi,\eta_t)] = \int_{\mathcal{X}} \mathbb{E}_\eta[D(\xi,\eta_t)]\mathrm{d}\mu_N(\eta) = \int_{\mathcal{X}} \mathbb{E}_\xi[D(\xi_t,\eta)]\mathrm{d}\mu_N(\eta) = \mathbb{E}_\xi\left[\mathbb{E}_{\mu_N}[D(\xi_t,\eta)]\right] \le \mathbb{E}_\xi[C^{|\xi_t|}] = C^{|\xi|},$$

where the last equality follows from the fact that the number of particles is conserved.

Theorem 2.2. [Hydrodynamic scaling] *Let $\bar\rho_0, \bar\rho_1 \in C_b^2(\mathbb{R};\mathbb{R}_+)$ be two macroscopic profiles, and let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of compatible initial conditions. Let $\mathbb{P}_N$ be the law of the measure-valued process*

$$\{X^N(t)\colon\ t \geq 0\}, \qquad X^N(t) := (X_0^N(t), X_1^N(t)),$$

induced by the initial measure μ_N. Then, for any $T, \delta > 0$ and $g \in C_c^\infty(\mathbb{R})$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\langle \mathsf{X}_i^N(t), g\rangle - \int_{\mathbb{R}} \mathrm{d}x\, \rho_i(x,t) g(x)\right| > \delta\right) = 0, \qquad i \in I,$$

where ρ_0 and ρ_1 are the unique continuous and bounded strong solutions of the system

$$\begin{cases} \partial_t\rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t\rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1), \end{cases} \tag{2.3}$$

with initial conditions

$$\begin{cases} \rho_0(x,0) = \bar\rho_0(x), \\ \rho_1(x,0) = \bar\rho_1(x). \end{cases} \tag{2.4}$$

Proof. The proof is standard and goes through the steps that we sketch below. (We omit the technical details, which can be found in the literature [10, 16, 41].)

First of all, note that the macroscopic equation (2.3) can be straightforwardly identified by computing the action of the rescaled generator $L^N = L_{\epsilon,\Upsilon/N^2}$ on cylindrical functions $f_i(\eta) := \eta_i(x)$ with $i \in I$, namely,

$$(L^N f_i)(\eta) = \epsilon^i\,[\eta_i(x+1) - 2\eta_i(x) + \eta_i(x-1)] + \frac{\Upsilon}{N^2}\,[\eta_{1-i}(x) - \eta_i(x)]\,,$$

and hence, for any $g \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\int_0^{tN^2} \mathrm{d}s\, L^N(\langle \mathsf{X}_i^N(s), g\rangle) &= \int_0^{tN^2} \mathrm{d}s\, \frac{\epsilon^i}{N}\sum_{x\in\mathbb{Z}} \eta_i(x,s)\, \tfrac{1}{2}\,[g((x+1)/N) - 2g(x/N) + g((x-1)/N)] \\ &\quad + \int_0^{tN^2} \mathrm{d}s\, \frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\, \frac{\Upsilon}{N^2}\,[\eta_{1-i}(x,s) - \eta_i(x,s)],\end{aligned}$$

where we applied the generator of simple random walk to the test function using reversibility w.r.t. the counting measure. By the regularity of g, we have

$$\int_0^{tN^2} \mathrm{d}s\, L^N(\langle \mathsf{X}_i^N(s), g\rangle) = \int_0^t \mathrm{d}s\, \langle \mathsf{X}_i^N(rN^2), \epsilon\Delta g\rangle + \int_0^{tN^2} \mathrm{d}s\, \frac{\Upsilon}{N^2}\,[\langle \mathsf{X}_{1-i}^N(s), g\rangle - \langle \mathsf{X}_i^N(s), g\rangle] + o(N^{-2}),$$

which is the discrete counterpart of the weak formulation of the right-hand side of (2.3), i.e.,

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}x\, \rho_i \Delta g + \Upsilon \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}x\, (\rho_{1-i} - \rho_i) g.$$

Thus, as a first step, we show that

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Bigg(\sup_{t\in[0,T]}\Bigg|&\langle \mathsf{X}_i^N(t), g\rangle - \langle \mathsf{X}_i^N(t), g\rangle - \int_0^t \mathrm{d}s\, \langle \mathsf{X}_i^N(rN^2), \epsilon^i\Delta g\rangle \\ &- \int_0^{tN^2} \mathrm{d}s\, \frac{\Upsilon}{N^2}\,[\langle \mathsf{X}_{1-i}^N(s) - \mathsf{X}_i^N(s), g\rangle]\Bigg| > \delta\Bigg) = 0.\end{aligned}$$

In order to prove the above convergence, we employ Dynkin's formula for Markov processes, which gives that the process defined by

$$M_i^N(g,t) := \langle \mathsf{X}_i^N(t), g\rangle - \langle \mathsf{X}_i^N(0), g\rangle - \int_0^{tN^2} \mathrm{d}s\, L^N(\langle \mathsf{X}_i^N(s), g\rangle)$$

is a martingale w.r.t. the natural filtration generated by the process $\{\eta_t :\ t \geq 0\}$ and with predictable quadratic variation expressed in terms of the carré du champ, i.e.,

$$\langle M_i^N(g,t), M_i^N(g,t)\rangle = \int_0^t \mathrm{d}s\, \mathbb{E}_{\mu_N}\left[\Gamma_i^N(g,s)\right]$$

with

$$\Gamma_i^N(g,s) = L^N(\langle \mathsf{X}_i^N(s), g\rangle)^2 - \langle \mathsf{X}_i^N(s), g\rangle\, L^N(\langle \mathsf{X}_i^N(s), g\rangle).$$

By Chebyshev's inequality and Doob's martingale inequality, we have

$$\begin{aligned}
&\mathbb{P}_{\mu_N}\Bigg(\sup_{t\in[0,T]}\Bigg|\langle \mathsf{X}_0^N(s), g\rangle - \langle \mathsf{X}_0^N(s), g\rangle \\
&\qquad - \int_0^t \mathrm{d}s\, \langle \mathsf{X}_0^N(rN^2), \epsilon\Delta g\rangle - \int_0^{tN^2} \mathrm{d}s\, \frac{\Upsilon}{N^2}\left[\langle \mathsf{X}_1^N(s), g\rangle - \langle \mathsf{X}_0^N(s), g\rangle\right]\Bigg| > \delta\Bigg) \\
(2.5)\quad &\leq \frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[\sup_{t\in[0,T]}\left|M_i^N(g,s)\right|^2\right] \leq \frac{4}{\epsilon^2}\mathbb{E}_{\mu_N}\left[\left|M_i^N(g,T)\right|^2\right] = \frac{4}{\epsilon^2}\mathbb{E}_{\mu_N}\left[\langle M_i^N(g,T), M_i^N(g,T)\rangle^2\right] \\
&= \frac{4}{\delta^2 N^2}\mathbb{E}_{\mu_N}\left[\int_0^{N^2T}\mathrm{d}s \sum_{x\in\mathbb{Z}^d} \eta_i(x,s)(1+\Upsilon\eta_i(x\pm1,s))\left[g\left(\tfrac{x\pm1}{N}\right) - g\left(\tfrac{x}{N}\right)\right]\right] \\
&\quad + \frac{4\Upsilon}{\delta^2 N^4}\mathbb{E}_{\mu_N}\left[\int_0^{N^2T}\mathrm{d}s \sum_{x\in\mathbb{Z}^d}(\eta_i(x,s) + \eta_{1-i}(x,s) + 2\Upsilon\eta_i(x,s)\eta_{1-i}(x,s))\, g^2\left(\tfrac{x}{N}\right)\right],
\end{aligned}$$

where in the last equality we explicitly computed the carré du champ. Let $k \in \mathbb{N}$ be such that the support of g is in $[-k,k]$. Then, by the regularity of g, we have that (2.5) is bounded by

$$\begin{aligned}
(2.6)\quad &\frac{4}{\delta^2N^2}(N^2T)(2k+1)N\|\nabla g\|_\infty \sup_{x,\in\mathbb{Z},\, s\in[0,N^2T]} \mathbb{E}_{\mu_N}\left[\eta_i(x,s)(1+\Upsilon\eta_i(x+1,s))\right] \\
&+ \frac{4\Upsilon}{\delta^2N^4}(N^2T)(2k+1)N\|g\|_\infty \sup_{x,\in\mathbb{Z},\, s\in[0,N^2T]} \mathbb{E}_{\mu_N}\left[\eta_i(x,s) + \eta_{1-i}(x,s) + 2\Upsilon\eta_i(x,s)\eta_{1-i}(x,s)\right],
\end{aligned}$$

and using Assumption 2.1(ii) we obtain the desired convergence.

The proof is concluded after performing the following two steps:

(i) Tightness of the sequence of distributions of the processes $\{X_i^N\}_{N\in\mathbb{N}}$, denoted by $\{Q_N\}_{N\in\mathbb{N}}$;

(ii) Concurrence of limit points: all limit points coincide and are supported by the unique path $X_i(t,\mathrm{d}x) = \rho_i(x,t)\,\mathrm{d}x$ with ρ_i the unique weak and strong solution of (2.3).

While for (i) we provide an explanation, we skip the proof of (ii) because it is standard and is based on PDE arguments (we refer to [41, Lemmas 8.6–8.7] for further details).

Tightness of the sequence $\{Q_N\}_{N\in\mathbb{N}}$ follows from the compact containment condition, i.e., for any $\delta > 0$ and $t > 0$ there exists a compact set $K \subset M$ such that $\mathbb{P}_{\mu_N}(X_i^N \in K) > 1 - \delta$, and from the equi-continuity, i.e., $\limsup_{N\to\infty} \mathbb{P}_{\mu_N}(\omega(X_i^N, \delta, T) \geq \mathfrak{e}) \leq \mathfrak{e}$ for $\omega(\alpha,\delta,T) := \sup\{d_M(\alpha(s),\alpha(t)) :\ s,t \in [0,T], |s-t| \leq \delta\}$ with d_M the metric on Radon measure defined by

$$d_M(\nu_1,\nu_2) := \sum_{j\in\mathbb{N}} 2^{-j}\left(1 \wedge \left|\int_{\mathbb{R}}\phi_j \mathrm{d}\nu_1 - \int_{\mathbb{R}}\phi_j\mathrm{d}\nu_2\right|\right)$$

for an appropriately chosen sequence of functions $(\phi_j)_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$. We refer to [41, Section A.10] for details on the above metric and to the proof of [41, Lemma 8.5] for the proof of the equi-continuity condition. We conclude by proving the compact containment condition. To that end, let us define

$$K := \left\{\nu \in M \textit{ such that } \exists\, k \in \mathbb{N} \textit{ such that } \nu[\ell,\ell+1] \leq (2\ell+1)\ell^2\ \forall\, \ell \in [k,\infty] \cap \mathbb{N}\right\}.$$

By [41, Proposition A.25], K is a pre-compact subset of M. Via the Markov inequality it follows that

$$\mathbb{P}_{\mu_N}\left(X_i^N([-\ell,\ell]) \geq (2\ell+1)\ell^2\right) \leq \frac{1}{(2\ell+1)\ell^2}\mathbb{E}_{\mu_N}\left[X_i^N([-\ell,\ell])\right] = \frac{1}{(2\ell+1)\ell^2}\sum_{x\in[-\ell,\ell]\cap\frac{\mathbb{Z}}{N}}\mathbb{E}_{\mu_N}\left[\eta_i(x,tN^2)\right],$$

and using Assumption 2.1(ii) we obtain that $\mathbb{P}_{\mu_N}\left(X_i^N([-\ell,\ell]) \geq (2\ell+1)\ell^2\right) \leq \frac{\theta}{\ell^2}$. A Borel-Cantelli argument gives that $Q_N(K) = 1$ for all $N \in \mathbb{N}$. □

Remark 2.3. [Total density] (i) If ρ_0, ρ_1 are smooth enough and satisfy (2.3) then, by taking extra derivatives, we see that the total density ρ satisfies the *thermal telegrapher equation*

(2.7) $\partial_t\left(\partial_t\rho + 2\gamma\rho\right) = -\epsilon\Delta(\Delta\rho) + (1+\epsilon)\Delta\left(\partial_t\rho + \rho\right),$

which is second order in ∂_t and fourth order in ∂_x (see [30, 31] for a derivation). Note that (2.7) shows that the total density does not satisfy the usual diffusion equation. This fact will be investigated in detail in the next section, where we will analyse the non-Fick property of ρ.
(ii) If $\epsilon = 1$, then (2.7) simplifies to the *heat equation* $\partial_t\rho = \Delta\rho$.
(iii) If $\epsilon = 0$, then (2.7) reads

$$\partial_t\left(\partial_t\rho + 2\lambda\rho\right) = \Delta\left(\partial_t\rho + \rho\right),$$

which is known as the *strongly damped wave equation*. The term $\partial_t(2\lambda\rho)$ is referred to as frictional damping, the term $\Delta(\partial_t\rho)$ as Kelvin-Voigt damping (see [9]). ♠

2.2 Existence, uniqueness and representation of the solution

The existence and uniqueness of the solution of the system (2.3) can be proved by standard PDE arguments (see e.g. [39]). Below we recall some known results that have a more probabilistic interpretation.

Stochastic representation of the solution. The system in (2.3) fits in the realm of switching diffusions (see e.g. [42]) widely studied in the mathematical finance literature. Indeed, let $\{i_t\colon t\geq 0\}$ be the pure jump process on the state space $I = \{0,1\}$ that switches at rate Υ, and whose generator acting on bounded functions $g\colon I\to\mathbb{R}$ is

$$(Ag)(i) := \Upsilon(g(1-i) - g(i)), \qquad i\in I.$$

Let $\{X_t\colon t\geq 0\}$ be the stochastic process on $\mathbb{R}$ solving the stochastic differential equation

$$\mathrm{d}X_t = \psi(i_t)\,\mathrm{d}W_t,$$

where $\{W_t\colon t\geq 0\}$ is standard Brownian motion and $\psi\colon I\to\{D_0, D_1\}$ is given by

$$\psi := D_0\,\mathbf{1}_{\{0\}} + D_1\,\mathbf{1}_{\{1\}},$$

with $D_0 = 1$ and $D_1 = \epsilon$. Let $\mathcal{L} = \mathcal{L}_{\epsilon,\Upsilon}$ be the generator defined by

$$(\mathcal{L}f)(x,i) := \lim_{t\downarrow 0}\frac{1}{t}\,\mathbb{E}_{x,i}[f(X_t,i_t) - f(x,i)]$$

for $f\colon\mathbb{R}\times I\to\mathbb{R}$ such that $f(\cdot,i)\in C_b^2(\mathbb{R})$. Then, via a standard computation (see e.g. [26, Eq.(4.4)]), it follows that

$$\begin{aligned}(\mathcal{L}f)(x,i) &= \psi(i)(\Delta f)(x,i) + \Upsilon[f(x,1-i) - f(x,i)]\\ &= \begin{cases}\Delta f(x,0) + \Upsilon\,[f(x,1) - f(x,0)], & i=0,\\ \epsilon\Delta f(x,1) + \Upsilon\,[f(x,0) - f(x,1)], & i=1.\end{cases}\end{aligned}$$

We therefore have the following result that corresponds to [26, Chapter 5, Section 4, Theorem 4.1](see also [42, Theorem 5.2]).

Theorem 2.4. [Stochastic representation of the solution] *Suppose that* $\bar\rho_i\colon\mathbb{R}\to\mathbb{R}$ *for* $i\in I$ *are continuous and bounded. Then* (2.3) *has a unique solution given by*

$$\rho_i(x,t) = \mathbb{E}_{(x,i)}[\bar\rho_{i_t}(X_t)], \qquad i\in I.$$

Note that if there is only one particle in the system (1.2), then we are left with a single random walk, say $\{Y_t\colon t\geq 0\}$, whose generator, denoted by A, acts on bounded functions $f\colon\mathbb{Z}\times I\to\mathbb{R}$ as

$$\mathsf{A}f(y,i) = \psi(i)\left[\sum_{z\sim y}[f(z,i) - f(y,i)]\right] + \Upsilon\,[f(y,1-i) - f(y,i)].$$

After we apply the generator to the function $f(y,i) = y$, we get

$$(Af)(y,i) = 0,$$

i.e., the position of the random walk is a martingale. Computing the quadratic variation via the carré du champ, we find

$$\mathsf{A}(Y_t^2) = \psi(i_t)[(Y_t+1)^2 - Y_t^2] + \psi(i_t)[(Y_t-1)^2 - Y_t^2] = 2\psi(i_t).$$

Hence the predictable quadratic variation is given by

$$\int_0^t \mathrm{d}s\, 2\psi(i_s).$$

Note that for $\epsilon = 0$ the latter equals the total amount of time the random walk is fast up to time t.

When we diffusively scale the system (scaling the reaction term as before), then the quadratic variation becomes

$$\int_0^{tN^2} \mathrm{d}s\, \psi(i_{N,s}) = \int_0^t \mathrm{d}r\, \psi(i_r).$$

Thus, we have the following invariance principle statement:

Given the path of the process $\{i_t\colon\ t \geq 0\}$,

$$\lim_{N\to\infty} \frac{Y_{N^2 t}}{N} = W_{\int_0^t \mathrm{d}r\, \sqrt{\psi(i_r)}},$$

where $\{W_t\colon\ t \geq 0\}$ is standard Brownian motion.

This fact is consistent with the classical invariance principle, i.e., after diffusive scaling, conditional on the trajectory of the switching process, the single-particle motion converges to a time-changed Brownian motion.

Thus, if we knew the path of the process $\{i_r\colon\ r \geq 0\}$, then we could express the solution of the system (2.3) in terms of a time-changed Brownian motion. However, even though $\{i_r\colon\ r \geq 0\}$ is a simple flipping process, we cannot say much explicitly about the random time $\int_0^t \mathrm{d}r\, \sqrt{\psi(i_r)}$. We therefore look for a simpler formula, where the relation to a Brownian motion with different velocities is more explicit. We achieve this by looking at the resolvent of the generator $\mathcal{L}$ in the following proposition.

Proposition 2.5. [Resolvent] *Let $f\colon\ \mathbb{R}\times I \to \mathbb{R}$ be a bounded and smooth function. Then, for $\epsilon \in (0,1]$ and $i \in I$,*

$$\begin{aligned}
&(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x,i) \\
&= \int_0^\infty \mathrm{d}t\, \frac{1}{\epsilon^i} \mathrm{e}^{-t\ell_\epsilon(\Upsilon,\lambda)} \Big(\cosh\big(t\tfrac{\Upsilon c_\epsilon(\lambda)}{2\epsilon}\big)\, \mathbb{E}_x[f(W_t,i)] + (-1)^i \tfrac{(\lambda+1)(1-\epsilon)}{c_\epsilon(\lambda)} \sinh\big(t\tfrac{\Upsilon c_\epsilon(\lambda)}{2\epsilon}\big)\, \mathbb{E}_x[f(W_t,i)]\Big) \\
&+ \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\ell_\epsilon(\Upsilon,\lambda)} \Big(2\sinh\big(t\tfrac{\Upsilon c_\epsilon(\lambda)}{2\epsilon}\big)\, \mathbb{E}_x[f(W_t,1-i)]\Big),
\end{aligned} \tag{2.8}$$

where $c_\epsilon(\lambda) := \sqrt{(1-\epsilon)^2(\lambda+1)^2 + \frac{4}{\epsilon}}$ and $\ell_\epsilon(\lambda) := \frac{\sigma}{2\epsilon}$, while for $\epsilon = 0$,

$$(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x,i) = \int_0^\infty \mathrm{d}t\, \frac{2\mathrm{e}^{-\Upsilon t \frac{\lambda(\lambda+2)}{\lambda+1}}}{(\lambda+1)^i} \left(\mathbb{E}_x[f(W_t,0)] + \frac{1}{\lambda+1}\mathbb{E}_x[f(W_t,1)]\right). \tag{2.9}$$

Proof. The proof is split into two parts.

Case $\epsilon > 0$. We can split the generator $\mathcal{L}$ as

$$\mathcal{L} = \psi(i)\tilde{\mathcal{L}} = \psi(i)\left(\Delta + \frac{1}{\psi(i)}A\right) = \psi(i)(\Delta + \tilde{A}),$$

i.e., we decouple X_t and i_t in the action of the generator. We can now use the Feynman-Kac formula to express the resolvent of the operator $\mathcal{L}$ in terms of the operator $\tilde{\mathcal{L}}$. Denoting by $\tilde{\mathbb{E}}$ the expectation of the process with generator $\tilde{\mathcal{L}}$ and looking at the function

$$f(x,i) = \bar{\rho}_0(x)\, \mathbf{1}_{\{0\}}(i) + \bar{\rho}_1(x)\, \mathbf{1}_{\{1\}}(i),$$

we have, for $\lambda \in \mathbb{R}$,

$$(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x) = \left(\frac{\lambda \boldsymbol{I}}{\psi} - \tilde{\mathcal{L}}\right)^{-1} \left(\frac{f(x,i)}{\psi(i)}\right) = \int_0^\infty \mathrm{d}t\ \tilde{\mathbb{E}}_{(x,i)}\left[\mathrm{e}^{-\int_0^t \mathrm{d}s\, \frac{\lambda}{\psi(i_s)}}\ \frac{f(X_t, i_t)}{\psi(i_t)}\right],$$

and by the decoupling of X_t and i_t under $\tilde{\mathcal{L}}$, we get

$$\begin{aligned}
&(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x,i) \\
&= \int_0^\infty \mathrm{d}t\ \tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda \int_0^t \mathrm{d}s\, \frac{1}{\psi(i_s)}}\ \frac{\mathbf{1}_{\{0\}}(i_t)}{\psi(i_t)}\right] \mathbb{E}_x[\bar{\rho}_0(W_t)] + \int_0^\infty \mathrm{d}t\ \tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda \int_0^t \frac{1}{\psi(i_s)}}\ \frac{\mathbf{1}_{\{1\}}(i_t)}{\psi(i_t)}\right] \mathbb{E}_x[\bar{\rho}_1(W_t)] \\
&= \int_0^\infty \mathrm{d}t\ \tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda \int_0^t \mathrm{d}s\, \frac{1}{\psi(i_s)}}\ \mathbf{1}_{\{0\}}(i_t)\right] \mathbb{E}_x[\bar{\rho}_0(W_t)] + \frac{1}{\epsilon}\int_0^\infty \mathrm{d}t\ \tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda \int_0^t \mathrm{d}s\, \frac{1}{\psi(i_s)}}\ \mathbf{1}_{\{1\}}(i_t)\right] \mathbb{E}_x[\bar{\rho}_1(W_t)].
\end{aligned}$$

Defining

$$A := \begin{bmatrix} -\Upsilon & \Upsilon \\ \Upsilon & -\Upsilon \end{bmatrix}, \qquad \psi_\epsilon := \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix},$$

and using again the Feynman-Kac formula, we have

$$(\lambda \boldsymbol{I} - \mathcal{L})^{-1} \begin{bmatrix} f(x,0) \\ f(x,1) \end{bmatrix} = \int_0^\infty \mathrm{d}t\, K_\epsilon(t,\lambda) \begin{bmatrix} \mathbb{E}_x[f(W_t,0)] \\ \mathbb{E}_x[f(W_t,1)] \end{bmatrix}$$

with $K_\epsilon(t,\lambda) = \mathrm{e}^{t\psi_\epsilon^{-1}(-\lambda \boldsymbol{I} + A)}\psi^{-1}$.

Using the explicit formula for the exponential of a 2×2 matrix (see e.g. [3, Corollary 2.4]), we obtain

$$\begin{aligned}
(2.10)\quad &\mathrm{e}^{t\psi_\epsilon^{-1}(-\lambda \boldsymbol{I}+A)} \\
&= \mathrm{e}^{-\frac{\Upsilon t(\lambda+1)}{2}\left(\frac{1+\epsilon}{\epsilon}\right)} \begin{bmatrix} \cosh(c_\epsilon(t,\lambda)) + \frac{\Upsilon t(\lambda+1)}{2c_\epsilon(t,\lambda)}\frac{1-\epsilon}{\epsilon}\sinh(c_\epsilon(t,\lambda)) & \frac{t\Upsilon}{c_\epsilon(t,\lambda)}\sinh(c_\epsilon(t,\lambda)) \\ \frac{\Upsilon t}{\epsilon c_\epsilon(t,\lambda)}\sinh(c_\epsilon(t,\lambda)) & \cosh(c_\epsilon(t,\lambda)) - \frac{\Upsilon t(\lambda+1)}{2c_\epsilon(t,\lambda)}\frac{1-\epsilon}{\epsilon}\sinh(c_\epsilon(t,\lambda)) \end{bmatrix},
\end{aligned}$$

with $c_\epsilon(t,\lambda) = \frac{1}{2}\frac{t\Upsilon}{\epsilon}\sqrt{(1-\epsilon)^2(\lambda+1)^2 + \frac{4}{\epsilon}}$, from which we obtain (2.8).

<u>Case $\epsilon = 0$.</u> For the case $\epsilon = 0$ (sleeping particles on top), we derive $K_0(t,\lambda)$ by taking the limit for $\epsilon \downarrow 0$ in the previous expression, i.e., $K_0(t,\lambda) = \lim_{\epsilon\downarrow 0} K_\epsilon(t,\lambda)$. We thus have that $K_0(t,\lambda)$ is equal to

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \mathrm{e}^{-\frac{\Upsilon t(\lambda+1)}{2}\left(\frac{1+\epsilon}{\epsilon}\right)} \begin{bmatrix} \cosh(c_\epsilon(t,\lambda)) + \frac{\Upsilon t(\lambda+1)}{2c_\epsilon(t,\lambda)}\frac{1-\epsilon}{\epsilon}\sinh(c_\epsilon(t,\lambda)) & \frac{t\sigma}{\epsilon c_\epsilon(t,\lambda)}\sinh(c_\epsilon(t,\lambda) \\ \frac{\sigma t}{\epsilon c_\epsilon(t,\lambda)}\sinh(c_\epsilon(t,\lambda)) & \frac{1}{\epsilon}\cosh(c_\epsilon(t,\lambda)) - \frac{\sigma t(\lambda+1)}{2c_\epsilon(t,\lambda)}\frac{1-\epsilon}{\epsilon^2}\sinh(c_\epsilon(t,\lambda)) \end{bmatrix} \\
&= 2\mathrm{e}^{-\Upsilon t\frac{\lambda(\lambda+2)}{\lambda+1}} \begin{bmatrix} 1 & \frac{1}{\lambda+1} \\ \frac{1}{\lambda+1} & \frac{1}{(\lambda+1)^2} \end{bmatrix},
\end{aligned}$$

from which (2.9) follows. □

Remark 2.6. [Symmetric layers] Note that for $\epsilon = 1$ we have

$$(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x,i) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t}\left(\frac{1+\mathrm{e}^{-2t}}{2}\,\mathbb{E}_x[f(W_t,i)] + \frac{1-\mathrm{e}^{-2t}}{2}\,\mathbb{E}_x[f(W_t,1-i)]\right).$$

♠

We conclude this section by noting that the system in (2.3) was studied in detail in [30, 31]. By taking Fourier and Laplace transforms and inverting them, it is possible to deduce explicitly the solution, which is expressed in terms of solutions to the classical heat equation. More precisely, using formula [31, Eq.2.2], we have that, for $\gamma = 1$,

$$\begin{aligned}
(2.11)\quad \rho_0(x,t) &= \mathrm{e}^{-\Upsilon t}\,\mathbb{E}_x[\bar{\rho}_0(B_=(t))] \\
&+ \frac{\Upsilon}{1-\epsilon}\mathrm{e}^{-\Upsilon t}\int_{\epsilon t}^t \mathrm{d}s\left(\left(\frac{s-\epsilon t}{t-\epsilon}\right)^{1/2} I_1(\upsilon(s))\,\mathbb{E}_x[\bar{\rho}_0(B_0(s))] + I_0(\upsilon(s))\,\mathbb{E}_x[\bar{\rho}_1(B_1(s))]\right)
\end{aligned}$$

and

$$\begin{aligned}
(2.12)\quad \rho_1(x,t) &= \mathrm{e}^{-\Upsilon t}\mathbb{E}_x[\bar{\rho}_1(B_1(\epsilon t))] \\
&+ \frac{\Upsilon}{1-\epsilon}\mathrm{e}^{-\sigma t}\int_{\epsilon t}^t \mathrm{d}s\left(\left(\frac{s-\epsilon t}{t-\epsilon}\right)^{-1/2} I_1(\upsilon(s))\,\mathbb{E}_x[\bar{\rho}_1(B_1(s)] + I_0(\upsilon(s))\,\mathbb{E}_x[\bar{\rho}_0(B_0(s))]\right),
\end{aligned}$$

where $\upsilon(s) = \frac{2\sigma}{1-\epsilon}((t-s)(s-\epsilon t))^{1/2}$, $I_0(\cdot)$ and $I_1(\cdot)$ are the modified Bessel functions, and B_0 and B_1 are two independent standard Brownian motions.

3 The system with boundary reservoirs

In this section we consider a finite version of the switching interacting particle systems introduced in Definition 1.1 to which boundary reservoirs are added. Section 3.1 defines the model. Section 3.2 identifies the dual and the stationary measures. Section 3.3 derives the non-equilibrium profile, both for the microscopic system and the macroscopic system, and offers various simulations. Section 3.4 shows that for certain choices of the rates there can be a flow of particles uphill, i.e., against the gradient imposed by the reservoirs, both for the microscopic system and the macroscopic system.

3.1 Model

We consider the same system as in Definition 1.1, but restricted to $V := \{1, \dots, N\} \subset \mathbb{Z}$. In addition we set $\hat{V} := V \cup \{L, R\}$ and attach a *left-reservoir* to L and a *right-reservoir* to R, both for fast and slow particles. To be more precise, there are four reservoirs (see Fig. 4):

i) For the fast particles, a left-reservoir at L injects fast particles at $x = 1$ at rate $\rho_{L,0}(1 + \sigma\eta_0(1, t))$ and a right-reservoir at R injects fast particles at $x = N$ at rate $\rho_{R,0}(1 + \sigma\eta_0(N, t))$. The left-reservoir absorbs fast particles at rate $1 + \sigma\rho_{L,0}$, while the right-reservoir does so at rate $1 + \sigma\rho_{R,0}$.

ii) For the slow particles, a left-reservoir at L injects slow particles at $x = 1$ at rate $\rho_{L,1}(1 + \sigma\eta_1(1, t))$ and a right-reservoir at R injects slow particles at $x = N$ at rate $\rho_{R,1}(1 + \sigma\eta_1(N, t))$. The left-reservoir absorbs fast particles at rate $1 + \sigma\rho_{L,1}$, while the right-reservoir does so at rate $1 + \sigma\rho_{R,1}$.

Inside V, the particles move as before.

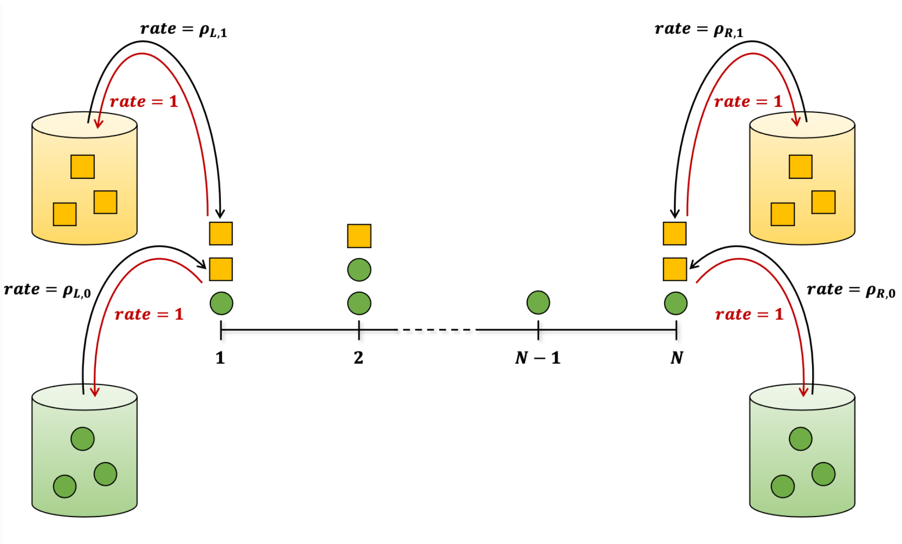

Figure 4: Case $\sigma = 0$, $\epsilon > 0$ with boundary reservoirs.

For $i \in I$, $x \in V$ and $t \geq 0$, let $\eta_i(x, t)$ denote the number of particles in layer i at site x at time t. For $\sigma \in \{-1, 0, 1\}$, the Markov process $\{\eta(t) \colon\ t \geq 0\}$ with

$$\eta(t) = \{\eta_0(x, t), \eta_1(x, t)\}_{x \in V}$$

has state space

$$\mathcal{X} = \begin{cases} I^V \times I^V, & \sigma = -1, \\ \mathbb{N}_0^V \times \mathbb{N}_0^V, & \sigma = 0, 1, \end{cases}$$

and generator

(3.1) $L := L_{\epsilon,\gamma,N} = L^{\text{bulk}} + L^{\text{res}}$

with

(3.2) $L^{\text{bulk}} := L_0^{\text{bulk}} + \epsilon L_1^{\text{bulk}} + \gamma L_{0\updownarrow 1}^{\text{bulk}}$

acting on bounded cylindrical function $f\colon \mathcal{X} \to \mathbb{R}$ as

$$(L_0^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N-1} \Big\{ \eta_0(x)(1+\sigma\eta_0(x+1)) \left[f(\eta_0 - \delta_x + \delta_{x+1}, \eta_1) - f(\eta_0, \eta_1) \right] + \eta_0(x+1)(1+\sigma\eta_0(x)) \left[f(\eta_0 - \delta_{x+1} + \delta_x, \eta) - f(\eta_0, \eta_1) \right] \Big\},$$

$$(L_1^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N-1} \Big\{ \eta_1(x)(1+\sigma\eta_1(x+1)) \left[f(\eta_0, \eta_1 - \delta_x + \delta_{x+1}) - f(\eta_0, \eta_1) \right] + \eta_1(x+1)(1+\sigma\eta_1(x)) \left[f(\eta_0, \eta_1 - \delta_{x+1} + \delta_x) - f(\eta_0, \eta_1)) \right] \Big\},$$

$$(L_{0\updownarrow 1}^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N} \Big\{ \eta_0(x)(1+\sigma\eta_1(x)) \left[f(\eta_0 - \delta_x, \eta_1 + \delta_x) - f(\eta_0, \eta_1) \right] + \eta_1(x)(1+\sigma\eta_0(x)) \left[f(\eta_0 + \delta_x, \eta_1 - \delta_x) - f(\eta_0, \eta_1)) \right] \Big\},$$

and

(3.3) $L^{\text{res}} := L_0^{\text{res}} + L_1^{\text{res}}$

with

$$\begin{aligned}(L_0^{\text{res}} f)(\eta) &= \eta_0(1)(1+\sigma\rho_{L,0}) \left[f(\eta_0 - \delta_1, \eta_1) - f(\eta_0, \eta_1) \right] + \rho_{L,0}(1+\sigma\eta_0(1)) \left[f(\eta_0 + \delta_1, \eta_1) - f(\eta_0, \eta_1) \right] \\ &+ \eta_0(N)(1+\sigma\rho_{R,0}) \left[f(\eta_0 - \delta_N, \eta_1) - f(\eta_0, \eta_1) \right] + \rho_{R,0}(1+\sigma\eta_0(N)) \left[f(\eta_0 + \delta_N, \eta) - f(\eta_0, \eta_1) \right],\end{aligned}$$

$$\begin{aligned}(L_1^{\text{res}} f)(\eta) &= \eta_1(1)(1+\sigma\rho_{L,1}) \left[f(\eta_0, \eta_1 - \delta_1) - f(\eta_0, \eta_1) \right] + \rho_{L,1}(1+\sigma\eta_1(1)) \left[f(\eta_0, \eta_1 + \delta_1) - f(\eta_0, \eta_1) \right] \\ &+ \eta_1(N)(1+\sigma\rho_{R,1}) \left[f(\eta_0, \eta_1 - \delta_N) - f(\eta_0, \eta_1) \right] + \rho_{R,1}(1+\sigma\rho_{R,N}) \left[f(\eta_0, \eta_1 + \delta_N) - f(\eta_0, \eta_1) \right].\end{aligned}$$

3.2 Duality and stationary measures

In [8] it was shown that a system of independent random walks on a finite set V, coupled with a left-reservoir and a right-reservoir, is dual to the same particle system but with the reservoirs replaced by absorbing sites. The same was proved in [24] for more general sets and for inhomogeneous rates. Our model is a particular instance of the case treated in [24, Remark 2.2]), because we can think of the rate as conductances attached to the edges.

More precisely, we consider the system where particles jump on two copies of

$$\hat{V} = V \cup \{L, R\}$$

and follow the same dynamics as before in V, but with the reservoirs at L and R absorbing. We denote by ξ the configuration

$$\xi = (\xi_0, \xi_1) := (\{\xi_0(x)\}_{x\in\hat{V}}, \{\xi_1(x)\}_{x\in\hat{V}}),$$

where $\xi_i(x)$ denotes the number of particles at site x in layer i. The state space is $\mathcal{X} = \mathbb{N}_0^{\hat{V}} \times \mathbb{N}_0^{\hat{V}}$, and the generator is

(3.4) $\hat{L} := \hat{L}_{\epsilon,\gamma,N} = \hat{L}^{\text{bulk}} + \hat{L}^{L,R}$

with

$$\hat{L}^{\text{bulk}} := \hat{L}_0^{\text{bulk}} + \epsilon\hat{L}_1^{\text{bulk}} + \gamma\hat{L}_{0\updownarrow 1}^{\text{bulk}}$$

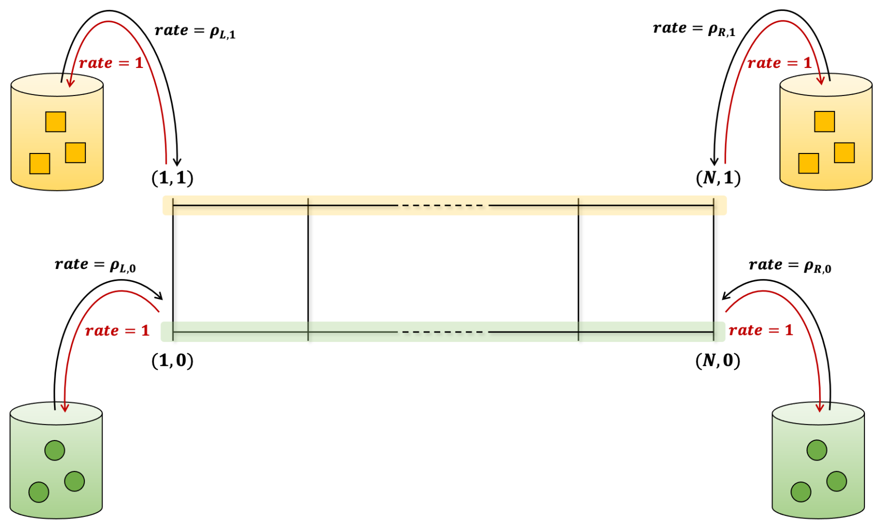

Figure 5: Case $\sigma = 0$, $\epsilon > 0$ with boundary reservoirs: two-layer representation.

acting on cylindrical functions $f\colon \mathcal{X} \to \mathbb{R}$ as

$$\begin{aligned}(L_0^{\text{bulk}} f)(\xi) = \sum_{x=1}^{N-1} \Big\{ &\xi_0(x)(1+\sigma\xi_0(x+1))\,[f(\xi_0-\delta_x+\delta_{x+1},\xi_1)-f(\xi_0,\xi_1)] \\ &+\xi_0(x+1)(1+\sigma\xi_0(x))\,[f(\xi_0-\delta_{x+1}+\delta_x,\xi_1)-f(\xi_0,\xi_1)]\Big\}, \\ (L_1^{\text{bulk}} f)(\xi) = \sum_{x=1}^{N-1} \Big\{ &\xi_1(x)(1+\sigma\xi_1(x+1))\,[f(\xi_0,\xi_1-\delta_x+\delta_{x+1})-f(\xi_0,\xi_1)] \\ &+\xi_1(x+1)(1+\sigma\xi_1(x))\,[f(\xi_0,\xi_1-\delta_{x+1}+\delta_x)-f(\xi_0,\xi_1)]\Big\}, \\ (\hat{L}_{0\updownarrow 1}^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N} \Big\{ &\xi_0(x)(1+\sigma\xi_1(x))\,[f(\xi_0-\delta_x,\xi_1+\delta_x)-f(\xi_0,\xi_1)] \\ &+\xi_1(x)(1+\sigma\xi_0(x))\,[f(\xi_0+\delta_x,\xi_1-\delta_x)-f(\xi_0,\xi_1)]\Big\},\end{aligned}$$

and

$$\hat{L}^{L,R} = \hat{L}_0^{L,R} + \hat{L}_1^{L,R}$$

acting as

$$\begin{aligned}(\hat{L}_0^{L,R} f)(\xi) &= \xi_0(1)\,[f(\xi_0-\delta_1,\xi_1)-f(\xi_0,\xi_1)] + \xi_0(N)\,[f(\xi_0-\delta_N,\xi_1)-f(\xi_0,\xi_1)], \\ (\hat{L}_1^{L,R} f)(\xi) &= \xi_1(1)\,[f(\xi_0,\xi_1-\delta_1)-f(\xi_0,\xi_1)] + \xi_1(N)\,[f(\xi_0,\xi_1-\delta_N)-f(\xi_0,\xi_1)].\end{aligned}$$

Proposition 3.1. [Duality] [8, Theorem 4.1] and [24, Proposition 2.3] *The Markov processes*

$$\begin{aligned}&\{\eta(t)\colon\ t\ge 0\}, \qquad \eta(t) = \{\eta_0(x,t),\eta_1(x,t)\}_{x\in V}, \\ &\{\xi(t)\colon\ t\ge 0\}, \qquad \xi(t) = \{\xi_0(x,t),\xi_1(x,t)\}_{x\in V},\end{aligned}$$

with generators L in (3.1) *and $\hat{L}$ in* (3.4) *are dual with duality function*

$$D(\xi,\eta) := \left(\prod_{i\in I} d_{(L,i)}(\xi_i(L))\right) \times \left(\prod_{x\in V} d(\xi_i(x),\eta_i(x))\right) \times \left(\prod_{i\in I} d_{(R,i)}(\xi_i(R))\right),$$

where $\xi = (\xi_0,\xi_1), \eta = (\eta_0,\eta_1)$ and, for $k,n\in\mathbb{N}$ and $i\in I$, $d(\cdot,\cdot)$ is given in (1.6) *and*

$$d_{(L,i)}(k) = (\rho_{L,i})^k\,, \qquad d_{(R,i)}(k) = (\rho_{R,i})^k\,.$$

3.3 Non-equilibrium stationary profile

Also the existence and uniqueness of the non-equilibrium steady state has been established in [24, Theorem 3.3] and apply to our setting. More precisely, we have the following.

Theorem 3.2. [Stationary measure] [24, Theorem 3.3(a)] *For $\sigma \in \{-1, 0, 1\}$ there exists a unique stationary measure μ_{stat} for $\{\eta(t) \colon t \geq 0\}$. Moreover, for $\sigma = 0$ and for any values of $\rho_{L,0}$, $\rho_{L,1}$, $\rho_{R,0}$, $\rho_{R,1}$,*

$$(3.5)\quad \mu_{stat} = \prod_{(x,i)\in V\times I} \nu_{(x,i)}, \qquad \nu_{(x,i)} = \text{Poisson}\,(\theta_{(x,i)}),$$

while μ_{stat} is in general not in product form for $\sigma \in \{-1, 1\}$, unless $\rho_{L,0} = \rho_{L,1} = \rho_{R,0} = \rho_{R,1}$, for which

$$(3.6)\quad \mu_{stat} = \prod_{(x,i)\in V\times I} \nu_{(x,i),\theta},$$

where $\nu_{(x,i),\theta}$ are given in (1.4).

3.3.1 Stationary microscopic profile and absorption probability

In this section we provide an explicit expression for the stationary microscopic density of each type of particles. To this end, let μ_{stat} be the unique non-equilibrium stationary measure of the process

$$\{\eta(t) \colon t \geq 0\}, \qquad \eta(t) := \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V},$$

and let $\theta_x := (\theta_0(x), \theta_1(x))$ be the stationary microscopic profile, i.e., for $x \in V$ and $i \in I$,

$$(3.7)\quad \theta_i(x) = \mathbb{E}_{\mu_{stat}}[\eta_i(x,t)].$$

Write $\mathbb{P}_\xi$ (and $\mathbb{E}_\xi$) to denote the law (and the expectation) of the dual Markov process

$$\{\xi(t) \colon t \geq 0\}, \qquad \xi(t) := \{\xi_0(x,t), \xi_1(x,t)\}_{x\in \hat{V}},$$

starting from $\xi = \{\xi_0(x), \xi_1(x)\}_{x\in\hat{V}}$. For $x \in V$, set

$$(3.8)\quad \begin{aligned} \vec{p}_x &:= \begin{bmatrix} \hat{p}(\delta_{(x,0)}, \delta_{(L,0)}) & \hat{p}(\delta_{(x,0)}, \delta_{(L,1)}) & \hat{p}(\delta_{(x,0)}, \delta_{(R,0)}) & \hat{p}(\delta_{(x,0)}, \delta_{(R,1)}) \end{bmatrix}^T, \\ \vec{q}_x &:= \begin{bmatrix} \hat{p}(\delta_{(x,1)}, \delta_{(L,0)}) & \hat{p}(\delta_{(x,1)}, \delta_{(L,1)}) & \hat{p}(\delta_{(x,1)}, \delta_{(R,0)}) & \hat{p}(\delta_{(x,1)}, \delta_{(R,1)}) \end{bmatrix}^T, \end{aligned}$$

where

$$(3.9)\quad \hat{p}(\xi, \tilde{\xi}) = \lim_{t\to\infty} \mathbb{P}_\xi(\xi(t) = \tilde{\xi}), \quad \xi = \delta_{(x,i)} \text{ for some } (x,i) \in V \times I, \text{ and } \tilde{\xi} \in \{\delta_{(L,0)}, \delta_{(L,1)}, \delta_{(R,0)}, \delta_{(R,1)}\},$$

and let

$$(3.10)\quad \vec{\rho} := \begin{bmatrix} \rho_{(L,0)} & \rho_{(L,1)} & \rho_{(R,0)} & \rho_{(R,1)} \end{bmatrix}^T.$$

Note that $\hat{p}(\delta_{(x,i)}, \cdot)$ is the probability of the dual process, starting from a single particle at site x at layer $i \in I$, of being absorbed at one of the four reservoirs. Using Proposition 3.1 and Theorem 3.2, we obtain the following.

Corollary 3.3. [Dual representation of stationary profile] *For $x \in V$, the microscopic stationary profile is given by*

$$(3.11)\quad \begin{aligned} \theta_0(x) &= \vec{p}_x \cdot \vec{\rho}, \\ \theta_1(x) &= \vec{q}_x \cdot \vec{\rho}, \end{aligned} \qquad x \in \{1, \ldots, N\},$$

where $\vec{p}_x, \vec{q}_x$ and $\vec{\rho}$ are as in (3.8)–(3.10).

We next compute the absorption probabilities associated to the dual process in order to obtain more explicit expression for the stationary microscopic profile θ_x. The absorption probabilities $\hat{p}$ of the dual process satisfy

$$(\hat{L}\hat{p})(\cdot,\tilde{\xi})(\xi) = 0,$$

where $\hat{L}$ is the dual generator defined in (3.4).

In matrix form, the above translates into the following systems of equations:

$$\begin{aligned}
\vec{p}_1 &= \frac{1}{2+\gamma}(\vec{p}_0 + \vec{p}_2) + \frac{\gamma}{2+\gamma}\vec{q}_1, \\
\vec{q}_1 &= \frac{\epsilon}{(1+\epsilon)+\gamma}\vec{q}_2 + \frac{1}{(1+\epsilon)+\gamma}\vec{q}_0 + \frac{\gamma}{(1+\epsilon)+\gamma}\vec{p}_1, \\
\vec{p}_x &= \frac{1}{2+\gamma}(\vec{p}_{x-1} + \vec{p}_{x+1}) + \frac{\gamma}{2+\gamma}\vec{q}_x, && x \in \{2,\dots,N-1\}, \\
\vec{q}_x &= \frac{\epsilon}{2\epsilon+\gamma}(\vec{q}_{x-1} + \vec{q}_{x+1}) + \frac{\gamma}{2\epsilon+\gamma}\vec{p}_x, && x \in \{2,\dots,N-1\}, \\
\vec{p}_N &= \frac{1}{2+\gamma}(\vec{p}_{N-1} + \vec{p}_{N+1}) + \frac{\gamma}{2+\gamma}\vec{q}_N, \\
\vec{q}_N &= \frac{\epsilon}{(1+\epsilon)+\gamma}\vec{q}_{N-1} + \frac{1}{(1+\epsilon)+\gamma}\vec{q}_{N+1} + \frac{\gamma}{(1+\epsilon)+\gamma}\vec{p}_N,
\end{aligned} \tag{3.12}$$

where

$$\vec{p}_0 := [\ 1\ \ 0\ \ 0\ \ 0\ ]^T, \qquad \vec{q}_0 := [\ 0\ \ 1\ \ 0\ \ 0\ ]^T,$$
$$\vec{p}_{N+1} := [\ 0\ \ 0\ \ 1\ \ 0\ ]^T, \quad \vec{q}_{N+1} := [\ 0\ \ 0\ \ 0\ \ 1\ ]^T.$$

We divide the analysis of the absorption probabilities into two cases: $\epsilon = 0$ and $\epsilon > 0$.

Case $\epsilon = 0$.

Proposition 3.4. [Absorption probability for $\epsilon = 0$] *Consider the dual process*

$$\{\xi(t)\colon\ t \geq 0\}, \qquad \xi(t) = \{\xi_0(x,t), \xi_1(x,t)\}_{x\in V},$$

with generator $\hat{L}_{\epsilon,\gamma,N}$ (see (3.4)) with $\epsilon = 0$. Then for the dual process, starting from a single particle, the absorption probabilities $\hat{p}(\cdot,\cdot)$ (see (3.9)) are given by

$$\begin{aligned}
\hat{p}(\delta_{(x,0)}, \delta_{(L,0)}) &= \frac{1+\gamma}{1+2\gamma}\left(\frac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \frac{1+2\gamma}{1+N+2N\gamma}\,x\right), \\
\hat{p}(\delta_{(x,0)}, \delta_{(L,1)}) &= \frac{\gamma}{1+2\gamma}\left(\frac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \frac{1+2\gamma}{1+N+2N\gamma}\,x\right), \\
\hat{p}(\delta_{(x,0)}, \delta_{(R,0)}) &= \frac{1+\gamma}{1+2\gamma}\left(\frac{-\gamma}{1+N+2N\gamma} + \frac{1+2\gamma}{1+N+2N\gamma}\,x\right), \\
\hat{p}(\delta_{(x,0)}, \delta_{(R,1)}) &= \frac{\gamma}{1+2\gamma}\left(\frac{-\gamma}{1+N+2N\gamma} + \frac{1+2\gamma}{1+N+2N\gamma}\,x\right),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\hat{p}(\delta_{(1,1)}, \delta_{(L,0)}) &= \frac{\gamma(N-\gamma+2N\gamma)}{(1+2\gamma)(1+N+2N\gamma)}, \quad \hat{p}(\delta_{(1,1)}, \delta_{(L,1)}) = \frac{1+N+(1+3N)\gamma-(1+2N)\gamma^2}{(1+2\gamma)(1+N+2N\gamma)}, \\
\hat{p}(\delta_{(1,1)}, \delta_{(R,0)}) &= \frac{\gamma(1+\gamma)}{(1+2\gamma)(1+N+2N\gamma)}, \quad \hat{p}(\delta_{(1,1)}, \delta_{(R,1)}) = \frac{\gamma^2}{(1+2\gamma)(1+N+2N\gamma)},
\end{aligned} \tag{3.14}$$

and

$$\hat{p}(\delta_{(x,1)}, \delta_{(\beta,i)}) = \hat{p}(\delta_{(x,0)}, \delta_{(\beta,i)}), \qquad x \in \{2,\dots,N-1\},\ (\beta,i) \in \{L,R\}\times I, \tag{3.15}$$

and

$$\begin{aligned}
\hat{p}(\delta_{(N,1)}, \delta_{(L,0)}) &= \hat{p}(\delta_{(1,1)}, \delta_{(R,0)}), \quad \hat{p}(\delta_{(N,1)}, \delta_{(L,1)}) = \hat{p}(\delta_{(1,1)}, \delta_{(R,1)}), \\
\hat{p}(\delta_{(N,1)}, \delta_{(R,0)}) &= \hat{p}(\delta_{(1,1)}, \delta_{(L,0)}), \quad \hat{p}(\delta_{(N,1)}, \delta_{(R,1)}) = \hat{p}(\delta_{(1,1)}, \delta_{(L,1)}).
\end{aligned} \tag{3.16}$$

Proof. Note that, for $\epsilon = 0$, from the linear system in (3.12) we get

$$\begin{aligned} \vec{p}_{x+1} - \vec{p}_x &= \vec{p}_x - \vec{p}_{x-1}, \\ \vec{q}_x &= \vec{p}_x, \end{aligned} \qquad x \in \{2, \ldots, N-1\}. \tag{3.17}$$

Thus, if we set $\vec{c} = \vec{p}_2 - \vec{p}_1$, then it suffices to solve the following 4 equations with 4 unknowns $\vec{p}_1, \vec{c}, \vec{q}_1$ and $\vec{q}_N$:

$$\begin{aligned} \vec{p}_1 &= \frac{1}{2+\gamma}(\vec{p}_0 + \vec{p}_1 + \vec{c}) + \frac{\gamma}{2+\gamma}\vec{q}_1, \\ \vec{q}_1 &= \frac{\epsilon}{(1+\epsilon)+\gamma}(\vec{p}_1 + \vec{c}) + \frac{1}{(1+\epsilon)+\gamma}\vec{q}_0 + \frac{\gamma}{(1+\epsilon)+\gamma}\vec{p}_1, \\ \vec{p}_1 + (N-1)\vec{c} &= \frac{1}{2+\gamma}(\vec{p}_1 + (N-2)\vec{c} + \vec{p}_{N+1}) + \frac{\gamma}{2+\gamma}\vec{q}_N, \\ \vec{q}_N &= \frac{\epsilon}{(1+\epsilon)+\gamma}(\vec{p}_1 + (N-2)\vec{c}) + \frac{1}{(1+\epsilon)+\gamma}\vec{q}_{N+1} + \frac{\gamma}{(1+\epsilon)+\gamma}((\vec{p}_1 + N\vec{c}). \end{aligned} \tag{3.18}$$

This gives the desired result. □

As a corollary, we obtain the stationary microscopic profile for the original process $\{\eta(t)\colon t \geq 0\}$, $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ when $\epsilon = 0$.

Theorem 3.5. [Stationary microscopic profile for $\epsilon = 0$]
The stationary microscopic profile $(\theta_0(x), \theta_1(x))_{x\in V}$ (see (3.7)) for the process $\{\eta(t)\colon t \geq 0\}$ with $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma,N}$ (see (3.1)) and $\epsilon = 0$ is given by

$$\begin{aligned} \theta_0(x) = \frac{1+\gamma}{1+2\gamma}&\Big[\Big(\tfrac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{L,0} + \Big(\tfrac{-\gamma}{1+N+2N\gamma} + \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{R,0}\Big] \\ &+ \frac{\gamma}{1+2\gamma}\Big[\Big(\tfrac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{(L,1)} + \Big(\tfrac{-\gamma}{1+N+2N\gamma} + \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{(R,1)}\Big] \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \theta_1(1) &= \frac{\gamma}{1+\gamma}\theta_0(1) + \frac{1}{1+\gamma}\rho_{(L,1)}, \\ \theta_1(x) &= \theta_0(x), \qquad x \in \{2, \ldots, N-1\}, \\ \theta_1(N) &= \frac{\gamma}{1+\gamma}\theta_0(N) + \frac{1}{1+\gamma}\rho_{(R,1)}. \end{aligned} \tag{3.20}$$

Proof. The proof directly follows from Corollary 3.3 and Proposition 3.4. □

Case $\epsilon > 0$. We next compute the absorption probability for the dual process and the stationary microscopic profile for the original process when $\epsilon > 0$.

Proposition 3.6. [Absorption probability for $\epsilon > 0$] *Consider the dual process*

$$\{\xi(t)\colon t \geq 0\}, \qquad \xi(t) = \{\xi_0(x,t), \xi_1(x,t)\}_{x\in V},$$

with generator $\hat{L}_{\epsilon,\gamma}$ (see (3.4)) with $\epsilon \in (0,\infty)$. Let $\hat{p}(\cdot,\cdot)$ (see (3.9)) be the absorption probabilities of the dual process starting from a single particle, and let $(\vec{p}_x, \vec{q}_x)_{x\in V}$ be as defined in (3.8). Then

$$\begin{aligned} \vec{p}_x &= \vec{c}_1\,x + \vec{c}_2 + \epsilon(\vec{c}_3\,\alpha_1^x + \vec{c}_4\,\alpha_2^x), \\ \vec{q}_x &= \vec{c}_1\,x + \vec{c}_2 - (\vec{c}_3\,\alpha_1^x + \vec{c}_4\,\alpha_2^x), \end{aligned} \qquad x \in V, \tag{3.21}$$

where α_1, α_2 are the two roots of the equation

$$\epsilon\alpha^2 - (\gamma(1+\epsilon) + 2\epsilon)\,\alpha + \epsilon = 0, \tag{3.22}$$

and $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ are vectors that depend on the parameters $N, \epsilon, \alpha_1, \alpha_2$ (see (A.4) for explicit expressions).

Proof. Applying the transformation

(3.23) $\vec{\tau}_x := \vec{p}_x + \epsilon\vec{q}_x, \qquad \vec{s}_x := \vec{p}_x - \vec{q}_x,$

we see that the system in (3.12) decouples in the bulk (i.e., the interior of V), and

(3.24) $\vec{\tau}_x = \frac{1}{2}(\vec{\tau}_{x+1} + \vec{\tau}_{x-1}), \qquad \vec{s}_x = \frac{\epsilon}{\gamma(1+\epsilon)+2\epsilon}(\vec{s}_{x+1} + \vec{s}_{x-1}), \qquad x \in \{2,\dots,N-1\}.$

The solution of the above system of recursion equations takes the form

(3.25) $\vec{\tau}_x = \vec{A}_1 x + \vec{A}_2, \quad \vec{s}_x = \vec{A}_3\alpha_1^x + \vec{A}_4\alpha_2^x,$

where α_1, α_2 are the two roots of the equation

(3.26) $\epsilon\alpha^2 - (\gamma(1+\epsilon)+2\epsilon)\,\alpha + \epsilon = 0.$

Rewriting the four boundary conditions in (3.12) in terms of the new transformations, we get

(3.27) $\begin{bmatrix} \vec{A}_1 & \vec{A}_2 & \vec{A}_3 & \vec{A}_4 \end{bmatrix} = (1+\epsilon)(M_\epsilon^{-1})^T,$

where M_ϵ is is given by

(3.28) $$M_\epsilon := \begin{bmatrix} 0 & 1 & \epsilon & \epsilon \\ 1-\epsilon & 1 & (\epsilon-1)\alpha_1 - \epsilon & (\epsilon-1)\alpha_2 - \epsilon \\ N+1 & 1 & \epsilon\alpha_1^{N+1} & \epsilon\alpha_2^{N+1} \\ N+\epsilon & 1 & -\alpha_1^N(\epsilon\alpha_1 + (1-\epsilon)) & -\alpha_2^N(\epsilon\alpha_2 + (1-\epsilon)) \end{bmatrix}.$$

Since $\vec{p}_x = \frac{1}{1+\epsilon}(\vec{\tau}_x + \epsilon\vec{s}_x)$ and $\vec{q}_x = \frac{1}{1+\epsilon}(\vec{\tau}_x - \vec{s}_x)$, by setting

$$\vec{c}_i = \frac{1}{1+\epsilon}\vec{A}_i, \quad i \in \{1,2,3,4\},$$

we get the desired identities. □

Without loss of generality, from here onwards, we fix the choices of the roots α_1 and α_2 of the quadratic equation in (3.22) as

(3.29) $$\alpha_1 = 1 + \frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right) - \sqrt{\left[1+\frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right)\right]^2 - 1}, \quad \alpha_2 = 1 + \frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right) + \sqrt{\left[1+\frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right)\right]^2 - 1}.$$

Note that, for any $\epsilon, \gamma > 0$, we have

(3.30) $\alpha_1\alpha_2 = 1.$

As a corollary, we get the expression for the stationary microscopic profile of the original process.

Theorem 3.7. [Stationary microscopic profile for $\epsilon > 0$]
The stationary microscopic profile $(\theta_0(x), \theta_1(x))_{x\in V}$ (see (3.7)) for the process $\{\eta(t)\colon\ t \geq 0\}$ with $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma,N}$ (see (3.1)) with $\epsilon > 0$ is given by

(3.31) $$\begin{aligned} \theta_0(x) &= (\vec{c}_1 \,.\, \vec{\rho})x + (\vec{c}_2 \,.\, \vec{\rho}) + \epsilon(\vec{c}_3 \,.\, \vec{\rho})\alpha_1^x + \epsilon(\vec{c}_4 \,.\, \vec{\rho})\alpha_2^x, \\ \theta_1(x) &= (\vec{c}_1 \,.\, \vec{\rho})x + (\vec{c}_2 \,.\, \vec{\rho}) - (\vec{c}_3 \,.\, \vec{\rho})\alpha_1^x - (\vec{c}_4 \,.\, \vec{\rho})\alpha_2^x, \end{aligned} \qquad x \in V,$$

where $(\vec{c}_i)_{1\leq i\leq 4}$ are as in (A.4), and

$$\vec{\rho} := \begin{bmatrix} \rho_{(L,0)} & \rho_{(L,1)} & \rho_{(R,0)} & \rho_{(R,1)} \end{bmatrix}^T.$$

Proof. The proof directly follows from Corollary 3.3 and Proposition 3.6. □

Remark 3.8. [Symmetric layers] For $\epsilon = 1$, the inverse of the matrix M_ϵ in the proof of Proposition 3.6 takes a simpler form. This is because for $\epsilon = 1$ the system is fully symmetric. In this case, the explicit expression of the stationary microscopic profile is given by

$$\begin{aligned}\theta_0(x) =& \frac{1}{2}\left(\frac{N+1-x}{N+1} + \frac{\alpha_2^{N+1-x} - \alpha_1^{N+1-x}}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{L,0} + \frac{1}{2}\left(\frac{x}{N+1} + \frac{\alpha_2^x - \alpha_1^x}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{R,0} \\ &+ \frac{1}{2}\left(\frac{N+1-x}{N+1} - \frac{\alpha_2^{N+1-x} - \alpha_1^{N+1-x}}{\alpha_2^{N+1}\alpha_1^{N+1}}\right)\rho_{(L,1)} + \frac{1}{2}\left(\frac{x}{N+1} - \frac{\alpha_2^x - \alpha_1^x}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{(R,1)}\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\theta_1(x) =& \frac{1}{2}\left(\frac{N+1-x}{N+1} - \frac{\alpha_2^{N+1-x} - \alpha_1^{N+1-x}}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{L,0} + \frac{1}{2}\left(\frac{x}{N+1} - \frac{\alpha_2^x - \alpha_1^x}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{R,0} \\ &+ \frac{1}{2}\left(\frac{N+1-x}{N+1} + \frac{\alpha_2^{N+1-x} - \alpha_1^{N+1-x}}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{(L,1)} + \frac{1}{2}\left(\frac{x}{N+1} + \frac{\alpha_2^x - \alpha_1^x}{\alpha_2^{N+1} - \alpha_1^{N+1}}\right)\rho_{(R,1)}.\end{aligned} \tag{3.33}$$

However, note that

$$\theta_0(x) + \theta_1(x) = 2[(\vec{c}_1.\vec{\rho})x + (\vec{c}_2.\vec{\rho})] - (1-\epsilon)[(\vec{c}_3\,.\vec{\rho})\alpha_1^x - (\vec{c}_4\,.\vec{\rho})\alpha_2^x],$$

which is linear in x only when $\epsilon = 1$, and

$$\theta_0(x) - \theta_1(x) = (1+\epsilon)[(\vec{c}_3\,.\vec{\rho})\alpha_1^x + (\vec{c}_4\,.\vec{\rho})\alpha_2^x]$$

which is purely exponential in x. ♠

3.3.2 Stationary macroscopic profile and boundary-value problem

Motivated by the hydrodynamic limit of the system without boundary (see Theorem 2.2), we may hope for a similar result for the finite-volume system with boundary reservoirs when this is scaled diffusively. Indeed, when space is scaled by $1/N$, time is speeded up by N^2, the switching rate γ scaled up such that $\gamma N^2 \to \Upsilon > 0$ and the system is started from a suitable initial distribution μ_N, we expect that the empirical distribution of the density of both types of particles in the system converges as $N \to \infty$ to a deterministic limit that is absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$ and the associated densities satisfy a PDE with fixed boundary conditions that are essentially imposed by the four reservoirs.

Assuming that the microscopic system with reservoirs do admit a hydrodynamic limit under the appropriate parameter scaling mentioned above, it should be the case that the two pointwise limits

$$\rho_0(y) := \lim_{N\to\infty} \theta_0^{(N)}(\lfloor yN \rfloor), \quad \rho_1(y) := \lim_{N\to\infty} \theta_1^{(N)}(\lfloor yN \rfloor)$$

exist for any $y \in [0,1]$, where $(\rho_0(\cdot), \rho_1(\cdot))$ represents the "stationary macroscopic density" of the two types of particles in the diffusive regime, where $(\theta_0^{(N)}(x), \theta_1^{(N)}(x))_{1\le x\le N}$ is the stationary microscopic profile of the microscopic system with switching rate γ_N such that $N^2\gamma_N \to \Upsilon > 0$. The following theorem indeed confirms this and provides the expression for the limit.

Theorem 3.9. [Stationary macroscopic profile] *Let $(\theta_0^{(N)}(x), \theta_1^{(N)}(x))_{1\le x\le N}$ be the stationary microscopic profile (see (3.7)) for the process $\{\eta(t)\colon\ t \ge 0\}$, $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma_N,N}$ (see (3.1)), where γ_N is such that $N^2\gamma_N \to \Upsilon$ as $N \to \infty$ for some $\Upsilon > 0$. Then, for each $y \in [0,1]$, the pointwise limits (see Fig. 6 for illustrations)*

$$\rho_0(y) := \lim_{N\to\infty} \theta_0^{(N)}(\lfloor yN \rfloor), \quad \rho_1(y) := \lim_{N\to\infty} \theta_1^{(N)}(\lfloor yN \rfloor) \tag{3.34}$$

exists and are given by

$$\begin{aligned}&\rho_0(y) = \rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y, && y \in [0,1], \\ &\rho_1(y) = \rho_0(y), && y \in (0,1), \\ &\rho_1(0) = \rho_{(L,1)},\ \rho_1(1) = \rho_{(R,1)},\end{aligned} \tag{3.35}$$

when $\epsilon = 0$, $\rho_{(L,1)} = \rho_{(L,0)}$ *and* $\rho_{(R,1)} = \rho_{(R,0)}$, *while*

$$\begin{aligned}\rho_0(y) =& \frac{\epsilon}{1+\epsilon}\left[\frac{\sinh\left[B_{\epsilon,\Upsilon}(1-y)\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}(\rho_{(L,0)}-\rho_{(L,1)}) + \frac{\sinh\left[B_{\epsilon,\Upsilon}\,y\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}(\rho_{(R,0)}-\rho_{(R,1)})\right] \\ &+ \frac{1}{1+\epsilon}\left[\rho_{(R,0)}\,y + \rho_{(L,0)}\,(1-y)\right] + \frac{\epsilon}{1+\epsilon}\left[\rho_{(R,1)}\,y + \rho_{(L,1)}\,(1-y)\right],\end{aligned} \tag{3.36}$$

$$\begin{aligned}\rho_1(y) =& \frac{1}{1+\epsilon}\left[\frac{\sinh\left[B_{\epsilon,\Upsilon}(1-y)\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}(\rho_{(L,1)}-\rho_{(L,0)}) + \frac{\sinh\left[B_{\epsilon,\Upsilon}\,y\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}(\rho_{(R,1)}-\rho_{(R,0)})\right] \\ &+ \frac{1}{1+\epsilon}\left[\rho_{(R,0)}\,y + \rho_{(L,0)}\,(1-y)\right] + \frac{\epsilon}{1+\epsilon}\left[\rho_{(R,1)}\,y + \rho_{(L,1)}\,(1-y)\right],\end{aligned} \tag{3.37}$$

when $\epsilon > 0$, *where* $\sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$ *and* $B_{\epsilon,\Upsilon} := \sqrt{\Upsilon(1+\frac{1}{\epsilon})}$.

Proof. **[SN: To be added.]** □

The following result tells us that for $\epsilon > 0$ the stationary macroscopic profiles satisfy a stationary PDE with fixed boundary conditions and also admit a stochastic representation in terms of an absorbing switching Brownian motion.

Theorem 3.10. **[Stationary boundary value problem]** *Consider the boundary value problem*

$$\begin{cases}0 = \Delta u_0 + \Upsilon(u_1 - u_0),\\ 0 = \epsilon\Delta u_1 + \Upsilon(u_0 - u_1),\end{cases} \tag{3.38}$$

with boundary conditions

$$\begin{cases}u_0(0) = \rho_{L,0},\ u_0(1) = \rho_{R,0},\\ u_1(0) = \rho_{L,1},\ u_1(1) = \rho_{R,1}.\end{cases} \tag{3.39}$$

Then the PDE admits a unique strong solution given by

$$u_i(y) = \rho_i(y), \qquad y \in [0,1], \tag{3.40}$$

where $(\rho_0(\cdot), \rho_1(\cdot))$ *are as defined in* (3.34)*. Furthermore,* $(\rho_0(\cdot), \rho_1(\cdot))$ *has the stochastic representation*

$$\rho_i(y) = \mathbb{E}_{(y,i)}[\psi_{i_\tau}(X_\tau)], \tag{3.41}$$

where $\{i_t\colon\ t \ge 0\}$ *is the pure jump process on state space* $I = \{0,1\}$ *that switches at rate* Υ*, the functions* $\psi_0, \psi_1\colon\ I \to \mathbb{R}_+$ *are defined as*

$$\psi_0 = \rho_{(L,0)}\,\mathbf{1}_{\{0\}} + \rho_{(R,0)}\,\mathbf{1}_{\{1\}}, \qquad \psi_1 = \rho_{(L,1)}\,\mathbf{1}_{\{0\}} + \rho_{(R,1)}\,\mathbf{1}_{\{1\}},$$

$\{X_t\colon\ t \ge 0\}$ *is the stochastic process* $[0,1]$ *that satisfies the SDE*

$$\mathrm{d}X_t = \psi(i_t)\,\mathrm{d}W_t$$

with $\{W_t\colon\ t \ge 0\}$ *a standard Brownian motion, the switching Brownian motion* $(X_t, i_t\colon\ t \ge 0)$ *is killed at the stopping time*

$$\tau := \inf\{t \ge 0:\ X_t \in I\},$$

and $\psi\colon\ I \to \{1, \epsilon\}$ *is given by* $\psi := \mathbf{1}_{\{0\}} + \epsilon\,\mathbf{1}_{\{1\}}$.

Proof. **[SN: To be added.]** □

Note that, as a result of the above theorem, it follows that the four absorption probabilities of the switching absorbed Brownian motion $(X_t, i_t)_{t\ge0}$ starting from $(y,i) \in [0,1] \times I$ are indeed the respective coefficients of $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}$ and $\rho_{(R,1)}$ appearing in the expression of $\rho_i(y)$.

As a corollary of Theorem 3.10 and the results in [31, Section 3], note that the time-dependent boundary-value problem

$$(3.42)\quad \begin{cases} \partial_t\rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t\rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1), \end{cases}$$

with initial conditions

$$(3.43)\quad \begin{cases} \rho_0(x,0) = \bar{\rho}_0(x), \\ \rho_1(x,0) = \bar{\rho}_1(x), \end{cases}$$

and boundary conditions

$$(3.44)\quad \begin{cases} \rho_0(0,t) = \rho_{L,0},\ \rho_0(1,t) = \rho_{R,0}, \\ \rho_1(0,t) = \rho_{L,1},\ \rho_1(1,t) = \rho_{R,1}, \end{cases}$$

admits a unique solution given by

$$(3.45)\quad \begin{cases} \rho_0(x,t) = \rho_0^{hom}(x,t) + \rho_0^{stat}(x), \\ \rho_1(x,t) = \rho_1^{hom}(x,t) + \rho_1^{stat}(x), \end{cases}$$

where

(3.46)
$$\rho_0^{hom}(x,t) = \mathrm{e}^{-\Upsilon t}h_0(x,t) + \frac{\Upsilon}{1-\epsilon}\mathrm{e}^{-\Upsilon t}\int_{\epsilon t}^{t}\mathrm{d}s\left(\left(\frac{s-\epsilon t}{t-\epsilon}\right)^{1/2} I_1(\eta(s))h_0(x,s) + I_0(\eta(s))h_1(x,s)\right),$$

(3.47)
$$\rho_1^{hom}(x,t) = \mathrm{e}^{-\Upsilon t}h_1(x,s) + \frac{\Upsilon}{1-\epsilon}\mathrm{e}^{-\Upsilon t}\int_{\epsilon t}^{t}\mathrm{d}s\left(\left(\frac{s-\epsilon t}{t-\epsilon}\right)^{-1/2} I_1(\eta(s))h_1(x,s) + I_0(\eta(s))h_0(x,s)\right),$$

$$(3.48)\quad \eta(s) = \frac{2\Upsilon}{1-\epsilon}((t-s)(s-\epsilon t))^{1/2},$$

$h_0(x,t)$, $h_1(x,t)$ are the solutions of

$$(3.49)\quad \begin{cases} \partial_t h_0 = \Delta h_0, \\ \partial_t h_1 = \Delta h_1, \\ h_0(x,0) = \bar{\rho}_0(x) - \rho_0^{stat}(x), \\ h_1(x,1) = \bar{\rho}_1(x) - \rho_1^{stat}(x), \\ h_0(0,t) = h_0(1,t) = h_1(0,t) = h_1(1,t) = 0, \end{cases}$$

and $\rho_0^{stat}(x)$, $\rho_1^{stat}(x)$ are given in (3.37).

We conclude this section by proving that the solution of the time-dependent boundary-value problem in (3.42) eventually converges to the stationary profile in (3.37).

Proposition 3.11. [Convergence to stationary profile] *Let $\rho_0^{hom}(x,t)$ and $\rho_1^{hom}(x,t)$ be given in* (3.46) *and* (3.47)*, respectively, i.e., the solutions of the boundary-value problem* (3.42) *with zero boundary conditions and initial conditions given by $\rho_0^{hom}(x,0) = \bar{\rho}_0(x) - \rho_0^{stat}(x)$ and $\rho_1^{hom}(x,0) = \bar{\rho}_1(x) - \rho_1^{stat}(x)$. Then, for any $k \in \mathbb{N}$,*

$$\lim_{t\to\infty} \|\rho_0^{hom}(x,t)\|_{C^k(0,1)} + \|\rho_1^{hom}(x,t)\|_{C^k(0,1)} = 0.$$

Proof. We show that

$$\lim_{t\to\infty} \|\rho_0^{hom}(x,t)\|_{L^2(0,1)} + \|\rho_1^{hom}(x,t)\|_{L^2(0,1)} = 0,$$

after which the results will follows via Sobolev embedding theorems.

Multiply the first equation of (3.42) by ρ_0 and the second equation by ρ_1. Integration by parts yields

$$(3.50)\quad \begin{cases} \partial_t \left(\int_0^1 \rho_0^2 \mathrm{d}x \right) = -\int_0^1 \mathrm{d}x\, |\partial_x \rho_0|^2 + \Upsilon \int_0^1 \mathrm{d}x\, (\rho_1\rho_0 - \rho_0^2), \\ \partial_t \left(\int_0^1 \mathrm{d}x\, \rho_1^2(x,t) \right) = -\epsilon \int_0^1 \mathrm{d}x\, |\partial_x \rho_1|^2 + \Upsilon \int_0^1 \mathrm{d}x\, (\rho_0\rho_1 - \rho_1^2). \end{cases}$$

Summing the two equations above and defining $E(t) := \int_0^1 \rho_0^2 + \rho_1^2 \mathrm{d}x$, we obtain

$$(3.51)\quad \partial_t E(t) = -\left(\int_0^1 \mathrm{d}x\, |\partial_x \rho_0|^2 + \epsilon \int_0^1 \mathrm{d}x\, |\partial_x \rho_1|^2 \right) - \Upsilon \int_0^1 \mathrm{d}x\, (\rho_0 - \rho_1)^2.$$

The Poincaré inequality implies $\int_0^1 \mathrm{d}x\, |\partial_x\rho_0|^2 + \epsilon \int_0^1 \mathrm{d}x\, |\partial_x\rho_1|^2 \geq C_p E(t)$, in particular, $\partial_t E(t) \leq -\epsilon C_p E(t)$, from which we obtain

$$E(t) \leq \mathrm{e}^{-C_p t} E(0),$$

and so the first part of the proof is concluded.

From [39, Theorem 2.1] it follows that

$$A := \begin{bmatrix} \Delta - \Upsilon & \Upsilon \\ \Upsilon & \epsilon\Delta - \Upsilon \end{bmatrix},$$

with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, generates a semigroup $\{S_t \colon t \geq 0\}$. If we set $\vec{\rho}(t) = S_t(\vec{\overline{\rho}} - \vec{\rho^{hom}})$, then by the semigroup property we have

$$\vec{\rho}(t) = S_{t-1}(S_{1/k})^k(\vec{\overline{\rho}} - \vec{\rho^{hom}}), \qquad t \geq 1,$$

and hence $A^k\vec{\rho}(t) = S_{t-1}(AS_{1/k})^k(\vec{\overline{\rho}} - \vec{\rho^{hom}})$. If we set $\vec{p} := (AS_{1/k})^k(\vec{\overline{\rho}} - \vec{\rho^{hom}})$, then we obtain, by [39, Theorem 5.2(d)],

$$\|A^k\vec{\rho}(t)\|_{L^2(0,1)} \leq \|S_{t-1}\vec{p}\|_{L^2(0,1)},$$

where $\lim_{t\to\infty} \|S_{t-1}\vec{p}\|_{L^2(0,1)}$ by the first part of the proof. The compact embedding

$$D(A^k) \hookrightarrow H^{2k}(0,1) \hookrightarrow C^k(0,1), \qquad k \in \mathbb{N},$$

concludes the proof. □

3.4 The stationary current

In this section we compute the average current in the non-equilibrium steady state that is induced by different densities at the boundaries. We consider the microscopic and macroscopic systems, respectively.

Microscopic system. We start by defining the notion of current. The microscopic currents are associated with the edges of the underlying two-layered graph. Since the particles move between the two layers at the same rate $\gamma > 0$, the current in the vertical direction is zero and therefore we only consider the current in the horizontal direction. In our setting, we denote by $\mathcal{J}^0_{x,x+1}(t)$ and $\mathcal{J}^1_{x,x+1}(t)$ the instantaneous current through the horizontal edge $(x, x+1)$, $x \in V$, of the bottom layer, respectively, top layer at time t. We obviously have

$$\mathcal{J}^0_{x,x+1} = \eta_0(x,t) - \eta_0(x+1,t), \qquad \mathcal{J}^1_{x,x+1} = \epsilon[\eta_1(x,t) - \eta_1(x+1,t)].$$

We are interested in the stationary currents $J^0_{x,x+1}(t)$, respectively, $J^1_{x,x+1}(t)$, which are obtained as

$$(3.52)\quad J^0_{x,x+1} = \mathbb{E}_{stat}[\eta_0(x) - \eta_0(x+1)], \qquad J^1_{x,x+1} = \epsilon\mathbb{E}_{stat}[\eta_1(x) - \eta_1(x+1)],$$

where $\mathbb{E}_{stat}$ denotes expectation w.r.t. the unique invariant probability measure of the microscopic system $\{\eta(t) \colon t \geq 0\}$ with $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$. In other words, $J^0_{x,x+1}$ and $J^1_{x,x+1}$ give the average flux of particles of type 0 and type 1 across the bond $(x, x+1)$ due to diffusion.

Of course, the average number of particle at each site varies in time also as a consequence of the reaction term:

$$\frac{d}{dt}\mathbb{E}[\eta_0(x,t)] = \mathbb{E}[\mathcal{J}^0_{x-1,x}(t) - \mathcal{J}^0_{x,x+1}(t)] + \mathbb{E}[\eta_1(x,t)] - \mathbb{E}[\eta_0(x,t)],$$
$$\frac{d}{dt}\mathbb{E}[\eta_1(x,t)] = \mathbb{E}[\mathcal{J}^1_{x-1,x}(t) - \mathcal{J}^1_{x,x+1}(t)] + \mathbb{E}[\eta_0(x,t)] - \mathbb{E}[\eta_1(x,t)].$$

Summing these equations, we see that there is no contribution of the reaction part to the variation of the average number of particles at site x:

$$\frac{d}{dt}\mathbb{E}[\eta_0(x,t) + \eta_1(x,t)] = \mathbb{E}[\mathcal{J}_{x-1,x}(t) - \mathcal{J}_{x,x+1}(t)].$$

The sum

(3.53) $J_{x,x+1} = J^0_{x,x+1} + J^1_{x,x+1},$

with $J^0_{x,x+1}$ and $J^1_{x,x+1}$ defined in (3.52), will be called the *stationary current* between sites at $x, x+1$, $x \in V$, which is responsible for the variation of the total average number of particles at each site, regardless of their type.

Proposition 3.12. [Stationary microscopic current] *For $x \in \{2, \dots, N-1\}$ the stationary currents defined in* (3.52) *are given by*

(3.54) $J^0_{x,x+1} = -\vec{c}_1 \cdot \vec{\rho} - \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)]$

and

(3.55) $J^1_{x,x+1} = -\epsilon\vec{c}_1 \cdot \vec{\rho} + \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)],$

where $\vec{c}_1, \vec{c}_3, \vec{c}_4$ are the vectors defined in (A.4) *of Appendix A, and α_1, α_2 are defined in* (3.29). *As a consequence, the current $J_{x,x+1} = J^0_{x,x+1} + J^1_{x,x+1}$ is independent of x and is given by*

(3.56) $J_{x,x+1} = -(1+\epsilon)\left[C_1\left(\rho_{R,0} - \rho_{L,0}\right) + \epsilon\, C_2\left(\rho_{R,1} - \rho_{L,1}\right)\right],$

where

$$C_1 = \frac{[\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1) + \epsilon\,(\alpha_1^{N+1} - 1)]}{\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1)(N+1) + 2\epsilon\,(\alpha_1^{N+1} - 1)(N+\epsilon)},$$
$$C_2 = \frac{(\alpha_1^{N+1} - 1)}{\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1)(N+1) + 2\epsilon\,(\alpha_1^{N+1} - 1)(N+\epsilon)}. \qquad (3.57)$$

Proof. From (3.52) we have

(3.58) $J^0_{x,x+1} = \theta_0(x) - \theta_0(x+1), \qquad J^1_{x,x+1} = \epsilon[\theta_1(x) - \theta_1(x+1)],$

where $\theta_0(\cdot), \theta_1(\cdot)$ are the average microscopic profiles. Recall from (3.11) that $\theta_0(x) = \vec{p}_x \cdot \vec{\rho}$ and $\theta_1(x) = \vec{q}_x \cdot \vec{\rho}$, where $\vec{p}_x, \vec{q}_x, \vec{\rho}$ are defined in (3.8)–(3.10). Using (3.23) and (3.25), we get

$$J_{x,x+1} = (\vec{p}_x + \epsilon\,\vec{q}_x) \cdot \vec{\rho} - (\vec{p}_{x+1} + \epsilon\,\vec{q}_{x+1}) \cdot \vec{\rho} = -(\vec{\tau}_{x+1} - \vec{\tau}_x) \cdot \vec{\rho} = -\vec{A_1} \cdot \vec{\rho},$$

where $\vec{A_1}$ is the first row of the matrix $(1+\epsilon)M_\epsilon^{-1}$ with M_ϵ defined in (3.28). **[SN: Complete the proof.]** □

Macroscopic system. The microscopic current scale like $1/N$. The current of the macroscopic system can be obtained from the microscopic current by

(3.59) $J^0(y) = \lim_{N\to\infty} N J^0_{\lfloor yN\rfloor,\lfloor yN\rfloor+1}, \qquad J^1(y) = \lim_{N\to\infty} N J^1_{\lfloor yN\rfloor,\lfloor yN\rfloor+1}.$

Proposition 3.13. [Stationary macroscopic current] *For $y \in (0,1)$ the stationary currents defined in* (3.59) *are given by*

(3.60) $J^0(y) = ...$

and

(3.61) $J^1(y) = ...$

As a consequence, the current $J(y) = J^0(y) + J^1(y)$ is constant and is given by

(3.62) $J(y) = -\left[(\rho_{R,0} - \rho_{L,0}) + \epsilon\,(\rho_{R,1} - \rho_{L,1})\right].$

Proof. **[SN: Complete the proof.]** □

Remark 3.14. [Currents] Combining the expressions for the density profiles and the current, we see that

$$J^0(y) = -\frac{d\rho_0}{dx}(y), \qquad J^1(y) = -\epsilon\frac{d\rho_1}{dx}(y).$$

♠

3.5 Discussion: Fick's law and uphill diffusion

In this section we discuss the behaviour of the boundary-driven system as the parameter ϵ is varied. For simplicity we restrict our discussion to the macroscopic setting, although similar comments hold for the microscopic system as well.

In view of the previous results, we can rewrite the equations for the densities $\rho_0(y,t), \rho_1(y,t)$ as

$$\begin{cases} \partial_t\rho_0 = -\nabla J_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t\rho_1 = -\nabla J_1 + \Upsilon(\rho_0 - \rho_1), \\ J_0 = -\nabla\rho_0, \\ J_1 = -\epsilon\nabla\rho_1, \end{cases}$$

which are complemented with the boundary values

$$\begin{cases} \rho_0(0,t) = \rho_{L,0},\ \rho_0(1,t) = \rho_{R,0}, \\ \rho_1(0,t) = \rho_{L,1},\ \rho_1(1,t) = \rho_{R,1}. \end{cases}$$

We will be concerned with the total density $\rho = \rho_0 + \rho_1$, whose evolution equation does not contain the reaction part, and is given by

$$\begin{cases} \partial_t\rho = -\nabla J, \\ J = -\nabla(\rho_0 + \epsilon\rho_1), \end{cases} \tag{3.63}$$

with boundary values

$$\begin{cases} \rho(0,t) = \rho_L = \rho_{L,0} + \rho_{R,0}, \\ \rho(1,t) = \rho_R = \rho_{R,0} + \rho_{R,1}. \end{cases} \tag{3.64}$$

Non-validity of Fick's law. From (3.63) we immediately see that Fick's law of mass transport are satisfied if and only if $\epsilon = 1$. When we allow diffusion and reaction of slow and fast particles, i.e., $0 \le \epsilon < 1$, Fick's law breaks down, since the current associated to the total mass is not proportional to the gradient of the total mass. Rather, the current J is the sum of a contribution J^0 due to the diffusion of fast particles of type 0 and a contribution J^1 due to the diffusion of slow particles of type 1 (which is proportional to ϵ). Interestingly, the breaking of Fick's law opens up the possibility of several interesting phenomena.

Equal boundary densities with non-zero current. In a system with diffusion and reaction of slow and fast particles we may observe a *non-zero current when the total density has the same value at the two boundaries*. This is different from what is observed in standard diffusive systems driven by boundary reservoirs, where in order to have a stationary current it is necessary that the reservoirs have different chemical potentials, and therefore different densities at the boundaries.

Let us, for instance, consider the specific case when $\rho_{L,0} = \rho_{R,1} = 2$ and $\rho_{L,1} = \rho_{R,0} = 4$, which indeed implies equal densities at the boundaries $\rho_L = \rho_R = 6$. The density profiles and currents are displayed in figure 6 for two values of ϵ, which shows the comparison between the Fick-regime $\epsilon = 1$ (left panels) and the regime with very slow particles $\epsilon = 0.001$ (right panels). On the one hand, in the Fick-regime the profile of both types of particles interpolates between the boundary values, with a slightly non-linear shape that has been quantified precisely in **[SN: Insert equation.]** Furthermore, in the same regime $\epsilon = 1$, the total density profile is flat and the total current J vanishes because $J_1(y) = -J_2(y)$ for all $y \in [0, 1]$.

On the other hand, in the non-Fick regime $\epsilon = 0.001$, the computation in **[SN: Insert equation.]** yields a profile for the fast particles that (almost linearly) interpolates between the boundary values, whereas the profile for the slow particles is non-monotone: it has two bumps at the boundaries and in the bulk closely follows the other profile. As a consequence, the total density profile is not flat and has two bumps at the boundaries. Most strikingly, the total current is $J = -2$, since now the current of the bottom layer J_0 is dominating, while the current of the bottom layer J_1 is small (order ϵ).

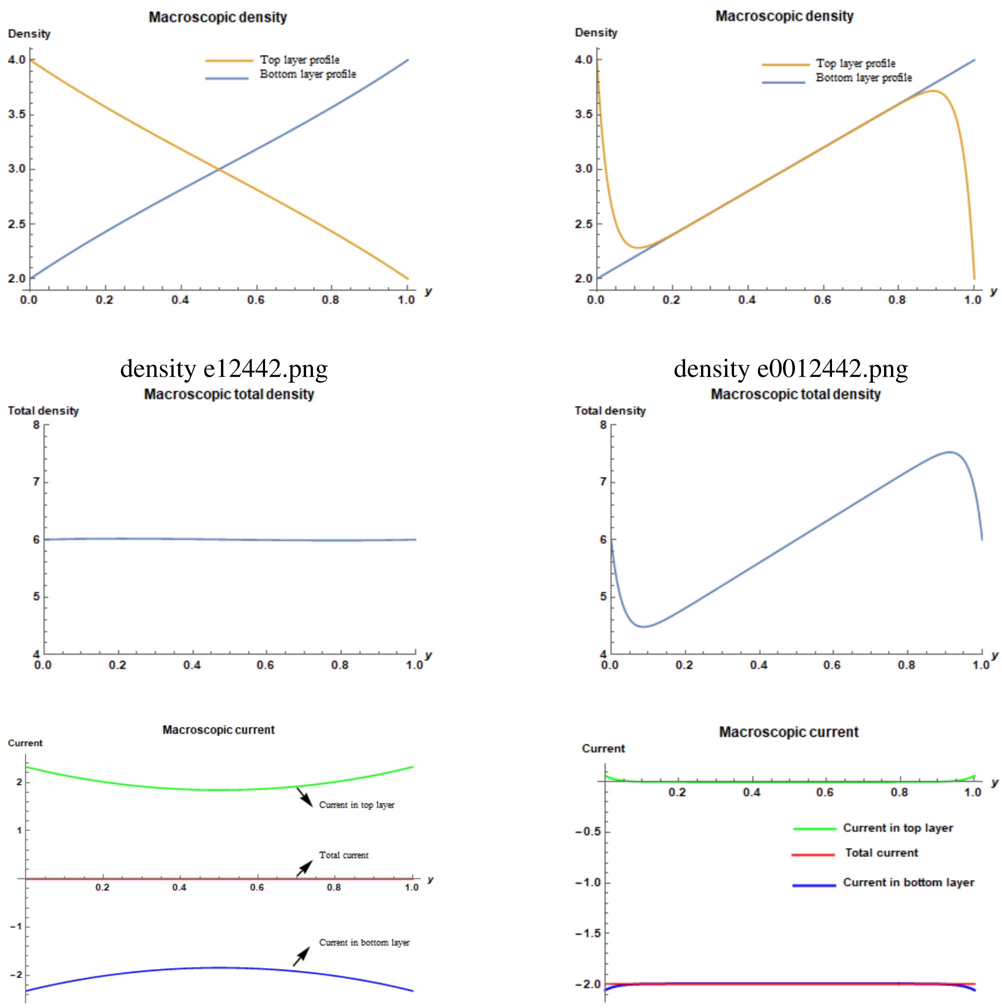

Figure 6: Macroscopic density profile and associated current. Here $\rho_{(L,0)} = 2, \rho_{(L,1)} = 4$ and $\rho_{(R,0)} = 4, \rho_{(R,1)} = 2, \Upsilon = 1$. For the left panel $\epsilon = 1$ and for the right panel $\epsilon = 0.001$.

Unequal boundary densities with uphill diffusion. By tuning the parameters $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}, \rho_{(R,1)}$ and ϵ, we can push the system into a regime where the total current $J < 0$ and the total densities are such that $\rho_R < \rho_L$, where $\rho_R = \rho_{(R,0)} + \rho_{(R,1)}$ and $\rho_L = \rho_{(L,0)} + \rho_{(L,1)}$. In this regime, *the current goes uphill*, since the total density of particles at the right is lower than at the left, yet the average current is negative.

For an illustration, consider the case when $\rho_{L,1} = 6, \rho_{R,0} = 4$ and $\rho_{L,0} = \rho_{R,1} = 2$, which implies $\rho_L = 8$ and $\rho_R = 6$ and thus $\rho_R < \rho_L$. The density profiles and currents are shown in figure 7 for two values of ϵ, in particular, a comparison between the Fick-regime $\epsilon = 1$ (left panels) and the regime of with very slow particles $\epsilon = 0.001$ (right panels). **[SF: Add more comments.]**

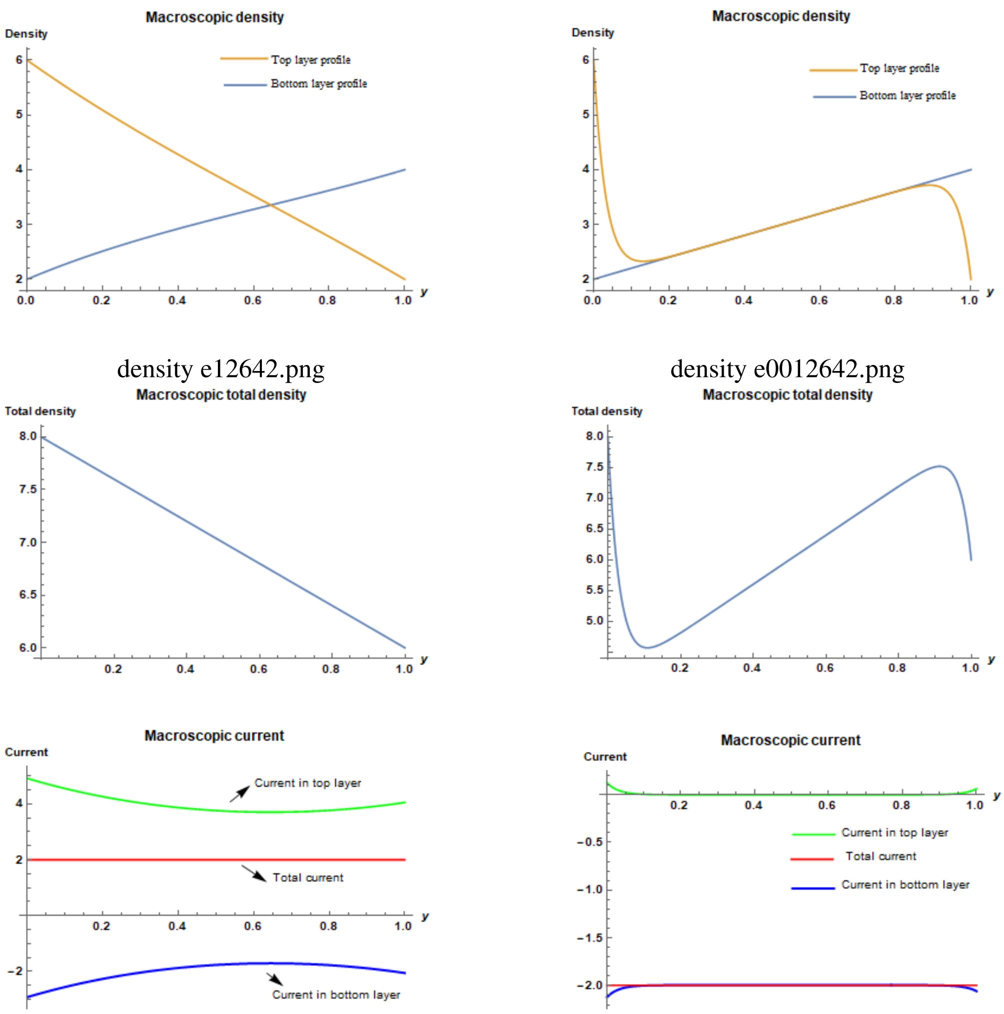

Figure 7: Macroscopic density profile and associated current. Here $\rho_{(L,0)} = 2, \rho_{(L,1)} = 6$ and $\rho_{(R,0)} = 4, \rho_{(R,1)} = 2, \Upsilon = 1$. For the left panel $\epsilon = 1$ and for the right panel $\epsilon = 0.001$.

The transition between downhill and uphill. We observe that in this example the change from downhill to uphill diffusion occurs at $\epsilon = 1/2$. The density profiles and currents are shown in figure 8 for two additional values of ϵ, one in the "mild" downhill regime $J > 0$ for $\epsilon = 0.75$ (left panels), the other in the "mild" uphill regime e $J > 0$ for $\epsilon = 0.25$(right panels). **[SF: Add more comments.]**

Identification of the uphill regime.

Definition 3.15. [Uphill diffusion] *For parameters $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}, \rho_{(R,1)}$ and $0 \leq \epsilon \leq 1$ we say the system has an uphill current in stationarity if the total current J and the difference between the total density of particles in the right and the left side of the system given by $\rho_R - \rho_L$ have the same sign, where it is understood that $\rho_R = \rho_{(R,0)} + \rho_{(R,1)}$ and $\rho_L = \rho_{(L,0)} + \rho_{(L,1)}$.*

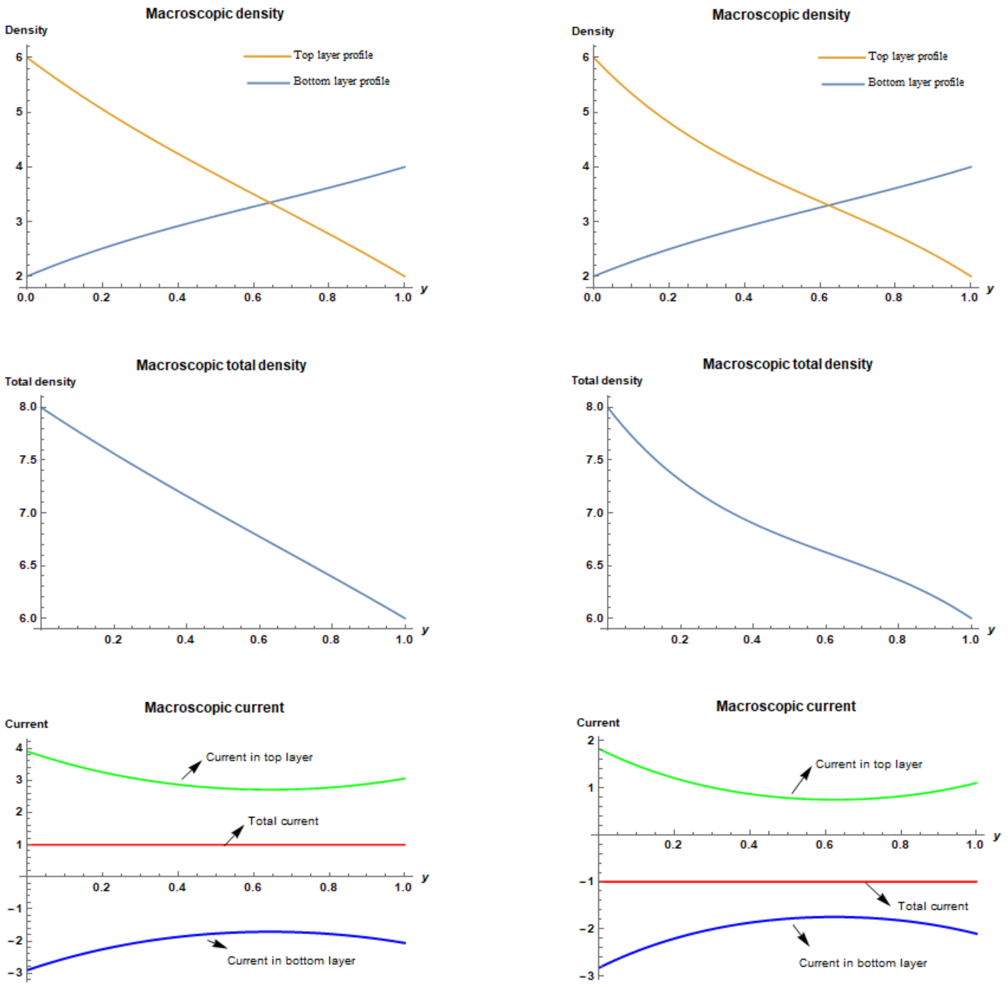

Figure 8: Macroscopic density profile and associated current in "mild" downhill and uphill regime. Here $\rho_{(L,0)} = 2, \rho_{(L,1)} = 6$ and $\rho_{(R,0)} = 4, \rho_{(R,1)} = 2, \Upsilon = 1$. For the left panel $\epsilon = 0.75$ and for the right panel $\epsilon = 0.25$.

Proposition 3.16. [Uphill regime] *Let* $a_0 := \rho_{(R,0)} - \rho_{(L,0)}$ *and* $a_1 := \rho_{(R,1)} - \rho_{(L,1)}$. *Then the macroscopic system admits an uphill current in stationarity if and only if the parameters satisfy the constraint*

(3.65) $a_0^2 + (1+\epsilon)\, a_0 a_1 + \epsilon a_1^2 < 0.$

If, furthermore, $\epsilon \in [0,1]$, *then:*

(i) either $a_0 + a_1 > 0$ *with* $a_0 < 0,\ a_1 > 0$ *or* $a_0 + a_1 < 0$ *with* $a_0 > 0,\ a_1 < 0$,

(ii) or $\epsilon \in [0, -\frac{a_0}{a_1}]$.

Proof. Note that, by (3.62), there is an uphill current if and only if $a_0 + a_1$ and $a_0 + \epsilon a_1$ have opposite signs. In other words, it happens if and only if

$$(a_0 + a_1)(a_0 + \epsilon\, a_1) = a_0^2 + (1+\epsilon)\, a_0 a_1 + \epsilon a_1^2 < 0.$$

The above constraint forces $a_0 a_1 < 0$. Further simplification reduces the parameter regime to the following four cases:

- $a_0 + a_1 > 0$ with $a_0 < 0,\ a_1 > 0$ and $\epsilon < -\frac{a_0}{a_1}$,
- $a_0 + a_1 < 0$ with $a_0 > 0,\ a_1 < 0$ and $\epsilon < -\frac{a_0}{a_1}$,
- $a_0 + a_1 > 0$ with $a_0 > 0,\ a_1 < 0$ and $\epsilon > -\frac{a_0}{a_1}$,
- $a_0 + a_1 < 0$ with $a_0 < 0,\ a_1 > 0$ and $\epsilon > -\frac{a_0}{a_1}$.

Thus, under the assumption $\epsilon \in [0,1]$, only the first two of the above four cases survive. □

3.6 The width of the boundary layer

We have seen that for $\epsilon = 0$ the microscopic density profile of the fast particles $\theta_0(x)$ linearly interpolates between $\rho_{L,0}$ and $\rho_{R,0}$, whereas the density profile of the dormant particles satisfies $\theta_1(x) = \theta_0(x)$ for all $x \in \{2, \dots, N-1\}$. In the macroscopic setting this produces a macroscopic profile $\rho_0(y) = \rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y$ while the bottom-layer profile develops two discontinuities at the boundaries

$$\rho_1(y) = [\rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y]\, \mathbf{1}_{(0,1)}(y) + \rho_{L,1} \mathbf{1}_{\{1\}}(y) + \rho_{R,1} \mathbf{1}_{\{0\}}(y).$$

For small but positive ϵ the curve is smooth and the discontinuity is turned into a boundary layer. In this section we investigate the width of the boundary layer as $\epsilon \downarrow 0$.

We define the width $\bar{y}(\epsilon)$ of the boundary layer as the point where the top-layer density profile (and therefore also the total density profile) significantly differs from the bulk linear profile of the case $\epsilon = 0$. Since the bulk profile is essentially linear, this amount to requiring that, for some constant $c > 0$,

(3.66) $$\left| \frac{d^2}{dy^2} \rho_1(y) \right| = c,$$

or equivalently, due to the relation $J_1(y) := -\epsilon\, \partial_y \rho_1$,

(3.67) $$\left| \frac{d}{dy} J_1(y) \right| = c\epsilon.$$

Recalling the expression of ρ_1 given in **[SF: Insert equation.]** we get the condition

(3.68) $$c\epsilon = \Upsilon(\rho_{(L,0)} - \rho_{(L,1)})\, \frac{\sinh\big[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}(1-y)\big]}{\sinh\big[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}\big]}.$$

Solving (3.68), we get

(3.69) $$\bar{y}(\epsilon) = 1 - \frac{1}{\sqrt{\Upsilon(1+\frac{1}{\epsilon})}} \sinh^{-1}\left[\frac{\epsilon \sinh\big[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}\big]}{\Upsilon|\rho_{(L,0)} - \rho_{(L,1)}|} \right].$$

Hence $\bar{y}(\epsilon) \asymp \sqrt{\epsilon}\log(1/\epsilon)$ as $\epsilon \downarrow 0$.

A Inverse of the boundary-layer matrix

The inverse of the matrix M_ϵ defined in (3.28) is given by (α_1 and α_2 are as in (3.29))

$$\text{(A.1)}\quad M_\epsilon^{-1} := \frac{1}{Z}\begin{bmatrix} -m_{13} & -m_{14} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31}(\alpha_2) & m_{32}(\alpha_2) & m_{33}(\alpha_2) & m_{34}(\alpha_2) \\ -m_{31}(\alpha_1) & -m_{32}(\alpha_1) & -m_{33}(\alpha_1) & -m_{34}(\alpha_1) \end{bmatrix},$$

where

(A.2)

$$\begin{aligned}
Z &:= \alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,[\alpha_2(1+N)(1-\epsilon)(\alpha_2^{N-1}-1) + 2\epsilon(N+\epsilon)(\alpha_2^{1+N}-1)],\\
m_{13} &:= \alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,[\alpha_2(1-\epsilon)(\alpha_2^{N-1}-1) + \epsilon(\alpha_2^{N+1}-1)],\\
m_{14} &:= \epsilon\,\alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,(\alpha_2^{N+1}-1),\\
m_{21} &:= (1+N)(1-\epsilon)^2(\alpha_2^{N-1}-\alpha_1^{N-1}) - \epsilon(1-\epsilon)^2(\alpha_2-\alpha_1)\\
&\qquad + \epsilon^2(1+2N+\epsilon)(\alpha_2^{N+1}-\alpha_1^{N+1}) + \epsilon(1-\epsilon)(2+3N+\epsilon)(\alpha_2^N-\alpha_1^N),\\
m_{22} &:= \epsilon\,[(1-\epsilon)(1+N)(\alpha_2^N-\alpha_1^N) + \epsilon(1+2N+\epsilon)(\alpha_2^{N+1}-\alpha_1^{N+1})],\\
m_{23} &:= \epsilon\,(1-\epsilon)[(N+\epsilon)(\alpha_2-\alpha_1) - (1-\epsilon)(\alpha_2^N-\alpha_1^N) - \epsilon(\alpha_2^{N+1}-\alpha_1^{N+1})],\\
m_{24} &:= -\epsilon(1-\epsilon)[(1+N)(\alpha_2-\alpha_1) + \epsilon\,(\alpha_2^{N+1}-\alpha_1^{N+1}),
\end{aligned}$$

and the polynomials $m_{31}(z), m_{32}(z), m_{33}(z), m_{34}(z)$ are defined as

$$\text{(A.3)}\quad \begin{aligned}
m_{31}(z) &:= -(1-\epsilon)^2\,z - \epsilon\,(1-\epsilon) + (1-\epsilon)(N+\epsilon)\,z^N - \epsilon(1-2N-3\epsilon)\,z^{N+1},\\
m_{32}(z) &:= -(1-\epsilon)(1+N)z^N - \epsilon\,(1-\epsilon) - \epsilon(1+2N+\epsilon)\,z^{N+1},\\
m_{33}(z) &:= (1-\epsilon)^2\,z^N + \epsilon\,(1-\epsilon)\,z^{N+1} - (1-\epsilon)(N+\epsilon)\,z + \epsilon(1-2N-3\epsilon),\\
m_{34}(z) &:= (1+N)(1-\epsilon)\,x + \epsilon\,(1-\epsilon)\,z^{N+1} + \epsilon(1+2N+\epsilon).
\end{aligned}$$

We remark that most of the terms appearing in the inverse simplify because of (3.30). We define the four vectors $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ as the respective rows of M_ϵ^{-1}, i.e.,

$$\text{(A.4)}\quad \begin{aligned}
\vec{c}_1 &:= (M_\epsilon^{-1})^T\vec{e}_1, \quad \vec{c}_2 := (M_\epsilon^{-1})^T\vec{e}_2,\\
\vec{c}_3 &:= (M_\epsilon^{-1})^T\vec{e}_3, \quad \vec{c}_4 := (M_\epsilon^{-1})^T\vec{e}_4,
\end{aligned}$$

where

$$\begin{aligned}
\vec{e}_1 &:= [\ 1\quad 0\quad 0\quad 0\ ]^T, \quad \vec{e}_2 := [\ 0\quad 1\quad 0\quad 0\ ]^T,\\
\vec{e}_3 &:= [\ 0\quad 0\quad 1\quad 0\ ]^T, \quad \vec{e}_4 := [\ 0\quad 0\quad 0\quad 1\ ]^T.
\end{aligned}$$

B Related models

In Appendices B.1–B.3 we put forward three related models, which we believe to be of interest for future investigation.

B.1 Multi-layer switching interacting particle systems

The two-layer system introduced in Definition 1.1 can be extended to a multi-layer system in the obvious manner. Fix $M \in N$, and consider jump rates $D = \{D_i\}_{0\le i\le M}$ for $M+1$ different types of particles. The configuration is $\{\eta(x)\}_{x\in\mathbb{Z}}$ with $\eta(x) = \{\eta_i(x)\}_{0\le i\le M}$, and the generator is

$$\text{(B.1)}\quad \begin{aligned}
(L_{D,\gamma}f)(\eta) = \sum_{i=0}^{M} D_i \sum_{|x-y|=1} &\Big\{\eta_i(x)\,[f((\eta_0,\dots,\eta_{i-1},\eta_i-\delta_x+\delta_y,\eta_{i+1},\dots,\eta_M)) - f(\eta)]\\
&\quad + \eta_0(y)\,[f((\eta_0+\delta_x-\delta_y,\eta_1)) - f(\eta)]\Big\}\\
+ \sum_{|i-j|=1}\sum_{x\in\mathbb{Z}} &\Big\{\gamma_{\{i,j\}}\,\eta_0(x)[f((\eta_0-\delta_x,\eta_1+\delta_x)) - f(\eta)]\\
&\quad + \eta_1(x)\,[f((\eta_0+\delta_x,\eta_1-\delta_x)) - f(\eta)]\Big\}.
\end{aligned}$$

Duality holds also for this model. The self-duality functions and the stationary measures are known as well (see [24]). It is straightforward to extend all our results for the two-layer model to the multi-level model.

B.2 Interacting particle systems with infinitely many dormant layers

We consider a simple exclusion process on $\mathbb{Z}$ with infinitely many dormant layers. On each layer at most one particle can go to sleep. We denote the configuration by $\eta = \{\eta(x)\}_{x\in\mathbb{Z}}$, where

$$\eta(x) = \{\eta(x,i)\}_{i\in\mathbb{N}} \in I^{\mathbb{N}},$$

with

$$\begin{aligned} \eta(x,0) &= \text{ number of active particles at site } x,\\ \eta(x,n) &= \text{ number of dormant particles on dormant layer} n \text{ at site } x, \qquad n\in\mathbb{N}. \end{aligned}$$

For $n \in \mathbb{N}$, we let s_n be the rate at which an active particle becomes dormant on layer n or a dormant particle on layer n become active. Note that $\eta(x,i) \in \{0,1\}$ for $i \in \{0\}\cup\mathbb{N}$, and the generator is

$$\begin{aligned} (Lf)(\eta) = &\sum_{x\sim y} \omega_{\{x,y\}}\Big\{\eta(x,0)(1-\eta(y,0))\,[f(\eta^{x,y}) - f(\eta)] + \eta(y,0)(1-\eta(x,0))\,[f(\eta^{y,x}) - f(\eta)]\Big\}\\ &+ \sum_{x\in\mathbb{Z}}\sum_{n\in\mathbb{N}} s_n\Big\{\eta(x,0)(1-\eta(x,n))\,[f(\eta^{(x,0),(x,n)}) - f(\eta)] + \eta(x,n)(1-\eta(x,0))\,[f(\eta^{(x,n),(x,0)}) - f(\eta)]\Big\}, \end{aligned}$$

where

$$\eta^{x,y}(z,i) = \begin{cases} \eta(z,i), & z\notin\{x,y\},\ i\in\mathbb{N},\\ \eta(x,0)-1, & z=x,\ i=0,\\ \eta(y,0)+1, & z=y,\ i=0, \end{cases}$$

and

$$\eta^{(x,0),(x,n)}(z,i) = \begin{cases} \eta(z,i), & z\notin\{x\},\ i\in\mathbb{N},\\ \eta(x,0)-1, & z=x,\ i=0,\\ \eta(x,n)+1, & z=x,\ i=n, \end{cases}$$

and

$$\eta^{(x,n),(x,0)}(z,i) = \begin{cases} \eta(z,i), & z\notin\{x\},\ i\in\mathbb{N},\\ \eta(x,0)+1, & z=x,\ i=0,\\ \eta(x,n)-1, & z=x,\ i=n. \end{cases}$$

B.3 Exclusion process with bossy dormant particles and discouragement

We next consider an exclusion process on $\mathbb{Z}$ where the dormant particles discourage active particles to jump on top of them. At any site there can be at most one active particle and at most one dormant particle. Let

$$\begin{aligned} \eta_0(x) &= \text{number of active particles at site } x,\\ \eta_1(x) &= \text{number of dormant particles at site } x. \end{aligned}$$

The configuration is $\eta = \{\eta(x)\}_{x\in\mathbb{Z}}$ with $\eta(x) = (\eta_0(x),\eta_1(x))$. The total number of particles at site x is $\eta_0(x)+\eta_1(x) \in \{0,1,2\}$. The state space is $\mathcal{X} = I^{\mathbb{Z}} \times I^{\mathbb{Z}}$. Given $\epsilon \in (0,1)$, the *exclusion process with bossy particles* is the Markov process on $\mathcal{X}$ with generator

$$\begin{aligned} (L^\epsilon f)(\eta) = &\sum_{x\sim y}\Big\{\eta_0(x)(1-\eta_0(y))(1-\nu\eta_1(y))\,[f(\eta_0-\delta_x+\delta_y,\eta_1) - f(\eta)]\\ &\qquad + \eta_0(y)(1-\eta_0(x))(1-\nu\eta_1(x)\,[f(\eta_0+\delta_x-\delta_y,\eta_1) - f(\eta)]\Big\}\\ &+ \sum_{x\in\mathbb{Z}}\Big\{\eta_0(x)(1-\eta_1(x))\,[f(\eta_0-\delta_x,\eta_1+\delta_x) - f(\eta)]\\ &\qquad + \eta_1(x)(1-\eta_0(x))\,[f(\eta_0+\delta_x,\eta_1-\delta_x) - f(\eta)]\Big\}. \end{aligned}$$

We refer to the case $\nu = 1$ as the exclusion process with bossy dormant particles and total discouragement, for which the generator is

$$
\begin{aligned}
(L^{\epsilon} f)(\eta) = \sum_{x\sim y} \Big\{ &\eta_0(x)(1-\eta_0(y))(1-\eta_1(y)) \, [f(\eta_0 - \delta_x + \delta_y, \eta_1) - f(\eta)] \\
&+ \eta_0(y)(1-\eta_0(x))(1-\eta_1(x) \, [f(\eta_0 + \delta_x - \delta_y, \eta_1) - f(\eta)] \Big\} \\
+ \sum_{x\in\mathbb{Z}} \Big\{ &\eta_0(x)(1-\eta_1(x)) \, [f(\eta_0 - \delta_x, \eta_1 + \delta_x) - f(\eta)] \\
&+ \eta_1(x)(1-\eta_0(x)) \, [f(\eta_0 + \delta_x, \eta_1 - \delta_x) - f(\eta)] \Big\}.
\end{aligned}
$$

For small ν we may attempt the approach followed in [16] to deal with a mixture of Glauber dynamics and Kawasaki dynamics, where the reaction term is a perturbation of the exclusion dynamics and an *approximative* duality is exploited. In our case, the perturbation has the form of an extra term in the rate of the exclusion dynamics. For the hydrodynamic limit, the parameter ν must be scaled.

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