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I.Adan, O.Boxma, J.Resing

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# FUNCTIONAL EQUATIONS WITH MULTIPLE RECURSIVE TERMS

IVO ADAN, ONNO BOXMA, AND JACQUES RESING

ABSTRACT. In this paper we study a functional equation for generating functions of the form $f(z) = g(z)\sum_{i=1}^{M} p_i f(\alpha_i(z)) + K(z)$, viz., a recursion with multiple recursive terms. We derive and analyze the solution of this equation for the case that the $\alpha_i(z)$ are commutative contraction mappings. The results are applied to a wide range of queueing, autoregressive and branching processes.

## 1. INTRODUCTION

In many applied probability models, in particular in queueing models, the following type of recursion describes the behavior of a key performance measure:

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{k,n} + Z_n,$$

where the involved random variables are nonnegative and integer-valued. Under certain independence assumptions and ergodicity conditions, this gives rise to the following type of functional equation for the probability generating function (pgf) $f(z)$ of the steady-state distribution of the $X_n$ process:

$$f(z) = g(z)f(\alpha(z)) + K(z). \tag{1}$$

An example of such a model is a branching process with immigration (BPI). In that case the function $f(z)$ represents the pgf of the steady-state distribution of the number of individuals and the function $\alpha(z)$ the pgf of the offspring distribution. If in all states except state 0 the pgf of the immigration distribution is given by $g(z)$ while in the special state 0 the pgf of the immigration distribution is given by $g_0(z)$, then the function $f(z)$ satisfies (1) with $K(z) = f(0)\,(g_0(z) - g(z))$. Remark that in the special case that $g_0(z) = g(z)$, we have that $K(z) = 0$.

The solution of (1) is given by an expression containing an infinite product and an infinite sum (see equation (13) later on in this paper) which is obtained after iteration of equation (1). Branching processes with immigration appear for example in the analysis of the $M/G/1$ queue with (single or multiple) gated vacations (see Takagi [18], section 2.5 of Chapter 2), the $M/G/1$ queue with permanent customers (see Boxma and Cohen [3]) and, a multi-type variant, in the analysis of polling systems (see Resing [15]).

In the case that the branching process with immigration evolves in an i.i.d. random environment (BPIRE) in which the environment can be in $M$ different states, the corresponding functional equation is of the form

$$f(z) = g(z)\sum_{i=1}^{M} p_i f(\alpha_i(z)) + K(z), \tag{2}$$

where $p_i$ is the probability that the environment is in state $i$ and $\alpha_i(z)$ is the pgf of the offspring distribution when the environment is in state $i$, $i = 1, 2, \ldots, M$.

In this paper we obtain the solution of functional equation (2) in the particular case that the functions $\alpha_1(z), \ldots, \alpha_M(z)$ are commutative contraction mappings on the closed unit disk. Although (2) with multiple recursive terms is a natural extension of (1) with only a single recursive term, it is hardly studied in the queueing literature, probably because the number of different terms after the $n$-th iteration of (2) grows exponentially. That is also the reason why in this paper we restrict ourselves to the case in which the functions $\alpha_1(z), \ldots, \alpha_M(z)$ are commutative.

In Adan, Hathaway and Kulkarni [1] a specific example of (2) was analysed in detail in the study of a queueing system with two classes of impatient customers. The main goal of the present paper is to give a general treatment of Equation (2) and to show how in the commutative case the growth of the number of iteration terms can be handled. An additional aim is to treat several queueing and branching-type examples in which (2) appears, also allowing complications like the occurrence of a pole in $g(z)$ and $K(z)$.

*Organization of the paper.* In Section 2 we solve the recursion (2), both for the homogeneous case where $K(z) \equiv 0$ (Subsection 2.1) and for the inhomogeneous case (Subsection 2.2). The results are applied in the subsequent sections. We start, in Section 3, with a particular queueing model. Its choice is motivated by the fact that it provides a relatively simple illustration of the theory for the homogeneous case, while still having a few features that deviate from the setting of Section 2. Section 4 considers a special case of the branching process with immigration in a random environment, also called random coefficient integer-valued autoregressive process of order 1, in which the offspring of an individual in each environmental state can only be equal to 0 or 1. In Section 5 we consider an integer-valued reflected autoregressive process, which may be viewed as a generalization of an embededded queue length process in the $M/G/1$ queue. Section 6 is devoted to another reflected autoregressive process, this time on $[0, \infty)$. Some topics for further research are mentioned in Section 7.

## 2. The recursion

In this section we study recursion (2) for the generating function $f(z)$ of a non-negative discrete random variable $X$ with $\mathbb{E}[X] < \infty$, where $g(z)$ and $K(z)$ are analytic functions (and not necessarily generating functions), $p_1, \ldots, p_M$ is a probability distribution, and $\alpha_1(z), \ldots, \alpha_M(z)$ are commutative contraction mappings on the closed unit disk, i.e., there is a constant $\kappa < 1$ such that $|\alpha_i(z) - \alpha_i(u)| \leq \kappa |z - u|$ and $\alpha_i(\alpha_j(z)) = \alpha_j(\alpha_i(z))$ for each $i$ and $j$. For example, the mappings $\alpha_i(z) = 1 - a_i + a_i z$ with $|a_i| < 1$ are contractions with $\kappa = \max(a_1, \ldots, a_M)$, and they commute, since

$$\alpha_i(\alpha_j(z)) = 1 - a_i a_j + a_i a_j z = \alpha_j(\alpha_i(z)).$$

Note that the commutative property implies that the contractions $\alpha_i(z)$ have the same fixed point $a$. In the example above, we have $a = 1$. Equation (2) is suitable to iteratively determine $f(z)$. We distinguish between the following two cases.

### 2.1. The homogeneous case $K(z) = 0$.

After $n$ iterations of the homogeneous equation

$$f(z) = g(z) \sum_{i=1}^{M} p_i f(\alpha_i(z)), \tag{3}$$

we obtain

$$f(z) = \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1} \ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) f(\alpha_{i_1,\ldots,i_M}(z)), \tag{4}$$

where $\alpha_{i_1,\ldots,i_M}(z) = \alpha_1^{i_1}(\alpha_2^{i_2}(\cdots(\alpha_M^{i_M}(z))\cdots))$ and $\alpha_i^n(z)$ is defined as the $n$-th iterate of $\alpha_i(z)$. In particular, $\alpha_{0,\ldots,0}(z) = z$. The functions $L_{i_1,\ldots,i_M}(z)$ are recursively defined by

$$L_{i_1,\ldots,i_M}(z) = \sum_{k=1}^{M} g(\alpha_{i_1,\ldots,i_k-1,\ldots i_M}(z)) L_{i_1,\ldots,i_k-1,\ldots,i_M}(z), \tag{5}$$

with $L_{0,\ldots,0}(z) = 1$ and $L_{i_1,\ldots,i_M}(z) = 0$ if one of the indices equals $-1$. These functions can be interpreted as follows. A path from $(0,\ldots,0)$ to $(i_1,\ldots,i_M)$ is defined as a sequence of grid points that starts in $(0,\ldots,0)$ and ends in $(i_1,\ldots,i_M)$ by only taking unit steps $(0,\ldots,1,\ldots,0)$ with 1 at position $k = 1,\ldots,M$. Figure 1 shows a path from $(0,0)$ to $(3,2)$. Clearly, there are $\binom{i_1+\cdots+i_M}{i_1,\ldots,i_M}$ paths leading from $(0,\ldots,0)$ to $(i_1,\ldots,i_M)$. Now assign weight $g(\alpha_{i_1,\ldots,i_k-1,\ldots i_M}(z))$ to a step from any grid point $(i_1,\ldots,i_k-1,\ldots,i_M)$ to $(i_1,\ldots,i_M)$ and define the weight of a path as the product of weights of all steps in that path (cf. Figure 1). Then $L_{i_1,\ldots,i_M}(z)$ can be interpreted as the total weight of all $\binom{i_1+\cdots+i_M}{i_1,\ldots,i_M}$ paths from $(0,\ldots,0)$ to $(i_1,\ldots,i_M)$.

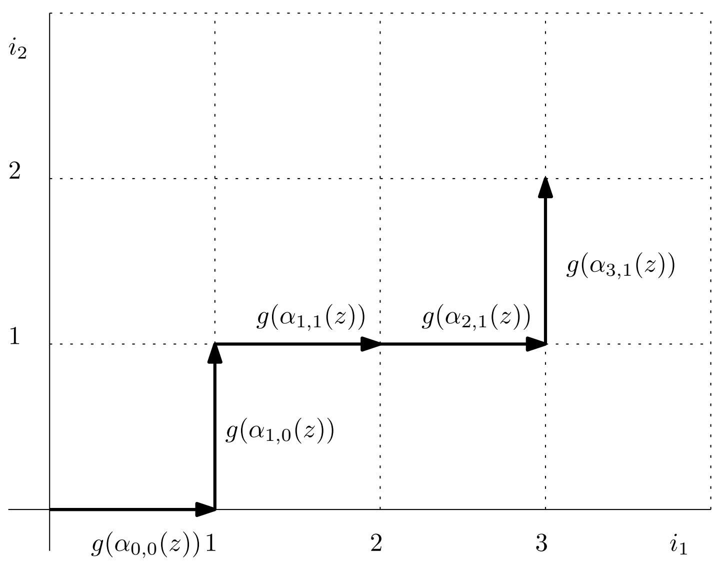


FIGURE 1. Path from $(0,0)$ to $(3,2)$ consisting of the sequence of grid points $(0,0),(1,0),(1,1),(2,1),(3,1),(3,2)$. Its path weight is the product of the step weights $g(\alpha_{0,0}(z))g(\alpha_{1,0}(z))g(\alpha_{1,1}(z))g(\alpha_{2,1}(z))g(\alpha_{3,1}(z))$.

To handle (4) we proceed as follows. Note that, when $i_1+\cdots+i_M = n$,

$$|\alpha_{i_1,\ldots,i_M}(z) - a| \le \kappa^n |z-a|, \tag{6}$$

and hence,

$$
\begin{aligned}
|f(\alpha_{i_1,\ldots,i_M}(z))-f(a)| &= |\int_{\alpha_{i_1,\ldots,i_M}(z)}^{a} f'(u)du| \\
&\leq |\alpha_{i_1,\ldots,i_M}(z)-a| \times \max_{[\alpha_{i_1,\ldots,i_M}(z),a]} |f'(u)| \\
&\leq |\alpha_{i_1,\ldots,i_M}(z)-a|\mathbb{E}[X] \\
&\leq \kappa^n |z-a|\mathbb{E}[X],
\end{aligned} \tag{7}
$$

where the integral is along the segment connecting $\alpha_{i_1,\ldots,i_M}(z)$ with $a$. Next we rewrite (4) as follows

$$
\begin{aligned}
f(z) &= \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) f(a) \\
&+ \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z)(f(\alpha_{i_1,\ldots,i_M}(z))-f(a)).
\end{aligned} \tag{8}
$$

For given $z$, the second term in the right-hand side of (8) converges to zero for $n\to\infty$. This can be seen as follows. By substituting $z=a$ in (3) we get $g(a)=1$. Hence, from (6), step weight $g(\alpha_{i_1,\ldots,i_M}(z))$ with $i_1+\cdots+i_M=n$ is close to $g(a)=1$ for all $n$ sufficiently large, say $|g(\alpha_{i_1,\ldots,i_M}(z))-1|<\epsilon<\kappa^{-1}-1$. In other words, the weight of sufficiently long paths grows at most with $1+\epsilon$ per step. So there is a constant $C$ such that for all $n$, the weight of a path from $(0,\ldots,0)$ to $(i_1,\ldots,i_M)$ with $i_1+\cdots+i_M=n$ is bounded by $C(1+\epsilon)^n$, implying that $L_{i_1,\ldots,i_M}(z)$ is bounded by $\binom{i_1+\cdots+i_M}{i_1,\ldots,i_M}C(1+\epsilon)^n$. Then, from (7), we can conclude that the second term in (8) is bounded by $C(1+\epsilon)^{n+1}\kappa^{n+1}|z-a|\mathbb{E}[X]$, which goes to zero for $n\to\infty$. We have thus proven the following theorem.

**Theorem 1.** *(Homogeneous case) The generating function $f(z)$ is given by*

$$
f(z)=\lim_{n\to\infty}\sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) f(a). \tag{9}
$$

**Remark 1.** The unknown $f(a)$ follows by substituting $z=1$ in (9) and using $f(1)=1$. This gives

$$
f(a)^{-1}=\lim_{n\to\infty}\sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(1).
$$

**Remark 2.** The first term in the right-hand side of (8) is a sum of $\binom{M+n}{M-1}$ terms. This number is $O(n^{M-1})$, which grows quickly (polynomially) in $n$ for already moderate values of $M$. However, this is not as quickly as in the non-commutative case, in which case we would have $M^{n+1}$ terms, growing exponentially in $n$.

**Remark 3.** Above we have seen that the second term in the right-hand side of (8) converges to zero geometrically fast (with rate $(1+\epsilon)\kappa$). Hence, the first term in the right-hand side of (8) will provide an accurate approximation for $f(z)$ already for small values of $n$.

2.2. **The inhomogeneous case $K(z) \neq 0$.** We now consider inhomogeneous Equation (2) and obtain after $n$ iterations,

$$f(z) = \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) f(\alpha_{i_1,\ldots,i_M}(z)) + \sum_{k=0}^{n} \sum_{i_1+\cdots+i_M=k} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) K(\alpha_{i_1,\ldots,i_M}(z)). \tag{10}$$

The first term of (10) is the same as in the homogeneous case. For given $z$, we proved that it converges if $g(a) = 1$. For convergence of the second term we need that either $K(\alpha_{i_1,\ldots,i_M}(z)) \to 0$ (Case 1) or $p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) \to 0$ (Case 2) as $i_1 + \cdots + i_M \to \infty$. In this subsection we successively consider both cases.

**Case 1**. Since $K(\alpha_{i_1,\ldots,i_M}(z)) \to K(a)$, in this case we basically assume that $K(a) = 0$. Note that, by substituting $z = a$ in (2), this assumption implies that $g(a) = 1$ if $f(a) \neq 0$, and thus that the first term in (10) converges. We now show that it also implies convergence of the second term. Similar to (7), we have for $i_1 + \cdots + i_M = n$,

$$|K(\alpha_{i_1,\ldots,i_M}(z))| = |K(\alpha_{i_1,\ldots,i_M}(z)) - K(a)| \leq \kappa^n |z-a| D, \tag{11}$$

where $D$ is the maximum value of $|K'(u)|$ in the closed disk with center $a$ and radius $|z-a|$ (intersected with the closed unit disk). Then the $k$-th term in the double sum appearing in (10) is bounded by $C(1+\epsilon)^k \kappa^k |z-a| D$, and hence, the double sum converges for $n \to \infty$. We conclude that the following holds.

**Theorem 2.** *(Inhomogeneous case) Provided $K(a) = 0$ and $f(a) \neq 0$, the generating function $f(z)$ is given by*

$$f(z) = \lim_{n\to\infty} \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) f(a) + \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_M=k} p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) K(\alpha_{i_1,\ldots,i_M}(z)). \tag{12}$$

**Remark 4.** In case $M = 1$, Equation (2) becomes

$$f(z) = g(z) f(\alpha_1(z)) + K(z),$$

yielding

$$f(z) = \prod_{n=0}^{\infty} g(\alpha_n(z)) f(a) + \sum_{k=0}^{\infty} \prod_{n=0}^{k-1} g(\alpha_n(z)) K(\alpha_k(z)). \tag{13}$$

The infinite product $\prod_{n=0}^{\infty} g(\alpha_n(z))$ converges iff $\sum_{n=0}^{\infty}(1 - g(\alpha_n(z))$ converges (cf. Chapter I of [19]). Since $g(\alpha_n(z))$ and $K(\alpha_k(z))$ converge geometrically fast to $g(a) = 1$ and $K(a) = 0$, respectively (cf. (11)), we conclude that both the infinite product and the infinite sum of products in (13) converge.

**Case 2.** We now assume that $p_1^{i_1}\ldots p_M^{i_M} L_{i_1,\ldots,i_M}(z) \to 0$ and show that the assumption $|g(a)| < 1$ is sufficient for convergence of this term to zero. For all $n$ sufficiently large, step weight $g(\alpha_{i_1,\ldots,i_M}(z))$ with $i_1 + \cdots + i_M = n$ is close to $g(a)$, say $|g(\alpha_{i_1,\ldots,i_M}(z)) - g(a)| < \epsilon$ with $|g(a)| + \epsilon < 1$. Hence, the weight of sufficiently long paths grows at most with $|g(a)| + \epsilon$ per step. So there is a constant $C$ such that for all $n$, the weight of a path from $(0, \ldots, 0)$ to

$(i_1, \dots, i_M)$ with $i_1 + \dots + i_M = n$ is bounded by $C(|g(a)| + \epsilon)^n$, implying that $L_{i_1,\dots,i_M}(z)$ is bounded by $\binom{i_1+\dots+i_M}{i_1,\dots,i_M} C(|g(a)| + \epsilon)^n$. Then, indeed, $p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(z) \to 0$ as $n \to \infty$. Further, since $K(\alpha_{i_1,\dots,i_M}(z)) \to K(a)$, these terms are bounded, say by constant $D$. Hence, we can conclude that the $k$-th term of the double sum in (10) is bounded by $C(|g(a)| + \epsilon)^k D$, and thus the double sum converges. The first term in (10) is bounded by $C(|g(a)| + \epsilon)^{n+1}$ which converges to zero as $n \to \infty$. This is summarized in the following theorem.

**Theorem 3.** *(Inhomogeneous case) Provided $|g(a)| < 1$, the generating function $f(z)$ is given by*

$$f(z) = \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_M=k} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(z) K(\alpha_{i_1,\dots,i_M}(z)). \tag{14}$$

**Remark 5.** In this section we studied Equation (2) for a generating function $f(z)$ of a non-negative discrete random variable $X$. It is readily seen that Theorems 1-3 are also valid in case $f(z)$ is the Laplace-Stieltjes transform (LST) of a non-negative continuous random variable $X$. Of course, then $\alpha_1(z, \dots, \alpha_M(z))$ are assumed to be contraction mappings on the closed positive half space (instead of the closed unit disk).

## 3. The $D_M/G/1$ shot-noise queue

In this section we consider the workload at arrival epochs for a specific queue. The analysis of the LST of the workload gives rise to a simple recursion that has the form (2), in the homogeneous variant. This section thus provides a relatively simple illustration of our theory for the homogeneous case – while still having a few features that deviate from the setting of Section 2. The model under consideration is the $D_M/G/1$ shot-noise queue; we refer to [7] for a recent survey on shot-noise queueing models. The $D_M/G/1$ shot-noise queue is a single server queue, in which the successive interarrival times $A_1, A_2, \dots$ of customers are i.i.d., with the distribution of a generic interarrival time $A$ being given by

$$\mathbb{P}(A = T_i) = p_i, \;\; i = 1, \dots, M. \tag{15}$$

The service requirements of successive customers $B_1, B_2, \dots$ are i.i.d. with finite mean, and with LST $\beta(\cdot)$; all interarrival times and service times are independent. The special feature of the model is that the server speed is workload-proportional (shot noise): when the workload is $x$, the service speed is $rx$. Let $X_n$ denote the workload just before the arrival of the $n$th customer. It is well known [2] that, in between arrivals, the workload decreases exponentially; hence

$$X_{n+1} = (X_n + B_n)\mathrm{e}^{-rA_{n+1}}, \;\; n = 1, 2, \dots \; . \tag{16}$$

It is readily verified that, due to the workload-proportional decrease, the steady-state distribution of the $\{X_n, n = 1, 2, \dots\}$ process exists. Stability conditions for queueing and storage models with more general workload-dependent decay have been discussed, a.o., by Cinlar and Pinsky [11] and Brockwell et al. [10].

Let $X$ denote a random variable with as distribution the steady-state distribution of the process $\{X_n, n = 0, 1, \dots\}$, with LST $\xi(\cdot)$; then

$$\xi(s) = \sum_{i=1}^{M} p_i \beta(a_i s) \xi(a_i s), \tag{17}$$

with $a_i := \mathrm{e}^{-rT_i}$, $i = 1, \dots, M$. Observe the differences with (3): we are now considering an LST instead of a generating function, and $\beta(a_i s)$ is *inside* the summation.

Let us first briefly consider the case of the $D/G/1$ shot-noise queue, i.e., $M = 1$. In that case, iterating (17) $n$ times gives, with $a = a_1$:

$$\xi(s) = \beta(as)\xi(as) = \beta(as)\beta(a^2 s)\xi(a^2 s) = ... = \xi(a^n s)\prod_{i=1}^{n}\beta(a^i s). \tag{18}$$

Now observe the following. Firstly, $\xi(a^n s) \to \xi(0) = 1$ for $n \to \infty$. Secondly, convergence is geometric, as

$$|1 - \xi(a^n s)| \le \int_0^\infty |1 - \mathrm{e}^{-a^n st}| \mathrm{d}\mathbb{P}(X < t) \le a^n s \int_0^\infty t \mathrm{d}\mathbb{P}(X < t) = a^n s\mathbb{E}[X]. \tag{19}$$

Thirdly, $\prod_{i=1}^\infty g_i$ converges iff $\sum_{i=1}^\infty (1 - g_i)$ converges (cf. Chapter I of [19]); hence the convergence of the product in (18) follows from

$$\sum_{i=1}^\infty |1 - \beta(a^i s)| \le \sum_{i=1}^\infty \int_0^\infty |1 - \mathrm{e}^{-a^i st}| \mathrm{d}\mathbb{P}(B < t) \le a^i s\mathbb{E}[B]. \tag{20}$$

We conclude that

$$\xi(s) = \prod_{i=1}^\infty \beta(a^i s). \tag{21}$$

Some thought will make it clear that the $i$-th term in this infinite product represents the contribution to $X$ from an arrival that occurred $i$ arrivals before the present one.

Let us now turn to the general $D_M/G/1$ shot-noise case, cf. (15). After $n$ iterations, (17) gives (very similarly to the analysis in Subsection 2.1)

$$\xi(s) = \sum_{i_1+\dots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s)\xi(a_1^{i_1} \dots a_M^{i_M} s), \tag{22}$$

where

$$L_{i_1,\dots,i_M}(s) = \beta(a_1^{i_1} \dots a_M^{i_M} s)\sum_{k=1}^M L_{i_1,\dots,i_k-1,\dots,i_M}(s), \tag{23}$$

$$L_{0,0,\dots,1,0,\dots,0}(s) = \beta(a_k s),\ k = 1, \dots, M,\ \text{with 1 on position } k. \tag{24}$$

Notice that an $(i_1, \dots, i_M)$ term corresponds to a contribution to the workload (just before an arrival epoch) from an arrival that occurred $i_1 + \dots + i_M$ arrivals before the present one, the total interval consisting of $i_k$ interarrival times of length $T_k$, $k = 1, \dots, M$, in any of $\binom{i_1+\dots+i_M}{i_1,\dots,i_M}$ orders. It is readily seen that

$$|L_{i_1,\dots,i_M}(s)| \le \binom{i_1 + \dots + i_M}{i_1, \dots, i_M}, \tag{25}$$

and hence the sum in (22) is bounded by one. Furthermore, letting $a_0 := \max(a_1, \dots, a_M)$ and observing that $a_0 = \mathrm{e}^{-r\min(T_1,\dots,T_M)} < 1$, it is seen in a similar way as above and as in

Section 2 that $\xi(a_1^{i_1} \dots a_M^{i_M} s)$ converges geometrically fast to $\xi(0) = 1$. Hence, rewriting (22) as

(26)
$$\xi(s) = \sum_{i_1+\dots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s) + \sum_{i_1+\dots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s)(\xi(a_1^{i_1} \dots a_M^{i_M} s) - 1),$$

we have the following theorem.

**Theorem 4.** *The LST of the steady-state workload just before arrival epochs in the $D_M/G/1$ shot-noise queue is given by*

$$\xi(s) = \lim_{n\to\infty} \sum_{i_1+\dots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s). \tag{27}$$

**Remark 6.** One could subsequently derive the steady-state workload LST *at an arbitrary epoch* by averaging over one arrival interval, and using a stochastic mean-value theorem.

## 4. The BPIRE or RCINAR(1) process

In this section we consider a branching process with immigration in a random environment (BPIRE process, see [14, 16]), also known as random coefficient integer-valued autoregressive process of order 1 (RCINAR(1) process, see [17, 20]). This process $\{X_n, n = 0, 1, \dots\}$ is defined as follows:

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{k,n} + Z_n, \tag{28}$$

where the $Z_n$ are nonnegative integer-valued random variables with finite mean. The $Y_{k,n}$ are Bernoulli random variables with parameter $\xi_n$, i.e., $\mathbb{P}(Y_{k,n} = 0) = 1-\xi_n$ and $\mathbb{P}(Y_{k,n} = 1) = \xi_n$; but we assume the special feature that the $\xi_n$ are themselves random variables, independent and identically distributed with $P(\xi_n = a_i) = p_i$, $i = 1, \dots, M$. Hence in generation $n$ it holds with probability $p_i$, $i = 1, \dots, M$, for all $Y_{k,n}$ that they are 1 with probability $a_i$ and 0 with probability $1 - a_i$. All $Z_j$ and $Y_{k,m}$ are also assumed to be independent. In Subsection 4.1 we consider the steady-state distribution of the process $\{X_n, n = 0, 1, \dots\}$ and in Subsection 4.2 we do this for the generalization in which the BPIRE process behaves differently at zero, i.e., when $X_n = 0$ for some $n$.

### 4.1. The steady-state case.

The generating function, $f(z)$, of the stationary distribution of the process $\{X_n, n = 0, 1, \dots\}$ satisfies the recursion

$$f(z) = g(z) \sum_{i=1}^{M} p_i f(1 - a_i + a_i z), \tag{29}$$

where $g(z)$ is the pgf of the random variable $Z_n$. Hence, we are in the homogeneous case (3) with contraction mappings $\alpha_i(z) = 1 - a_i + a_i z$. In this case, the functions $\alpha_{i_1,\dots,i_M}(z) = \alpha_1^{i_1}(\alpha_2^{i_2}(\cdots(\alpha_M^{i_M}(z))\cdots))$ are given by

$$\alpha_{i_1,\dots,i_M}(z) = 1 - \prod_{j=1}^{M} a_j^{i_j}(1 - z). \tag{30}$$

Define the functions $L_{i_1,\dots,i_M}(z)$ again recursively by (5) and use that the contraction mappings $\alpha_i(z)$ have fixed point $a=1$ in this case, and hence $f(a)=f(1)=1$. Theorem 1 now implies the following theorem.

**Theorem 5.** *The steady-state probability generating function $f(z)$ of the BPIRE process is given by*

$$f(z)=\lim_{n\to\infty}\sum_{i_1+\dots+i_M=n+1} p_1^{i_1}\dots p_M^{i_M} L_{i_1,\dots,i_M}(z). \tag{31}$$

**Remark 7.** In the special case that $p_1=1$ the recursion becomes

$$f(z)=g(z)f(1-a_1(1-z)), \tag{32}$$

yielding

$$f(z)=\prod_{j=0}^{\infty} g(1-a_1^j(1-z)). \tag{33}$$

In the special case that $a_2=\dots=a_M=0$ the recursion becomes

$$f(z)=g(z)\left[p_1 f(1-a_1(1-z)))+\sum_{i=2}^{M} p_i f(1)\right]=g(z)\left[p_1 f(1-a_1(1-z))+1-p_1\right], \tag{34}$$

and in this case the solution is given by

$$f(z)=\sum_{k=0}^{\infty}(1-p_1)p_1^k\prod_{j=0}^{k} g(1-a_1^j(1-z)). \tag{35}$$

4.2. **Deviating behaviour at zero.** In this subsection we assume that the BPIRE process behaves differently at zero, i.e., when $X_n=0$ for some $n$. In particular we assume that

$$X_{n+1}=V_n \text{ when } X_n=0, \tag{36}$$

with $V_0,V_1,\dots$ i.i.d. nonnegative integer random variables, with pgf $g_0(z)$. $V_n$ is assumed to be independent of all $Y_{i,m}$, $Z_m$ and $X_m$, $m=0,1,\dots,n$. It is readily seen that the steady state pgf $f(z)$ in this case satisfies the recursion

$$f(z)=g(z)\sum_{i=1}^{M} p_i f(1-a_i+a_i z)+f(0)[g_0(z)-g(z)]. \tag{37}$$

Hence we are in the inhomogeneous case of Equation (2) with $K(z):=f(0)[g_0(z)-g(z)]$. For future use we observe, by substituting $z=0$ in (37), that

$$f(0)=\frac{g(0)}{1+g(0)-g_0(0)}\sum_{i=1}^{M} p_i f(1-a_i). \tag{38}$$

We conclude that the following holds.

**Theorem 6.** *The probability generating function $f(z)$ is given by*

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\sum_{i_1+\dots+i_M=n+1} p_1^{i_1}\dots p_M^{i_M} L_{i_1,\dots,i_M}(z) \\ &+ \sum_{k=0}^{\infty}\sum_{i_1+\dots+i_M=k} p_1^{i_1}\dots p_M^{i_M} L_{i_1,\dots,i_M}(z) K(1-a_1^{i_1}\dots a_M^{i_M}(1-z)) \end{aligned} \tag{39}$$

*with $L_{i_1,\dots,i_M}(z)$ recursively defined by (5) with $L_{0,\dots,0}(z) = 1$ and $L_{i_1,\dots,i_M}(z) = 0$ if one of the indices equals $-1$. The constant $f(0)$, featuring in $K(z)$, is determined by substituting $z = 1 - a_i$ for $i = 1,\dots,M$ into (39), multiplying both sides by $p_i$, summing over $i$ and using (38).*

## 5. A reflected autoregressive process and an $M/G/1$ queue generalization

In this section we consider an integer-valued stochastic process $\{X_n,\ n = 0,1,\dots\}$ that is very similar to the autoregressive process of the previous section, but includes a negative component and has reflection at zero. It is determined by the following relation:

$$X_{n+1} = [\sum_{k=1}^{X_n} Y_{k,n} + Z_n - 1]^+,\ n = 0,1,\dots, \tag{40}$$

with $[x]^+ = \max(0,x)$, $Z_1, Z_2, \dots$ i.i.d. nonnegative integer-valued random variables with pgf $C(z)$ and where $Y_{k,n}$ are i.i.d. Bernoulli distributed random variables: $\mathbb{P}(Y_{k,n} = 1) = \xi_n$, $\mathbb{P}(Y_{k,n} = 0) = 1 - \xi_n$. In addition, we (again) assume the special feature that the $\xi_n$ are themselves random variables, independent and identically distributed with $\mathbb{P}(\xi_n = a_i) = p_i$, $i = 1,\dots,M$, where $\sum_{i=1}^M p_i = 1$. Hence in generation $n$ it holds with probability $p_i$, $i = 1,\dots,M$, for all $Y_{k,n}$ that they are 1 with probability $a_i$ and 0 with probability $1 - a_i$. All $Z_j$ and $Y_{k,m}$ are also assumed to be independent of each other and of all preceding $X_r$.

If all $Y_{k,n}$ are equal to one, then (40) can be interpreted as follows. Consider the number of waiting customers in the $M/G/1$ queue, *just after the beginning of the nth service*. Let $X_n$ denote this number, and let $Z_n$ denote the number of arrivals during the $n$th service. Then $X_{n+1} = [X_n + Z_n - 1]^+$. If, in addition, each of the $X_n$ customers becomes impatient with probability $1-a$ during the $n$th service and leaves, then the sequence $\{X_n\}$ satisfies (40) with $p_1 = 1$ and $a_1 = a$, i.e., with $\xi_n \equiv a$.

Without the maximum operator we have the defining recursion of an INAR(1) (integer-valued autoregressive) process, cf. Weiss [21]. We impose the stability condition that both $a_i < 1$ for all $i = 1,\dots,M$ and $\mathbb{E}[\log(1+Z)] < \infty$, cf. [6] for the case $M = 1$.

Below we show how the pgf $f(z)$ of the steady-state distribution of the $\{X_n,\ n = 0,1,\dots\}$ process can be obtained. It follows from (40), with $[x]^- = \min(0,x)$ and $X$ denoting a generic random variable with pgf $f(z)$, that

$$\begin{aligned} f(z) &= \mathbb{E}[z^{[\sum_{k=1}^X Y_k + Z - 1]^+}] = \mathbb{E}[z^{\sum_{k=1}^X Y_k + Z - 1}] + 1 - \mathbb{E}[z^{[\sum_{k=1}^X Y_k + Z - 1]^-}] \\ &= \frac{C(z)}{z}\sum_{i=1}^M p_i f(1 - a_i + a_i z) + 1 - \left[\mathbb{P}(\sum_{k=1}^X Y_k + Z - 1 \geq 0) + \frac{1}{z}\mathbb{P}(\sum_{k=1}^X Y_k + Z - 1 = -1)\right] \\ &= \frac{C(z)}{z}\sum_{i=1}^M p_i f(1 - a_i + a_i z) + (1 - \frac{1}{z})q_{-1}, \end{aligned} \tag{41}$$

where $q_{-1} := \mathbb{P}(\sum_{k=1}^X Y_k + Z - 1 = -1)$. After multiplication by $z$ and subsequent substitution of $z = 0$ we find:

$$q_{-1} = C(0)\sum_{i=1}^M p_i f(1 - a_i), \tag{42}$$

which makes sense probabilistically; the righthand side represents the probability that both $Z = 0$ and $\sum_{k=1}^{X} Y_k = 0$. We observe that (41) can be rewritten in the form

$$f(z) = g(z)\sum_{i=1}^{M} p_i f(1 - a_i + a_i z) + K(z),$$

where $g(z) = C(z)/z$ and $K(z) = q_{-1}(1 - \frac{1}{z})$. Hence we have the exact same form as (2). Furthermore, the fixed point of the iterates $\alpha_i(z) = 1 - a_i + a_i z$ is $z = 1$, and we have that $K(1) = 0$. Hence $f(z)$ is given by Theorem 2.

**Remark 8.** It should be noticed that now $g(z)$ and $K(z)$ have a pole at zero. The iterate $K(1 - a_1^{i_1} \dots a_M^{i_M}(1 - z))$ has a pole inside the unit circle if $a_1^{i_1} \dots a_M^{i_M} \in (\frac{1}{2}, 1)$. Typically this will only be the case for the first few iterations. In [6], where the case $M = 1$ is treated, it is shown that the singularities do not pose a real problem, as they are all removable singularities which are exactly compensated. In the present more general case this can be shown in a similar way, but that is beyond the scope of the present paper.

## 6. An $M/G/1$-type reflected autoregressive process

In this section we consider the following extension of a model of an autoregressive process, studied in [8]:

$$R_{n+1} = \max(A_n R_n + G_n, 0),\ n = 0, 1, ..., \tag{43}$$

where $R_0 = z$ and where, for $n = 0, 1, \cdots$, $G_n = Y_n - B_n$ with all $B_n$ independent random variables which are $\exp(\lambda)$ distributed, and all $Y_n$ non-negative i.i.d. random variables with distribution $F_Y(\cdot)$ and LST $\phi_Y(\cdot)$. In [8] one has $A_n \equiv a$ with $a \in (0, 1)$, but we now take $A_0, A_1, \dots$ i.i.d., with the following discrete distribution:

$$\mathbb{P}(A_1 = a_i) = p_i,\ i = 1, \dots, M,\ \text{with all } p_i > 0 \text{ and } \sum_{i=1}^{M} p_i = 1,\ \text{and all } a_i \in (0, 1). \tag{44}$$

In [8] both the transient and steady-state behavior of the $R_n$ process with $A_n \equiv a$ is studied, via a Wiener-Hopf technique (cf. [12]) that leads to a recursion. We apply the same tools in the extension defined by (43), (44). Below we first follow the approach of [8]. Introduce $U_n := \min(A_n R_n + G_n, 0)$ for $n = 0, 1, \dots$, and the transforms

$$R_z(r, s) := \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-sR_n} | R_0 = z],\ U_z(r, s) := \sum_{n=0}^{\infty} r^n \mathbb{E}[\mathrm{e}^{-sU_n} | R_0 = z]. \tag{45}$$

The first transform is analytic for Re $s \geq 0$ and the second one for Re $s \leq 0$. Observing that $1 + \mathrm{e}^x = \mathrm{e}^{\max(x,0)} + \mathrm{e}^{\min(x,0)}$ we have for $n = 0, 1, \dots$:

$$\mathrm{e}^{-sR_{n+1}} = \mathrm{e}^{-s(A_n R_n + G_n)} + 1 - \mathrm{e}^{-sU_n}.$$

Taking expectations and realizing that $R_n$, $A_n$ and $G_n$ are independent, we can write

$$\mathbb{E}[\mathrm{e}^{-sR_{n+1}} | R_0 = z] = \mathbb{E}[\mathrm{e}^{-sG_n}] \sum_{i=1}^{M} p_i \mathbb{E}[\mathrm{e}^{-sa_i R_n} | R_0 = z] + 1 - \mathbb{E}[\mathrm{e}^{-sU_n} | R_0 = z], \tag{46}$$

and hence, after multiplication of both sides by $r^{n+1}$ and summation, we obtain for Re $s = 0$:

$$R_z(r,s) - \mathrm{e}^{-sz} - r\phi_Y(s)\frac{\lambda}{\lambda - s}\sum_{i=1}^{M} p_i R_z(r, a_i s) = \frac{r}{1-r} - rU_z(r,s). \tag{47}$$

Restricting ourselves at this stage to Re $s = 0$ ensures that all terms are properly defined. Multiplying both sides by $\lambda - s$ one obtains:

$$(\lambda - s)R_z(r,s) - r\lambda\phi_Y(s)\sum_{i=1}^{M} p_i R_z(r, a_i s) = (\lambda - s)[\mathrm{e}^{-sz} + \frac{r}{1-r} - rU_z(r,s)]. \tag{48}$$

Because all $a_i < 1$, the steady-state distribution of the $\{R_n, n = 0, 1, \dots\}$ process always exists [13]. We shall restrict ourselves to the steady-state case. (The transient case can in principle be studied in a similar way. Here it should be observed that, with fixed point $a = 0$, we have $K(0) = 1 + \frac{r}{1-r} - rU_z(r, 0) = 1 \neq 0$. We are now in Case 2 of Subsection 2.2; $|r| < 1$ will guarantee the convergence of the corresponding $p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(z)$ as $i_1 + \dots + i_M = n \to \infty$.)

Let $R(s) = \mathbb{E}[\mathrm{e}^{-sR}]$, with $R$ a random variable with the steady-state distribution of the $R_n$ process. $U(s) = \mathbb{E}[\mathrm{e}^{-sU}]$ is similarly defined. After multiplying both sides of (48) by $1 - r$ and letting $r \to 1$, an Abelian theorem for generating functions implies that

$$(\lambda - s)R(s) - \lambda\phi_Y(s)\sum_{i=1}^{M} p_i R(a_i s) = (\lambda - s)[1 - U(s)]. \tag{49}$$

Now make the following observations:

- The lefthand side of (49) is analytic in Re $s > 0$, and continuous in Re $s \geq 0$.
- The righthand side of (49) is analytic in Re $s < 0$, and continuous in Re $s \leq 0$.
- $R(s)$ is for Re $s \geq 0$ bounded by 1, and hence the lefthand side of (49) behaves at most as a linear function in $s$ for large $s$, Re $s > 0$.
- $U(s)$ is for Re $s \leq 0$ bounded by 1, and hence the righthand side of (49) behaves at most as a linear function in $s$ for large $s$, Re $s < 0$.

Liouville's theorem [19] now implies that both sides of (49), in their respective half-planes, are equal to the same linear function in $s$. We focus on the lefthand side of (49):

$$(\lambda - s)R(s) - \lambda\phi_Y(s)\sum_{i=1}^{M} p_i R(a_i s) = C_0 + C_1 s, \ \text{Re } s \geq 0. \tag{50}$$

Substituting $s = 0$ we see that $C_0 = 0$. Taking $s \to \infty$ we see that $C_1 = -\mathbb{P}(R = 0)$, but that does not yet determine $C_1$. Taking $s = \lambda$ we observe that

$$C_1 = -\phi_Y(\lambda)\sum_{i=1}^{M} p_i R(a_i\lambda). \tag{51}$$

In fact, it is not hard to interpret this relation (replacing $C_1$ by $-\mathbb{P}(R = 0)$), using (43) and the fact that $\phi_Y(\lambda) = \mathbb{P}(B > Y)$ and $R(a_i\lambda) = \mathbb{P}(B > a_i R)$.

We can rewrite (50) as follows:

$$R(s) = H(s)\sum_{i=1}^{M} p_i R(a_i s) + K(s), \tag{52}$$

where

$$H(s) = \phi_Y(s)\frac{\lambda}{\lambda - s}, \;\; K(s) = C_1\frac{s}{\lambda - s}. \tag{53}$$

Equation (52) has exactly the same form as (2). Observe that the fixed point of the iterates $\alpha_i(z) = a_i z$ is $z = 0$, and that $K(0) = 0$. Hence Theorem 2 applies. It follows that

$$\begin{aligned} R(s) &= \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_M=k} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s) K(a_1^{i_1} \dots a_M^{i_M} s) \\ &+ \lim_{n\to\infty} \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(s). \end{aligned} \tag{54}$$

We finally need to determine the constant $C_1 = -\mathbb{P}(R = 0)$, that features in $K(s)$. This is done by taking $s = a_i\lambda$, for $i = 1, \dots, M$, in (54), and adding the resulting $M$ expressions, using (51):

$$\begin{aligned} C_1 &= -C_1\phi_Y(\lambda) \sum_{j=1}^{M} p_j \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_M=k} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(a_j\lambda) \frac{a_1^{i_1} \dots a_M^{i_M} a_j}{1 - a_1^{i_1} \dots a_M^{i_M} a_j} \\ &- \phi_Y(\lambda) \sum_{j=1}^{M} p_j \lim_{n\to\infty} \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(a_j\lambda), \end{aligned} \tag{55}$$

and hence

$$C_1 = -\frac{\phi_Y(\lambda)\sum_{j=1}^{M} p_j \;\lim_{n\to\infty} \sum_{i_1+\cdots+i_M=n+1} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(a_j\lambda),}{1 + \phi_Y(\lambda)\sum_{j=1}^{M} p_j \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_M=k} p_1^{i_1} \dots p_M^{i_M} L_{i_1,\dots,i_M}(a_j\lambda) \frac{a_1^{i_1} \dots a_M^{i_M} a_j}{1 - a_1^{i_1} \dots a_M^{i_M} a_j}}. \tag{56}$$

Just like in [8], the removable singularity $s = \lambda$ requires some extra care, but poses no real problems.

## 7. Conclusion and suggestions for further research

In this paper we have developed a method for treating recursions between random variables that lead to functional equations of the form (2). We have also presented several examples of branching processes, queueing processes and autoregressive processes, where such recursions and ensuing functional equations naturally occur. A brief (by no means exhaustive) collection of other queueing models which can be analysed with the approach of the present paper (and for which the special case of Equation (1) is treated in the following references) is (i) the $M/G/1$ queue with vacations [18], (ii) the globally gated polling model [5], (iii) the ASIP tandem model [4], and (iv) a vacation plus retrials model [9].

Several interesting research questions present themselves. We mention the following ones.

- In [1] a vector version of (2) for the LST of the virtual workload has been treated, for a specific queueing system with impatience. It would be interesting to study such a vector version in more generality. Interestingly, in [1], the mappings $\alpha_i(z)$ are of the form $\alpha_i(z) = z + \theta_i$. These commutative mappings, however, are no contractions.
- In this paper we have restricted ourselves to commutative contraction mappings. In the noncommutative case, one has an explosion of terms which no longer can be

grouped so neatly as in the analysis in Section 2. It would be interesting to investigate what can still be accomplished in the noncommutative case.

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Department of Industrial Engineering and Innovation Sciences, Eindhoven University of Technology, Eindhoven, The Netherlands
*Email address*: i.adan@tue.nl

Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands
*Email address*: o.j.boxma@tue.nl

Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands
*Email address*: j.a.c.resing@tue.nl