EURANDOM PREPRINT SERIES

2022-003

March 31, 2022

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ISSN 1389-2355

Approximation of the Distflow sequence V_n, satisfying $\Delta_2 V_n = k/V_n$, by sampling its continuous counterpart $f(t)$, satisfying $f''(t) = k/f(t)$

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Abstract.
It was shown recently in [1] that, for small positive a, $V_n - f_0(\frac{n}{N}\sqrt{a}) = o(1)$, $N \to \infty$, uniformly in $n = 0, 1, ..., N$. Here V_n, $n = 0, 1, \dots$, is defined recursively by $V_0 = 1$, $V_1 = 1 + k$; $\Delta_2 V_n = V_{n+1} - 2V_n + V_{n-1} = k/V_n$, $n = 1, 2, \dots$, with $k = a/N^2$, and $f_0(x)$, $x \geq 0$, satisfies $f_0(0) = 1$, $f_0'(0) = 0$; $f_0''(x) = 1/f_0(x)$, $x \geq 0$. In the present note, we present a refined convergence result that emerges when the initial condition $V_0 = 1$, $V_1 = 1 + kb$, with $b \in [1/2, 1]$, is coupled to an appropriately sampled and shifted version $f_0(\frac{n+\beta}{N}\sqrt{a})$ of f_0 with $\beta \in [0, 1/2]$.

1 Introduction

In [1] a particular model (Distflow model) is investigated for charging electric vehicles at N (large) charging stations connected by a power line, with the requirement that the ratio between the voltages at the root station and at the last station of the power line should stay below a level $1/(1-\Delta)$, where the tolerance Δ is small (typically of the order 0.1). Under the Distflow model, see [1], Subsection 2.3.1 and in particular (2.12–13), the normalized voltages V_n, $n = 0, 1, ..., N-1, N$, with V_n the voltage at the root station and V_0 the voltage at the last station of the power line, satisfy the recursion

$$V_0 = 1\ ,\quad V_1 = 1 + k_0\ ;\qquad V_{n+1} - 2V_n + V_{n-1} = \frac{k_n}{V_n}\ ,\quad n = 1, ..., N-1\ . \tag{1}$$

The k_n, comprising given charging rates p_n and a resistance value r, are normally small (of the order a/N^2 with $0 < a < 1/2$).

For analytically comparing the Distflow model to a linearized version (Linearized Distflow model, see [1], Subsection 2.3.2) of it, it is assumed that all k_n are equal to $k = a/N^2$, with $a \in (0, 1/2)$ independent of n. In [1], Section 5.3, (5.10–12) and Appendix C, a major effort is made to establish a relationship between the sequence V_n, $n = 0, 1, ...$, and the solution $f_0(x)$, $x \geq 0$, of the second-order boundary value problem

$$f_0''(x) = \frac{1}{f_0(x)}\ ,\quad x \geq 0\ ;\qquad f_0(0) = 1\ ,\quad f_0'(0) = 0\ . \tag{2}$$

In particular, it is shown that $V_n \to f_0(\frac{n}{N}\sqrt{a}) = f_0(n\sqrt{k})$ uniformly in $n = 0, 1, ..., N$ as $N \to \infty$ (with $a \in (0, 1/2)$ fixed). It is shown in [1], see Section 5.4 and Appendix D, that the f_0 of (2) can be related to the inverse of the imaginary error function (see Section 2 below for details), and is, therefore, analytically more tractable than the sequence V_n itself.

In [1], nothing has been said about the convergence speed in this main result (Proposition 5.1 in [1]). Moreover, the choice $f(t) = f_0(t\sqrt{k})$, $t \geq 0$, that satisfies $f'(0) = 0$, does not seem to account, as the continuous counterpart of the sequence V_n, $n = 0, 1, ...$, for the initial condition $V_0 = 1$, $V_1 = 1 + k$. In the present note, we shall consider, more generally, the sequences, recursively defined by

$$V_0 = 1\ ,\quad V_1 = 1 + kb\ ;\qquad V_{n+1} - 2V_n + V_{n-1} = \frac{k}{\sqrt{n}}\ ,\quad n = 1, 2, ...\ , \tag{3}$$

with fixed $b \in [1/2, 1]$ and, as before, $k = a/N^2$ with N large and fixed $a \in (0, 1/2)$. The sequences will be compared to the sequences of sample values $f_0((n+\beta)\sqrt{k})$, $n = 0, 1, \ldots$, of

$$f(t) = f_0((t+\beta)\sqrt{k}) \ , \qquad t \geq 0 \ , \tag{4}$$

where $\beta \in [0, 1/2]$ is fixed and f_0 is given by (2).

It will be shown that for any pair of values $b \in [1/2, 1]$, $\beta \in [0, 1/2]$

$$f(n) - \varepsilon_1 - V_n = O(nk) = O\Big(\frac{na}{N^2}\Big) \ , \qquad n = 0, 1, \ldots, N \ , \tag{5}$$

while the choice $\beta = b - 1/2$ yields the sharpening

$$f(n) - \varepsilon_1 - V_n = O(n^2k^2) = O\Big(\frac{n^2a^2}{N^4}\Big) \ , \qquad n = 0, 1, \ldots, N \ . \tag{6}$$

In (5) and (6) we have $\varepsilon_1 = f(1) - V_1 = O(k)$.

In the latter sharpening, we have $\beta = 0$ for the case that $b = 1/2$. In that case, we have $\varepsilon_1 = f(1) - V_1 \approx -\frac{1}{24}k^2$, see (29) below. Then (6) shows that $f(n) - V_n$ is very small, i.e., V_n is a very accurate estimate for $f_0(n\sqrt{k}) = f_0(\frac{n}{N}\sqrt{a})$. For instance, when $N = 2$, $a = 1$ (so that $k = a/N^2 = 1/4$), we compute

$$V_0 = 1 \, , \quad V_1 = 1 + \tfrac{1}{2}k = \tfrac{9}{8} \, , \quad V_2 = 2V_1 + \frac{k}{V_1} - V_0 = \tfrac{53}{36} = 1.4722222\ldots \ , \tag{7}$$

while $f_0(1)$ has the numerical value 1.4657576..., showing that V_2 is indeed an accurate estimate of $f_0(1)$. A similar exercise, with $N = 10$, $a = 0.1$, shows that the ensuing V_{10} estimates $f_0(1)$ with absolute accuracy 2.45×10^{-4}.

2 Preliminaries about f_0

By definition, we have for $x \geq 0$

$$f_0''(x) = \frac{1}{f_0(x)} \ ; \qquad f_0(0) = 1 \, , \quad f'(0) = 0 \ . \tag{8}$$

The function $f_0(x)$ is positive, increasing and convex. We have, see [1], Appendix D,

$$f_0(x) = \exp(U^2(x)) \ , \qquad \int_0^{U(x)} e^{v^2}\, dv = \frac{x}{\sqrt{2}} \, , \quad x \geq 0 \ . \tag{9}$$

Using the asympotics of the Dawson integral $\exp(-y^2) \int_0^y \exp(v^2)\,dv$, $y \to \infty$, see [2], it can be shown that $f_0(x) \sim x(2\ln x)^{1/2}$, $x \to \infty$.

There is the power series

$$f_0(x) = \sum_{l=0}^{\infty} c_l x^l = 1 + \tfrac{1}{2}\,x^2 - \tfrac{1}{24}\,x^4 + \tfrac{7}{720}\,x^6 - \tfrac{127}{40320}\,x^8 + \ldots \,. \tag{10}$$

Using (8), the c_l can be computed recursively via

$$\begin{aligned} &c_0 = 1\,, \quad c_1 = 0\,, \quad c_2 = 1/2\,; \\ &c_{l+2} = -\sum_{j=0}^{l-1} \frac{(j+2)(j+1)}{(l+2)(l+1)}\, c_{j+2}c_{l-j}\,, \quad l = 1, 2, \ldots\,. \end{aligned} \tag{11}$$

We have furthermore

$$c_{2k+1} = 0\,, \quad c_{2k+2} = \frac{(-1)^k\, 2^k\, C_{k+1}}{(k+1)\,\pi^{k+1/2}}\,, \quad k = 0, 1, \ldots\,, \tag{12}$$

see [3], in particular Table II, where the C_j are given numerically for $j = 1, 2, \ldots, 200$. The C_j are the power series coefficients in

$$\mathrm{inverf}(z) = \sum_{j=1}^{\infty} C_j\, z^{2j-1}\,, \qquad |z| < 1\,, \tag{13}$$

where $\mathrm{inverf}(z)$ is the inverse of the error function

$$\mathrm{erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-t^2}\, dt\,, \qquad w \in \mathbb{C}\,. \tag{14}$$

The function $f_0(z)$ is analytic in the whole complex plane, with the exception of the branch cuts $[i\sqrt{\frac{\pi}{2}}, i\infty)$ and $(-i\infty, -i\sqrt{\frac{\pi}{2}}]$. Hence, the power series in (10) has radius of convergence equal to $\sqrt{\frac{\pi}{2}}$. The C_j in (13) satisfy

$$C_j \sim \frac{1}{2j}\,\frac{1}{\sqrt{\ln(2j)}}\,, \qquad j \to \infty\,, \tag{15}$$

(private communication N.M. Temme, November 2020).

Lower and upper bounds for $f_0(x)$, sharp for larger values of x, can be obtained from corresponding bounds for Dawson's function, see [2], Section 4.

We have the following formulas for the derivatives of f_0 (also see [1], Appendix D, (D.4)):

$$f_0'(x) = (2\ln f_0(x))^{1/2}\ , \qquad f_0''(x) = \frac{1}{f_0(x)}\ , \tag{16}$$

$$f_0^{(3)}(x) = \frac{-f_0'(x)}{f_0^2(x)}\ , \qquad f_0^{(4)}(x) = \frac{2(f_0'(x))^2 - 1}{f_0^3(x)}\ , \tag{17}$$

$$f_0^{(5)}(x) = \frac{7f_0'(x) - 6(f_0'(x))^3}{f_0^4(x)}\ , \qquad f_0^{(6)}(x) = \frac{7 - 46(f_0'(x))^2 + 24(f_0'(x))^4}{f_0^5(x)}\ . \tag{18}$$

By Taylor's theorem, we have for $x > 0$

$$f_0(x) = 1 + \tfrac{1}{2}\,x^2\,f_0''(\xi_{x,2}) = 1 + \tfrac{1}{2}\,x^2 + \tfrac{1}{24}\,x^4\,f_0^{(4)}(\xi_{x,4})\ , \tag{19}$$

for some $\xi_{x,2}, \xi_{x,4} \in [0, x]$. Therefore, since

$$0 \le f_0''(\xi) \le 1\ , \qquad f_0^{(4)}(\xi) \ge -1\ , \qquad \xi \ge 0\ , \tag{20}$$

we have

$$1 \le 1 + \tfrac{1}{2}\,x^2 - \tfrac{1}{24}\,x^4 \le f_0(x) \le 1 + \tfrac{1}{2}\,x^2\ , \qquad 0 \le x \le \sqrt{12}\ . \tag{21}$$

For instance, $1 \le 1.4583... \le f_0(1) = 1.4657... \le 1.50$ (case $x = 1$).

The following table gives numerical values of $f_0(x)$ for $x = 0.00(0.05)1.05$

x	$f_0(x)$	x	$f_0(x)$
0.00	1.000000000	0.55	1.147682689
0.05	1.001249740	0.60	1.175007022
0.10	1.004995843	0.65	1.204458888
0.15	1.011229016	0.70	1.235986299
0.20	1.019933948	0.75	1.269536343
0.25	1.031089567	0.80	1.305059980
0.30	1.044669388	0.85	1.342490497
0.35	1.060641934	0.90	1.381787644
0.40	1.078971210	0.95	1.422894099
0.45	1.099617219	1.00	1.465757611
0.50	1.122536503	1.05	1.510326813

The numerical values of $f_0(x)$ were obtained by solving the second equation in (9) for $U(x) = (\ln f_0(x))^{1/2}$ using Newton's method. As initialization, we used the recursion in (3) with $b = 1/2$ and $b = 1$, respectively, and $k = 0.01$, to compute estimates V_n of $f_0(0.1n)$ and $f_0(0.1(n + 1/2))$, $n = 0, 1, ..., 10$, respectively. Also see Section 6, where we employ (10) for a similar purpose.

3 Comparing V_n and $f(n)$

We write the recursive relation in (3) as

$$V_{n+1} = V_n + (V_n - V_{n-1}) + \frac{k}{V_n}\ , \qquad n = 1, 2, \dots\ . \tag{22}$$

With $f(t) = f_0((t+\beta)\sqrt{k})$, $t \geq 0$, see (4), we have $f''(t) = k/f(t)$, $t \geq 0$, and so

$$\begin{aligned} f(n+1) &= f(n) + f'(n - \tfrac{1}{2}) + \int_{-1/2}^{1} f''(n+v)\, r(v)\, dv \\ &= f(n) + f'(n - \tfrac{1}{2}) + \int_{-1/2}^{1} \frac{k\, r(v)}{f(n+v)}\, dv\ , \qquad n = 1, 2, \dots\ , \end{aligned} \tag{23}$$

where

$$r(v) = \begin{cases} 1 & , \quad -1/2 \leq v \leq 0\ , \\ 1 - v & , \quad 0 \leq v \leq 1\ , \\ 0 & , \quad \text{otherwise}\ . \end{cases} \tag{24}$$

Observe that $r(v) \geq 0$, and that $\int_{-1/2}^{1} r(v)\, dv = 1$. We shall therefore compare for $n = 1, 2, \dots$

$$f(n) \text{ to } V_n\ , \quad f'(n - 1/2) \text{ to } V_n - V_{n-1}\ , \quad \int_{-1/2}^{1} \frac{k\, r(v)}{f(n+v)}\, dv \text{ to } \frac{k}{V_n}\ . \tag{25}$$

Set

$$\varepsilon_n = f(n) - V_n\ , \qquad n = 0, 1, \dots\ . \tag{26}$$

From the initial conditions $V_0 = 1$, $V_1 = 1 + kb$ in (3), the definition of $f(t)$ as $f_0((t+\beta)\sqrt{k})$ in (4), and the Taylor expansion of $f_0(x)$ in (10), we get

$$\varepsilon_0 = f_0(\beta\sqrt{k}) - 1 = \tfrac{1}{2}\,\beta^2 k - \tfrac{1}{24}\,\beta^4 k^2 + \tfrac{7}{720}\,\beta^6 k^3 - \dots\ , \tag{27}$$

$$\begin{aligned} \varepsilon_1 &= f_0((\beta+1)\sqrt{k}) - (1 + kb) \\ &= (\tfrac{1}{2}\,(\beta+1)^2 - b)\, k - \tfrac{1}{24}\,(\beta+1)^4 k^2 + \tfrac{7}{720}\,(\beta+1)^6 k^3 - \dots\ . \end{aligned} \tag{28}$$

For later use, we also mention that

$$\varepsilon_1 - \varepsilon_0 = (\beta + \tfrac{1}{2} - b)\, k - \tfrac{1}{24}\, ((\beta+1)^4 - \beta^4)\, k^2 + \tfrac{7}{720}\, ((\beta+1)^6 - \beta^6)\, k^3 - \ldots \,. \quad (29)$$

Next, for $n = 1, 2, \ldots$

$$f'(n - \tfrac{1}{2}) - (V_n - V_{n-1}) = f'(n + \tfrac{1}{2}) - (f(n) - f(n-1)) + \varepsilon_n - \varepsilon_{n-1}\,. \quad (30)$$

Finally, for $n = 1, 2, \ldots$

$$\begin{aligned}
&\int_{-1/2}^{1} \frac{k\, r(v)}{f(n+v)}\, dv - \frac{k}{V_n} \\
&= \frac{k}{f(n)} - \frac{k}{V_n} + \int_{-1/2}^{1} k\, r(v) \Big[\frac{1}{f(n+v)} - \frac{1}{f(n)}\Big]\, dv \\
&= -\frac{k\, \varepsilon_n}{f(n)\, V_n} + \int_{-1/2}^{1} r(v)\, [f''(n+v) - f''(n)]\, dv \\
&= -\frac{k\, \varepsilon_1}{f(n)\, V_n} - \frac{k(\varepsilon_n - \varepsilon_1)}{f(n)\, V_n} + \int_{-1/2}^{1} r(v)\, [f''(n+v) - f''(n)]\, dv\,. \quad (31)
\end{aligned}$$

Thus, we conclude from (22), (23), (26), (30) and (31) that for $n = 1, 2, \ldots$

$$\begin{aligned}
f(n+1) &= f(n) + f'(n - \tfrac{1}{2}) + \int_{-1/2}^{1} \frac{k\, r(v)}{f(n+v)}\, dv \\
&= V_n + (V_n - V_{n-1}) + \frac{k}{V_n} \\
&\quad + (f(n) - V_n) + (f'(n - \tfrac{1}{2}) - (V_n - V_{n-1})) \\
&\quad + \Big(\int_{-1/2}^{1} \frac{k\, r(v)}{f(n+v)}\, dv - \frac{k}{V_n}\Big) \\
&= V_{n+1} + \varepsilon_n + (f'(n - \tfrac{1}{2}) - (f(n) - f(n-1) + \varepsilon_n - \varepsilon_{n-1})
\end{aligned}$$

$$
\begin{aligned}
& + \Big(- \frac{k\,\varepsilon_1}{f(n)\,V_n} - \frac{k(\varepsilon_n - \varepsilon_1)}{f(n)\,V_n}\Big) \\
& + \int\limits_{-1/2}^{1} r(v)\,[f''(n+v) - f''(n)]\,dv \\
= \; & V_{n+1} + \varepsilon_{n+1} \ , \qquad (32)
\end{aligned}
$$

where

$$\varepsilon_{n+1} = 2\varepsilon_n - \varepsilon_{n-1} + \tau_n \ , \qquad (33)$$

with

$$
\begin{aligned}
\tau_n \;=\; & - \frac{k\,\varepsilon_1}{f(n)\,V_n} + \Big(f'(n - \tfrac{1}{2}) - (f(n) - f(n-1)) \\
& \qquad + \int\limits_{-1/2}^{1} r(v)\,[f''(n+v) - f''(n)]\,dv\Big) \\
& - \frac{k(\varepsilon_n - \varepsilon_1)}{f(n)\,V_n} \ . \qquad (34)
\end{aligned}
$$

We find from (33)

$$(\varepsilon_{i+1} - \varepsilon_i) - (\varepsilon_i - \varepsilon_{i-1}) = \tau_i \ , \qquad i = 1, 2, ... \ , \qquad (35)$$

and so, by summation over $i = 1, 2, ..., j$,

$$\varepsilon_{j+1} - \varepsilon_j = \varepsilon_1 - \varepsilon_0 + \sum_{i=1}^{j} \tau_i \ , \qquad j = 1, 2, ... \ , \qquad (36)$$

and, by summation over $j = 1, 2, ..., n$,

$$\varepsilon_{n+1} - \varepsilon_1 = n(\varepsilon_1 - \varepsilon_0) + \sum_{j=1}^{n} \sum_{i=1}^{j} \tau_i \ , \qquad n = 1, 2, ... \ . \qquad (37)$$

In the next section, we shall put effort in estimating the quantities that occur in the expression (34) for τ_n. We thus obtain estimates and bounds for $\varepsilon_{n+1} - \varepsilon_1$ that can be used to get an approximation of

$$V_{n+1} = -\varepsilon_{n+1} + f(n+1) = -\varepsilon_1 + f(n+1) - (\varepsilon_{n+1} - \varepsilon_1) \ , \qquad (38)$$

with $n = 1, 2, ..., N-1$.

4 Estimating τ_n

We recall that we have k of the form a/N^2 with $a \in (0, 1/2)$ and $N \to \infty$. We consider the quantity, see (34)

$$Q_n := f'(n - \tfrac{1}{2}) - (f(n) - f(n-1)) + \int_{-1/2}^{1} r(v)\,[f''(n+v) - f''(n)]\,dv\ . \quad (39)$$

With $f(t) = f_0((t+\beta)\sqrt{k})$, the analyticity properties of f_0, as noted in Section 2, we can safely use Taylor expansions of $f(n - \frac{1}{2} \pm \frac{1}{2})$ and $f''(n+v)$. Thus we have

$$\begin{aligned} f(n - \tfrac{1}{2} \pm \tfrac{1}{2}) &= f(n - \tfrac{1}{2}) + f'(n - \tfrac{1}{2})(\pm\tfrac{1}{2}) + \tfrac{1}{2}\,f''(n - \tfrac{1}{2})(\pm\tfrac{1}{2})^2 \\ &\quad + \tfrac{1}{6}\,f^{(3)}(n - \tfrac{1}{2})(\pm\tfrac{1}{2})^3 + \tfrac{1}{24}\,f^{(4)}(n - \tfrac{1}{2})(\pm\tfrac{1}{2})^4 \\ &\quad + \tfrac{1}{120}\,f^{(5)}(n - \tfrac{1}{2})(\pm\tfrac{1}{2})^5 + \ldots\ , \end{aligned} \quad (40)$$

and we obtain

$$f'(n - \tfrac{1}{2}) - (f(n) - f(n-1)) = -\tfrac{1}{24}\,f^{(3)}(n - \tfrac{1}{2}) - \tfrac{1}{1920}\,f^{(5)}(n - \tfrac{1}{2}) - \ldots\ . \quad (41)$$

We have similarly from

$$f''(n+v) - f''(n) = f^{(3)}(n)\,v + \tfrac{1}{2}\,f^{(4)}(n)\,v^2 + \tfrac{1}{6}\,f^{(5)}(n)\,v^3 + \ldots \quad (42)$$

that

$$\int_{-1/2}^{1} r(v)\,[f''(n+v) - f''(n)]\,dv = \tfrac{1}{24}\,f^{(3)}(n) + \tfrac{1}{16}\,f^{(4)}(n) + \tfrac{11}{1920}\,f^{(5)}(n)\ \ldots \quad (43)$$

where it has been used that

$$\int_{-1/2}^{1} v^j\,r(v)\,dv = \tfrac{1}{24}\,,\ \tfrac{1}{8}\,,\ \tfrac{11}{320}\ , \qquad j = 1, 2, 3\ . \quad (44)$$

Thus we get for Q_n in (39) the expression

$$\begin{aligned} &-\tfrac{1}{24}\,f^{(3)}(n - \tfrac{1}{2}) - \tfrac{1}{1920}\,f^{(5)}(n - \tfrac{1}{2}) - \ldots \\ &+ \tfrac{1}{24}\,f^{(3)}(n) + \tfrac{1}{16}\,f^{(4)}(n) + \tfrac{11}{1920}\,f^{(5)}(n) + \ldots\ . \end{aligned} \quad (45)$$

We shall now argue that we can neglect the terms involving $f^{(5)}$ in (45), compared to the combined terms involving $f^{(3)}$ and $f^{(4)}$, at the expense of a relative error of the order $\sqrt{k}$. We have by the mean value theorem

$$\begin{aligned} & \tfrac{-1}{24}\, f^{(3)}(n-\tfrac{1}{2}) + \tfrac{1}{24}\, f^{(3)}(n) + \tfrac{1}{16}\, f^{(4)}(n) \\ = \; & \tfrac{1}{48}\, f^{(4)}(\xi_n) + \tfrac{1}{16}\, f^{(4)}(n) = \tfrac{1}{12}\, f^{(4)}(\vartheta_n) \;, \end{aligned} \tag{46}$$

with numbers $\xi_n, \vartheta_n \in [n-\frac{1}{2}, n]$. Also,

$$\tfrac{-1}{1920}\, f^{(5)}(n-\tfrac{1}{2}) + \tfrac{11}{1920}\, f^{(5)}(n) \text{ is of the order } \tfrac{1}{192}\, f^{(5)}(\eta_n) \;, \tag{47}$$

with a number $\eta_n \in [n-\frac{1}{2}, n]$. Thus we should bound

$$\frac{\frac{1}{192}\, f^{(5)}(\eta_n)}{\frac{1}{12}\, f^{(4)}(\vartheta_n)} = \frac{1}{16}\, \frac{f^{(5)}(\eta_n)}{f^{(4)}(\vartheta_n)} \approx \frac{1}{16}\, \frac{f^{(5)}(n)}{f^{(4)}(n)} \;. \tag{48}$$

We observe that

$$f^{(j)}(t) = k^{\frac{1}{2}j}\, f_0^{(j)}((t+\beta)\sqrt{k}) \;. \tag{49}$$

With $x = (n+\beta)\sqrt{k} \le \sqrt{a} \le \frac{1}{2}\sqrt{2}$, and the explicit expressions for $f_0^{(j)}$ in (16–18), we should bound

$$\begin{aligned} \frac{\sqrt{k}\, f_0^{(5)}(x)}{f_0^{(4)}(x)} &= \frac{\sqrt{k}}{16 f_0(x)}\, \frac{7 f_0'(x) - 6 (f_0'(x))^3}{2 (f_0'(x))^2 - 1} \\ &= -\frac{\sqrt{k}}{16 f_0(x)}\, \frac{7t - 6t^3}{1-2t^2} \;, \qquad 0 \le x \le \tfrac{1}{2}\sqrt{2} \;, \end{aligned} \tag{50}$$

where we have set $t = f_0'(x)$. The function $f_0(x)$ varies gently between 1 and $f_0(\frac{1}{2}\sqrt{2}) = 1.2406$ when $x \in [0, \frac{1}{2}\sqrt{2}]$ and is therefore harmless. On the other hand, $t = f_0'(x)$ increases from 0 at $x = 0$ to 0.6567 at $x = \frac{1}{2}\sqrt{2}$, and is therefore, because of the denominator $1 - 2t^2$ in the second line of (50), of chief importance in (50) when x is close to $\frac{1}{2}\sqrt{2} = 0.7071$. We evaluate

$$\frac{1}{16 f_0(x)}\, \frac{7t - 6t^3}{1-2t^2} \Big|_{x=\frac{1}{2}\sqrt{2}} = 1.0616 \;. \tag{51}$$

We conclude that the ratio in (48) is bounded between 0 and $-c\sqrt{k}$ with c of the order unity.

We thus approximate the quantity Q_n in (39) by $\frac{1}{12}\, f^{(4)}(\vartheta_n)$ at the expense of a relative error $O(\sqrt{k})$. By (49) while the expression in (17) for $f_0^{(4)}(x)$ shows that $-1 \le f_0^{(4)}(x) \le 0$ when $0 \le x \le \frac{1}{2}\sqrt{2}$, we conclude that

$$Q_n \approx \tfrac{1}{12}\, f^{(4)}(\vartheta_n) = O(k^2) \;. \tag{52}$$

We proceed by considering the quantity $-k\,\varepsilon_1/f(n)\,V_n$ in (34). We have $f(n) \geq 1$, $V_n \geq 1$, and so by (28)

$$\frac{-k\,\varepsilon_1}{f(n)\,V_n} = -\,\frac{\frac{1}{2}\,(\beta+1)^2 - b}{f(n)\,V_n}\,k^2 + O(k^3)\ , \tag{53}$$

where the first quantity at the right-hand side has modulus $\leq |\frac{1}{2}\,(\beta+1)^2 - b|\,k^2$; since $\beta \in [0, 1/2]$, $b \in [1/2, 1]$, we have that $\frac{1}{2}\,(\beta+1)^2 - b \in [-\frac{1}{2}\,,\,\frac{5}{8}]$. We conclude from (34), (39) and (52), (53)

$$\tau_n = -\,\frac{k(\varepsilon_n - \varepsilon_1)}{f(n)\,V_n} + O(k^2)\ , \tag{54}$$

the O holding uniformly in $n = 1, 2, ..., N-1$ as $N \to \infty$.

The contribution to the double series $\sum_{j=1}^{n}\,\sum_{i=1}^{j}\,\tau_i$ of the O-term in (54) is $O(k^2 n^2) = O(n^2 a^2/N^4)$. In the next section it will be indicated that this is enough to establish the first main result in (5) from (37).

In the case that $\beta = b - 1/2$ (the condition under which the second main result, see (6), is to be established), we must be more precise about the $O(k^2)$-term in (54). This $O(k^2)$-term arises as

$$\sigma_i := \tfrac{1}{12}\,f^{(4)}(\vartheta_i) - \frac{k\,\varepsilon_1}{f(i)\,V_i} \tag{55}$$

from (52) and (53), and gives rise to a contribution

$$\sum_{j=1}^{n}\,\sum_{i=1}^{j}\,\sigma_i \tag{56}$$

to the double series $\sum_{j=1}^{n}\,\sum_{i=1}^{j}\,\tau_i$ in (37). Using that

$$\varepsilon_1 = (\tfrac{1}{2}\,(\beta+1)^2 - b)\,k + O(k^2) = \tfrac{1}{2}\,(b-\tfrac{1}{2})^2\,k + O(k^2) \tag{57}$$

when $\beta = b - 1/2$, we have

$$\sigma_i \approx \tfrac{1}{12}\,f_0^{(4)}(x_i)\,k^2 - \frac{(b-1/2)^2}{f(i)\,V_i}\,k^2\ , \tag{58}$$

where $x_i = (\vartheta_i + \beta)\sqrt{k}$ is close to $i\sqrt{k}$. We then approximate

$$\sum_{j=1}^{n}\sum_{i=1}^{j} f_0^{(4)}(x_i) = \sum_{i=1}^{n}(n+1-i)\, f_0^{(4)}\Big(\frac{i+\beta}{N}\sqrt{a}\Big)$$

$$\approx \sum_{i=0}^{n}(n-i)\, f_0^{(4)}\Big(\frac{i\sqrt{a}}{N}\Big) = \frac{N^2}{a}\sum_{i=0}^{n}\frac{\sqrt{a}}{N}\Big(\frac{n\sqrt{a}}{N}-\frac{i\sqrt{a}}{N}\Big) f_0^{(4)}\Big(\frac{i\sqrt{a}}{N}\Big)$$

$$\approx \frac{N^2}{a}\int_0^y (y-x)\, f_0^{(4)}(x)\, dx\ , \qquad y = \frac{n\sqrt{a}}{N}\ . \tag{59}$$

By partial integration

$$\int_0^y (y-x)\, f_0^{(4)}(x)\, dx = (y-x)\, f_0^{(3)}(y)\Big|_0^y + \int_0^y f_0^{(3)}(x)\, dx$$

$$= 0 + f_0^{(2)}(x)\Big|_0^y = \frac{1}{f_0(y)} - 1\ , \tag{60}$$

where it has been used that $f^{(3)}(0) = 0$, $f_0^{(2)}(x) = 1/f_0(x)$. Thus

$$\sum_{j=1}^{n}\sum_{i=1}^{j} f_0^{(4)}(x_i) \approx -\frac{N^2}{a}\Big(1-\frac{1}{f_0(y)}\Big)\ , \qquad y = \frac{n\sqrt{a}}{N}\ . \tag{61}$$

We also approximate

$$\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{1}{f(i)\,V_i} \approx \sum_{i=1}^{n}\frac{n+1-i}{f_0^2(\frac{i+\beta}{N}\sqrt{a})}$$

$$\approx \frac{N^2}{a}\int_0^y \frac{y-x}{f_0^2(x)}\, dx\ , \qquad y = \frac{n\sqrt{a}}{N}\ . \tag{62}$$

Hence, from (61) and (62), we get

$$\sum_{j=1}^{n}\sum_{i=1}^{j}\sigma_i \approx -\tfrac{1}{12}k^2\frac{N^2}{a}\Big(1-\frac{1}{f_0(y)}\Big) - (b-\tfrac{1}{2})^2 k^2\frac{N^2}{a}\int_0^y \frac{y-x}{f_0^2(x)}\, dx$$

$$= \; -\tfrac{1}{12}\,k^2n^2\,\frac{1}{y^2}\Big(1-\frac{1}{f_0(y)}\Big)-(b-\tfrac{1}{2})^2\,k^2n^2\,\frac{1}{y^2}\int\limits_0^y\frac{y-x}{f_0^2(x)}\,dx\;,$$

$$y=\frac{n\,\sqrt{a}}{N}\;. \tag{63}$$

The second line of (63) can be further approximated by using that, see (10),

$$f_0(x)=1+\tfrac{1}{2}\,x^2-\tfrac{1}{24}\,x^4+\ldots\;. \tag{64}$$

This gives

$$\frac{1}{y^2}\Big(1-\frac{1}{f_0(y)}\Big)=\tfrac{1}{2}\,(1-\tfrac{7}{12}\,y^2+\ldots)\,,\;\;\frac{1}{y^2}\int\limits_0^y\frac{y-x}{f_0^2(x)}\,dx=\tfrac{1}{2}\,(1-\tfrac{1}{6}\,y^2+\ldots)\;, \tag{65}$$

and we obtain

$$\sum_{j=1}^{n}\;\sum_{i=1}^{j}\sigma_i\approx-\tfrac{1}{24}\,k^2n^2(1-\tfrac{7}{12}\,y^2+\ldots)-\tfrac{1}{2}\,(b-\tfrac{1}{2})^2\,k^2n^2(1-\tfrac{1}{6}\,y^2+\ldots)\;,$$

$$y=\frac{n\,\sqrt{a}}{N}\;. \tag{66}$$

5 Proof of the main results

We start with the proof of (5) that takes the form

$$\varepsilon_n-\varepsilon_1=O(nk)\;,\qquad 0,1,\ldots,N\;. \tag{67}$$

The cases $n=0,1$ are trivial or settled by (29), so we may restrict to the cases $n=2,3,\ldots,N$. According to (37) and (54), we have for $n=1,2,\ldots,N-1$

$$\varepsilon_{n+1}-\varepsilon_1=n(\varepsilon_1-\varepsilon_0)-k\sum_{j=1}^{n}\;\sum_{i=1}^{j}\frac{\varepsilon_i-\varepsilon_1}{f(i)\,V_i}+O\Big(\frac{n^2a^2}{N^4}\Big)\;. \tag{68}$$

We have

$$\varepsilon_1-\varepsilon_0=(\beta+\tfrac{1}{2}-b)\,k+O(k^2)\;,\qquad\frac{n^2a^2}{N^4}=k\,\frac{n^2a}{N^2}\le ka\;. \tag{69}$$

Now, by induction in (68) using that $f(i)$, $V_i\ge 1$, we have that $\varepsilon_{n+1}-\varepsilon_1=O(nk)$. In the induction step we use that

$$k\sum_{j=1}^{n}\;\sum_{i=1}^{j}\frac{O((i-1)\,k)}{f(i)\,V_i}=O(k^2n^3)=O\Big(nk\,\frac{n^2a^2}{N^2}\Big)\;. \tag{70}$$

This already shows (5).

The above argument applies to both the case that $\beta + \frac{1}{2} - b = 0$ (which we consider in detail later), and the case that $\beta + \frac{1}{2} - b$ is not close to 0 which we consider now. In the latter case, see (69), the $O(\frac{n^2a^2}{N^4})$-term in (68) is at least a factor $\frac{na}{N^2}$ smaller than $|n(\varepsilon_1 - \varepsilon_0)|$. Ignoring this smaller term, we see by induction that $\varepsilon_{n+1} - \varepsilon_1$ has the same sign as $\varepsilon_1 - \varepsilon_0$, and that $|\varepsilon_{n+1} - \varepsilon_1| \leq n\,|\varepsilon_1 - \varepsilon_0|$. Therefore,

$$k \sum_{j=1}^{n} \sum_{i=1}^{j} \frac{\varepsilon_i - \varepsilon_1}{f(i)\,V_i} \quad \text{has the same sign as} \quad \varepsilon_1 - \varepsilon_0 \ , \tag{71}$$

and

$$\begin{aligned} \Big| k \sum_{j=1}^{n} \sum_{i=1}^{j} \frac{\varepsilon_i - \varepsilon_1}{f(i)\,V_i} \Big| \;&\leq\; k\,|\varepsilon_1 - \varepsilon_0| \sum_{j=1}^{n} \sum_{i=1}^{j} (i-1) \\ &=\; \tfrac{1}{6}\,kn(n^2-1)\,|\varepsilon_1 - \varepsilon_0| \leq \tfrac{1}{6}\,n\,|\varepsilon_1 - \varepsilon_0|\,\frac{n^2a}{N^2} \ . \end{aligned} \tag{72}$$

Thus we find that for $n = 1, 2, ..., N-1$

$$\varepsilon_{n+1} - \varepsilon_1 = n(\varepsilon_1 - \varepsilon_0)(1 - \delta_n) \ , \qquad 0 \leq \delta_n \leq \tfrac{1}{6}\,\frac{n^2}{N^2}\,a \ , \tag{73}$$

a result that yields a sharpening of (5) in the sense that we are more precise about the implicit constant in the O of (5).

We now consider the case that $\beta + \frac{1}{2} - b = 0$. Then

$$\varepsilon_1 - \varepsilon_0 = -\tfrac{1}{24}\,((\beta+1)^4 - \beta^4)\,k^2 + O(k^3) \ , \tag{74}$$

and so $n(\varepsilon_1 - \varepsilon_0)$ is much smaller than $\frac{n^2a^2}{N^4}$, see (68). In (55)–(66), we have been much more precise about the $O(\frac{n^2a^2}{N^4})$-term in (68), with (63) or (66) as final result. The second term at the right-hand side of (68) is even smaller than $n(\varepsilon_1 - \varepsilon_0)$, compare the developments to obtain (73) by incorporating this second term in the case that $\beta + \frac{1}{2} - b$ is away from 0. Ignoring this smaller term, we get

$$\varepsilon_{n+1} - \varepsilon_1 \approx n(\varepsilon_1 - \varepsilon_0) - C\,k^2n^2 \ , \qquad n = 1, ..., N-1 \ , \tag{75}$$

where $C = C(y)$ is given by

$$\begin{aligned} C \;&=\; \frac{1}{12y^2}\Big(1 - \frac{1}{f_0(y)}\Big) + \frac{(b-1/2)^2}{y^2} \int_0^y \frac{y-x}{f_0^2(x)}\,dx \\ &=\; \tfrac{1}{24}\,(1 - \tfrac{7}{12}\,y^2 + ...) + \tfrac{1}{2}(b - \tfrac{1}{2})^2\,(1 - \tfrac{1}{6}\,y^2 + ...)\,, \qquad y = \frac{n\,\sqrt{a}}{N} \ . \end{aligned} \tag{76}$$

Thus, (75) gives a sharp form of the result (6), in the sense that we are more precise about the implicit constant in the O of (6), also including the lower-order term $n(\varepsilon_1 - \varepsilon_0)$. Observe that the case $b = 1/2$ is special for then the second term on the second line of (76) vanishes.

6 Numerical illustration

We recall (38), so that we have for $n = 1, 2, ..., N$

$$V_n = -\varepsilon_n + f(n) = -\varepsilon_1 + f(n) - (\varepsilon_n - \varepsilon_1) \tag{77}$$

(the case $n = 1$ in (77), not covered by (38), holds trivially). Accordingly, we can consider both $f(n)$ and $-\varepsilon_1 + f(n)$ as an approximation of V_n. We present in this section numerical results for the following 4 cases:

$$(b = \tfrac{1}{2}\,,\ \beta = 0)\,,\quad (b = \tfrac{1}{2}\,,\ \beta = \tfrac{1}{2})\,,\quad (b = 1\,,\ \beta = 0)\,,\ (b = 1\,,\ \beta = \tfrac{1}{2})\ , \tag{78}$$

see (3) and (4), and we choose

$$a = 0.25\,,\quad N = 10\,,\quad k = \frac{a}{N^2} = 0.0025\,,\quad \sqrt{k} = 0.05\ . \tag{79}$$

We thus require for the $f(n)$ in (77) the numerical values of $f(n) = f_0((n + \beta)\sqrt{k})$ with $\beta = 0, 1/2$; $\sqrt{k} = 0.05$ and $n = 0, 1, ..., 10$. In the table below we display these values.

n	$f_0(0.05n)$	$f_0(0.05(n + 1/2))$
0	1.000000000	1.000312484
1	1.001249740	1.002811183
2	1.004995843	1.007802364
3	1.011229016	1.015273698
4	1.019933948	1.025206954
5	1.031089567	1.037578307
6	1.044669388	1.052358702
7	1.060641934	1.069514393
8	1.078971210	1.089007264
9	1.099617219	1.110795538
10	1.122536503	1.134834231

The numerical evaluatoin of $f_0(x)$ is done by solving the second equation in (9) for $U(x) = (\ln f_0(x))^{1/2}$, where the 5 terms of the series on the right-hand side of (10) are used to get a high-accuracy approximation for $f_0(x)$ that can be used for initialization of a Newton iteration to compute $U(x)$.

Case $b = 1/2$, $\beta = 0$. We evaluate V_n, $n = 0, 1, ..., 10$, according to the recursion

$$V_0 = 1\,,\;\; V_1 = 1+\tfrac{1}{2}\,k = 1.00125\,;\quad V_{n+1} = V_n + \frac{0.0025}{V_n} - V_{n-1}\,,\;\; n = 1, 2, ..., 9\,. \tag{80}$$

Furthermore, $f(n) = f_0(0.05n)$, see first column in the above table. Since $\beta - b + 1/2 = 0$ in this case, the result (75), yielding an approximation of $\varepsilon_{n+1} - \varepsilon_1$, is relevant. We have in the present case

$$\varepsilon_0 = 0\,,\;\; \varepsilon_1 = -0.000000260\,,\;\; \varepsilon_1 - \varepsilon_0 = -0.000000260\,. \tag{81}$$

Since ε_1 is extremely small in this case, we can consider both $f(n)$ and $-\varepsilon_1 + f(n)$, see (77), as an approximation of V_n. The other case in (78) that has $\beta - b + 1/2 = 0$ is $(b = 1\,,\, \beta = 1/2)$, and this case happens to have a large value of ε_1 so that then the approximation $-\varepsilon_1 + f(n)$ of V_n is more appropriate. To treat the two cases with $\beta - b + 1/2 = 0$ on an equal footing, we choose in both cases $-\varepsilon_1 + f(n)$ as an approximation of V_n. Thus in the table below, we display V_n and $-\varepsilon_1 + f(n)$, and we consider $\varepsilon_n - \varepsilon_1$ as the error in approximating V_n by $-\varepsilon_1 + f(n)$. This latter error is approximated by

$$\widehat{\varepsilon_n - \varepsilon_1} = (\varepsilon_1 - \varepsilon_0)(n-1) - C\,k^2(n-1)^2\,, \tag{82}$$

see (75), where we take $C = 1/25$ for convenience (the actual C as given by (76) varies between 1/24 and 1/27.5 when $n = 0, 1, ..., 10$).

n	V_n	$-\varepsilon_1 + f(n)$	$\varepsilon_n - \varepsilon_1$	$\widehat{\varepsilon_n - \varepsilon_1}$
2	1.004996879	1.004996103	-0.000000776	-0.000000510
3	1.011231328	1.011229276	-0.000002052	-0.000001520
4	1.019938010	1.019934208	-0.000003802	-0.000003030
5	1.031095822	1.031089827	-0.000005995	-0.000005040
6	1.044678238	1.044669648	-0.000008590	-0.000007550
7	1.060653736	1.060642194	-0.000011542	-0.000010560
8	1.078986271	1.078971470	-0.000014801	-0.000014070
9	1.099635796	1.099617479	-0.000018317	-0.000018080
10	1.122558800	1.122536763	-0.000022037	-0.000022590

Case $b = 1/2$, $\beta = 1/2$. We compute V_n, $n = 0, 1, ..., 10$, according to the recursion in (80). Furthermore, $f(n) = f_0(0.05(n+1/2))$. Since $\beta - b + 1/2 =$

$1/2 \neq 0$ in this case, the result (73), yielding an approximation of $\varepsilon_{n+1} - \varepsilon_1$, is relevant. We have

$$\varepsilon_0 = 0.000312484\,, \quad \varepsilon_1 = 0.0001561183\,, \quad \varepsilon_1 - \varepsilon_0 = 0.0001248699\ . \qquad (83)$$

The right-hand side of (73) shows that the error ε_n grows approximately as $\varepsilon_1 + (n-1)(\varepsilon_1 - \varepsilon_0)$, with ε_1 and $\varepsilon_1 - \varepsilon_0$ of comparable magnitude. In the table below, we display V_n and $f(n)$, together with the quantity ε_n as the error in approximating V_n by $f(n)$. This latter error $\varepsilon_n = \varepsilon_1 + (\varepsilon_n - \varepsilon_1)$ is approximated by

$$\hat{\varepsilon}_n = \varepsilon_1 + (n-1)(\varepsilon_1 - \varepsilon_0) \qquad (84)$$

in accordance with (73), where we have replaced δ_{n-1} by 0.

n	V_n	$f(n)$	ε_n	$\hat{\varepsilon}_n$
2	1.004995843	1.007802364	0.002806521	0.002809882
3	1.011229016	1.015273698	0.004044682	0.004058581
4	1.019933948	1.025206954	0.005273006	0.005307280
5	1.031089567	1.037578307	0.006488740	0.006555979
6	1.044669388	1.052358702	0.007689314	0.007804678
7	1.060641934	1.069514393	0.008872459	0.009053377
8	1.078971210	1.089007264	0.010036054	0.010302076
9	1.099617219	1.110795538	0.011178319	0.011550775
10	1.122536503	1.134834231	0.012299201	0.012799474

Case $b = 1$, $\beta = 0$. We evaluate V_n, $n = 0, 1, ..., 10$, according to the recursion

$$V_0 = 1\,, \quad V_1 = 1 + k = 1.0025\,; \qquad V_{n+1} = V_n + \frac{0.0025}{V_n} - V_{n-1}\,, \quad n = 1, 2, ..., 9\ . \qquad (85)$$

Furthermore, we have $f(n) = f_0(0.05n)$. Since $\beta + 1/2 - b = -1/2 \neq 0$ in this case, the result (73), yielding an approximation of $\varepsilon_{n+1} - \varepsilon_1$, is relevant. We have

$$\varepsilon_0 = 0\,, \quad \varepsilon_1 = -0.001250260\,, \ \varepsilon_1 - \varepsilon_0 = -0.01250260\ . \qquad (86)$$

In the table below, we display V_n and $f(n)$, together with the quantity ε_n as the error in approximating V_n by $f(n)$. This latter error is approximated as in (84).

n	V_n	$f(n)$	ε_n	$\hat{\varepsilon}_n$
2	1.007493766	1.004995843	-0.002497923	-0.002500520
3	1.014968936	1.011229016	-0.003739920	-0.003750780
4	1.024907236	1.019933948	-0.004973288	-0.005001040
5	1.037284781	1.031089567	-0.006195214	-0.006251300
6	1.052072465	1.044669388	-0.007403077	-0.007501560
7	1.069236411	1.060641934	-0.008594477	-0.008751820
8	1.088738474	1.078971210	-0.009767264	-0.010002080
9	1.110536773	1.099617219	-0.010919554	-0.011252340
10	1.134586235	1.122536503	-0.012049732	-0.012502600

Case $b = 1$, $\beta = 1/2$. We compute V_n, $n = 0, 1, ..., 10$, according to the recursion in (83), and we have $f(n) = f_0(0.05(n+1/2))$. Since $\beta + 1/2 - b = 0$, we use the result (75). Thus, we approximate V_n by $-\varepsilon_1 + f(n)$, at the expense of an error $\varepsilon_n - \varepsilon_1$. The latter error is approximated by

$$\widehat{\varepsilon_n - \varepsilon_1} = (n-1)(\varepsilon_1 - \varepsilon_0) - \tfrac{1}{6}\,(n-1)^2\,k^2 \ , \tag{87}$$

according to (75), where we have taken $C = 1/6$ (the actual C, see (63) and (66), varies between $\frac{1}{6} = \frac{1}{24} + \frac{1}{2}\,(b - 1/2)^2$ at $y = 0$ and $1/6.385$ at $y = 1/2$). We have

$$\varepsilon_0 = 0.000312484\,, \quad \varepsilon_1 = 0.000311183\,, \quad \varepsilon_1 - \varepsilon_0 = 0.000001301 \ . \tag{88}$$

This gives the following table

n	V_n	$-\varepsilon_1 + f(n)$	$\varepsilon_n - \varepsilon_1$	$\widehat{\varepsilon_n - \varepsilon_1}$
2	1.007493766	1.007491181	-0.000002585	-0.000002342
3	1.014968936	1.014962515	-0.000006421	-0.000006768
4	1.024907236	1.024895771	-0.000011465	-0.000013278
5	1.037284781	1.037267124	-0.000017657	-0.000021870
6	1.052072465	1.052047519	-0.000024946	-0.000032546
7	1.069236411	1.069203210	-0.000033201	-0.000045306
8	1.088738474	1.088696081	-0.000042393	-0.000060148
9	1.110536773	1.110484355	-0.000052410	-0.000077074
10	1.134586235	1.134523048	-0.000063187	-0.000096084

Observations. The tables for the two cases with $\beta - b + 1/2$ not close to 0, cases ($b = 1/2\,,\ \beta = 1/2$) and ($b = 1\,,\ \beta = 0$) in (78), show that the error $\varepsilon_n = f(n) - V_n$ is well approximated by the linear function $\hat{\varepsilon}_n = \varepsilon_1 + (n - 1)(\varepsilon_1 - \varepsilon_0)$ on the range $n = 2, 3, ..., 10$. In these cases the term $n(\varepsilon_1 - \varepsilon_0)$

with $\varepsilon_1 - \varepsilon_0 = (\beta + 1/2 - b)\, k + O(k^2)$, is dominant in the expression at the right-hand side of (68). The residual error is due to the deleted two other terms at the right-hand side of (68), and manifest themselves mainly for larger n.

In the two cases where $\beta - b + 1/2 = 0$, cases ($b = 1/2\,, \beta = 0$) and ($b = 1\,, \beta = 1/2$) in (78), we have that $\varepsilon_1 - \varepsilon_0 = O(k^2)$, causing the term $n(\varepsilon_1 - \varepsilon_0)$ to be dominated by the term $O(n^2a^2/N^4)$ at the right-hand side of (68). It is now necessary to be more precise about the implicit constant in the O-term. In estimating this implicit constant, there are several places where approximations had to be made, such as deletion of the terms involving $f^{(5)}$ in (45) (leading to a relative error of order $\sqrt{k}$), and replacement of two double series, see (59) and (62), by Riemann integrals with integration ranges that are slightly shifted to achieve a convenient form of the end result. As a result the estimated error $\widehat{\varepsilon_n - \varepsilon_1}$ of the error $\varepsilon_n - \varepsilon_1 = f(n) - \varepsilon_1 - V_n$ cannot be expected to be as accurate as in the cases with $\beta - b + 1/2$ away from 0. This is evident from the two tables for the cases with $\beta - b + 1/2 = 0$: in the table for the case $b = 1$, $\beta = 1/2$, we have $|\widehat{\varepsilon_n - \varepsilon_1}|$ is about 50% larger than $|\varepsilon_n - \varepsilon_1|$ for $n = 10$.

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