Gerber-Shiu metrics for a bivariate perturbed risk process

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Abstract

We consider a two-dimensional risk model with simultaneous Poisson arrivals of claims. Each claim of the first input process is at least as large as the corresponding claim of the second input process. In addition, the two net cumulative claim processes share a common Brownian motion component. For this model we determine the Gerber-Shiu metrics, covering the probability of ruin of each of the two reserve processes before an exponentially distributed time along with the ruin times and the undershoots and overshoots at ruin.

KEYWORDS: Cramér-Lundberg model ∘ Brownian perturbation ∘ multivariate risk ∘ ruin probability ∘ Gerber-Shiu metrics

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1 Introduction

The existing ruin theory literature has a strong focus on the univariate setting featuring a single reserve process. In practice, however, the position of an insurance firm is often described by multiple, typically correlated, reserve processes. Multivariate ruin turns out to be a challenging topic that can be dealt with explicitly only when imposing additional assumptions. Most notably, a certain *ordering* between the individual net cumulative claim processes, say $\boldsymbol{Y}(t) \equiv (Y_1(t), \dots, Y_d(t))$ for some $d \in \mathbb{N}$, needs to be imposed. The ambition of this paper is to, in this multivariate context, not only assess the likelihood of ruin, but to also provide insight into 'the way ruin occurs'. Indeed, through the so-called *Gerber-Shiu metrics* [14] we probabilistically describe the corresponding ruin times, undershoots, and overshoots. Throughout this paper we consider the case $d = 2$, but all results can be extended to higher dimensions, albeit at the price of the underlying analysis complicating seriously.

Model — In this paper we analyze two net cumulative claim processes, say $Y_1(t)$ and $Y_2(t)$, in which claims arrive simultaneously, according to a Poisson process with rate $\lambda > 0$. These claims $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots$ are 2-dimensional, componentwise non-negative i.i.d. random vectors that are distributed as the generic random vector $\boldsymbol{B} = (B^{(1)}, B^{(2)})$. We throughout assume the entries of this vector to be *ordered* in the sense that

$$\mathbb{P}\big(B^{(1)} \geqslant B^{(2)}\big) = 1, \tag{1}$$

where $B^{(i)}$ is a generic claim size corresponding to the net cumulative claim process $Y_i(t)$, for $i = 1, 2$. Above we mentioned that claims of both components arrive simultaneously, but by the ordering condition (1) those of the $Y_2(t)$ process can be void: observe that at any claim arrival in the $Y_2(t)$ process there is a claim arrival in the $Y_1(t)$ process, but not vice versa.

A specific feature of this paper is that we consider a *perturbed* risk model, in the sense that both processes $Y_1(t)$ and $Y_2(t)$ in addition contain a Brownian component. This component is common to both processes, and takes the form $Y_W(t) = -rt + \sigma W(t)$, where $r \geqslant 0$ can be interpreted as the premium rate (which we assume is the same for both processes), and $W(t)$ is a standard Brownian motion that is independent of the claim arrival process. The i-th net cumulative claim process thus becomes

$$Y_i(t) := Y_W(t) + \sum_{j=1}^{N(t)} B_j^{(i)},$$

for $i = 1, 2$, with $N(t)$ denoting a Poisson process with rate λ. It requires an elementary computation to verify that the corresponding bivariate Laplace exponent is given by

$$\varphi(\boldsymbol{\alpha}) := \log \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{Y}(1)} = \frac{\sigma^2}{2}(\alpha_1 + \alpha_2)^2 + r\, \mathbf{1}^\top \boldsymbol{\alpha} - \lambda(1 - b(\boldsymbol{\alpha})), \tag{2}$$

with

$$b(\boldsymbol{\alpha}) := \mathbb{E}\, e^{\alpha_1 B^{(1)} - \alpha_2 B^{(2)}}$$

denoting the bivariate Laplace-Stieltjes transform (LST) of the random vector $\boldsymbol{B}$. We let $X_i(t)$ be the reserve level process $u_i - Y_i(t)$ pertaining to the i-th net cumulative

claim process, with $u_1, u_2 > 0$ denoting the corresponding initial surpluses. Observe that the construction is such that the two individual net cumulative claim processes are ordered: $Y_1(t) \geqslant Y_2(t)$ for any $t \geqslant 0$, almost surely; the reserve level processes $X_1(t)$ and $X_2(t)$ are not necessarily ordered.
The case of the net cumulative claim processes being represented by the superposition of a compound Poisson process and a Brownian motion is particularly relevant in light of the findings of [1]. These imply that, under mild conditions, one can arbitrarily closely approximate any spectrally-positive Lévy process (i.e., Lévy process without any negative jumps) by such a superposition.

Quantities of interest — The Gerber-Shiu metrics provide insight into the two individual reserve level process at ruin. Concretely, with $\boldsymbol{u} = (u_1, u_2)^\top \in [0,\infty)^2$ and $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t))^\top \in \mathbb{R}^2$, the focus lies on capturing a probabilistic description of the joint distribution of the random vectors

$$\boldsymbol{\tau}(\boldsymbol{u}) := \begin{pmatrix} \tau_1(u_1) \\ \tau_2(u_2) \end{pmatrix}, \quad \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-) := \begin{pmatrix} Y_1(\tau_1(u_1)-) \\ Y_2(\tau_2(u_2)-) \end{pmatrix}, \quad \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})) := \begin{pmatrix} Y_1(\tau_1(u_1)) \\ Y_2(\tau_2(u_2)) \end{pmatrix};$$

here $\tau_i(u_i)$ is the ruin time corresponding to the net cumulative claim process $Y_i(t)$, being defined as the smallest positive t such that $Y_i(t) > u_i$, for $i = 1, 2$. Our analysis enables us to probabilistically analyze these ruin times, jointly with the values of the reserve process *immediately before* ruin $X_i(\tau_i(u_i)-) = u_i - Y_i(\tau_i(u_i)-)$, typically referred to as the *undershoot*, and the values of the reserve process *at* ruin $X_i(\tau_i(u_i)) = u_i - Y_i(\tau_i(u_i))$, representing the corresponding *overshoot*, for $i = 1, 2$.
We characterize the underlying sextuple law (for the two ruin times, two undershoots, and two overshoots) by identifying its corresponding transform. Concretely, our aim is to identify, for $\boldsymbol{u} \geqslant 0$ and $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \boldsymbol{0}$, $\boldsymbol{\gamma}_3 \leqslant \boldsymbol{0}$ (where $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2})^\top$ for $i = 1, 2, 3$), the object

$$\begin{aligned} p(\boldsymbol{u}) &\equiv p(\boldsymbol{u}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) \\ &:= \mathbb{E}\big(e^{-\boldsymbol{\gamma}_1^\top \boldsymbol{\tau}(\boldsymbol{u}) - \boldsymbol{\gamma}_2^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-)) - \boldsymbol{\gamma}_3^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})))} 1\{\boldsymbol{\tau}(\boldsymbol{u}) \leqslant T_\beta \boldsymbol{1}\}\big). \end{aligned}$$

Here T_β is an exponentially distributed time with parameter $\beta > 0$ sampled independently from everything else; by $\{\boldsymbol{\tau}(\boldsymbol{u}) \leqslant T_\beta \boldsymbol{1}\}$ we mean that *both* ruin times $\tau_1(u_1)$ and $\tau_2(u_2)$ should be smaller than the random variable T_β. As $\tau_i(u_i)$ and $X_i(\tau_i(u_i)-)$ are non-negative whereas $X_i(\tau_i(u_i))$ is non-positive, we let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \boldsymbol{0}$ and $\boldsymbol{\gamma}_3 \leqslant \boldsymbol{0}$.
In the sequel we frequently suppress the parameters $\beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$, as these are held constant throughout the analysis. Our work succeeds in finding $p(\boldsymbol{u})$ in terms of its transform, for $\boldsymbol{\alpha} \geqslant \boldsymbol{0}$ and $\beta > 0$,

$$\pi(\boldsymbol{\alpha}) \equiv \pi(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) := \int_0^\infty \int_0^\infty e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}}\, p(\boldsymbol{u}, \beta, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)\, \mathrm{d}u_1\, \mathrm{d}u_2.$$

Related literature — We proceed by a (non-exhaustive) literature review, covering papers on multivariate risk for models of Cramér-Lundberg type, papers that focus on adding a Brownian perturbation, papers on the identification of corresponding Gerber-Shiu metrics, and additional related papers from the queueing literature.

Early papers on two-dimensional risk processes with some form of ordering of claim sizes are [2, 3]. These works consider the joint ruin problem in the case of an insurer that splits claims and premia according to fixed proportions (which is a special case of our 'ordered claim assumption' (1)). A key observation is that the resulting two-dimensional ruin problem can be viewed as a one-dimensional crossing problem over a piece-wise linear barrier. Badescu et al. [5] extended the model of [2, 3] by allowing additional claim arrivals at the first insurer. Gong et al. [15] consider a discounted penalty function, adding Gerber-Shiu metrics (in that they include the surplus at ruin), and provide recursive integral formulas for that penalty function.

The present paper was inspired by the work by Badila et al. [6]. There, under the 'ordered claim assumption' (1), the two-dimensional risk model with simultaneous Poisson arrivals is treated in detail. Compared to our objectives, two aspects are lacking: in the first place the focus lies on ruin probabilities only, meaning that the evaluation of the Gerber-Shiu metrics was not included, and in the second place there is no Brownian perturbation. Importantly, [6] in addition establishes a bivariate duality result, which links the bivariate ruin model to a queueing model with two parallel queues and simultaneous Poisson arrivals at both queues – Assumption (1) translates into requiring that the service times at the first queue are always at least as large as the corresponding ones at the second queue. In particular, the infinite-horizon survival probabilities of the risk model are identified with the steady-state two-dimensional workload distribution in the queueing model. It is also shown in [6] how this analysis can be extended to a setting with more than two insurers/queues. Another contribution of [6] is that it points out that the the stationary distribution of the two-dimensional workload process in the queueing model *without ordering of the service requirements* can be found by using a boundary value technique. For this particular queueing model that was shown, in increasing level of generality, in [4, 9, 8]. The implication of the duality is that one can thus also handle the two-dimensional risk model without the ordering constraint being in place – at least, in case one is interested in the transform of the ruin probabilities only (i.e., not covering Gerber-Shiu metrics).

Gerber [12] has generalized the classical one-dimensional Cramér-Lundberg model by adding an independent Brownian motion component to the net cumulative claim process. For this perturbed model, Dufresne and Gerber [10] analyze the probability of ruin, as well as the separate probabilities of ruin caused by a claim or by the Brownian component. Wang [23] has derived explicit expressions for the latter two probabilities in the case of exponentially distributed claim sizes. Tsai and Wilmot [22] have added the Gerber-Shiu metrics to the model of [12, 10]. Generalizing a result of Gerber and Landry [13], they derive a defective renewal equation for the surplus process (expected discounted penalty function, involving both the deficit at ruin and the surplus immediately before ruin). This equation is then used to study the asymptotic behavior of the ruin probability when the initial capital tends to infinity. Li et al. [18] study the *finite-time* expected discounted penalty function (Gerber-Shiu) for the Cramér-Lundberg model with perturbations by solving a second-order partial integro-differential equation with boundary conditions. For the same model, Liu et al. [19] consider the joint distribution of the time to ruin and the number of claims until ruin.

Briefly returning to the dual two-dimensional queueing model that is being studied in [6], it is worthwhile to mention that this queueing model is there shown to be equivalent to a particular tandem fluid queue, viz., two queues in series in which the outflow of the first queue is a fluid, and with simultaneous Poisson arrivals with generic service time $B^{(2)}$ at the first queue and $B^{(1)} - B^{(2)}$ in the second queue. In turn, tandem fluid queues are equivalent to particular priority queues with preemptive resume priorities, as discussed in e.g. [16, 21].

Main contributions — This paper generalizes the results in [6] for a two-dimensional risk model with ordered claim sizes in three ways: we consider the probability of ruin before a finite time-horizon (rather than the all-time ruin probability), we add a Brownian component, and we consider the discounted penalty function with the Gerber-Shiu metrics. Our main results are Theorem 2 for the unperturbed case, and Theorem 3 for the case *with* a Brownian component. These theorems uniquely characterize, in terms of the transform $\pi(\alpha)$, the probability of ruin before an exponential time, along with the distributions of the ruin times, the undershoots at ruin, and the overshoots at ruin. The two-dimensional case without the Brownian component was briefly considered in [20, §7.5]. Importantly, as it turns out, the unperturbed model has to be treated rather differently than the perturbed model, in the sense that one cannot simply let the 'diffusion parameter' σ go to 0 in the perturbed model.

Organization — In Section 2 we provide a concise discussion of the univariate case, thus introducing some of the main tools that are used for the bivariate case. Section 3 provides some general results for that bivariate case, applying to both the unperturbed and perturbed risk model. Section 4 is devoted to the bivariate unperturbed risk model, while the bivariate perturbed risk model is analyzed in Section 5.

2 Univariate case

In this section we present a compact analysis of the univariate case. These univariate results will play a role in the bivariate case, but, as we will see, some are interesting in their own right, in particular the results for the case of the perturbed risk model, i.e., the model with $\sigma^2 > 0$.
In this univariate setting, we can use a somewhat simpler notation than the one used throughout the remainder of the paper. We let $Y(t)$ be a compound Poisson process with drift (characterized by the arrival rate $\lambda > 0$, the LST of the claim-size distribution $b(\alpha)$, and the premium rate $r > 0$) perturbed by a Brownian motion. Concretely, the net cumulative claim process is defined by

$$Y(t) := Y_W(t) + \sum_{j=1}^{N(t)} B_j,$$

where $Y_W(t) = -rt + \sigma W(t)$, with $W(t)$ a standard Brownian motion (or: Wiener process), $N(t)$ denoting a Poisson process with rate λ, and σ a non-negative parameter. With the firm's initial capital being u, the evolution of the reserve level is given by $X(t) := u - Y(t)$. It is readily checked that, for $\alpha \geqslant 0$,

$$\varphi_W(\alpha) := \log \mathbb{E}\, e^{-\alpha Y_W(1)} = \frac{\sigma^2}{2}\alpha^2 + r\alpha.$$

Denote by $\bar{Y}_W(t)$ and $\underline{Y}_W(t)$ the *running maximum* process and *running minimum* process, respectively, corresponding to $Y_W(t)$:

$$\bar{Y}_W(t) := \sup_{s\in[0,t]} Y_W(s), \quad \underline{Y}_W(t) := \inf_{s\in[0,t]} Y_W(s).$$

Recalling that T_β is an exponentially distributed random variable, sampled independently of everything else, it is a well-known result that $\bar{Y}_W(T_\beta)$ and $-\underline{Y}_W(T_\beta)$ have exponential distributions with parameters

$$x_\beta^+ := \frac{r+w}{\sigma^2}, \quad x_\beta^- := \frac{-r+w}{\sigma^2},$$

respectively, where $w := \sqrt{r^2+2\beta\sigma^2}$; see e.g. [20, Exercise 1.5.(ii)]. By a time-reversibility argument it follows that $\bar{Y}_W(T_\beta) - Y_W(T_\beta)$ and $-\underline{Y}_W(T_\beta)$ are identically distributed, while it is an implication of the Wiener-Hopf decomposition that $\bar{Y}_W(T_\beta)$ and $\bar{Y}_W(T_\beta) - Y_W(T_\beta)$ are independent. Observe that $x_\beta^+ x_\beta^- = 2\beta/\sigma^2$.

Define the *ruin time* by

$$\tau(u) := \inf\{t>0 : X(t)<0\} = \inf\{t>0 : Y(t)>u\},$$

so that the finite-horizon ruin probability is given by $p(u,t) := \mathbb{P}(\tau(u)\leqslant t)$. In this section we study the transform of this finite-time ruin probability, jointly with the corresponding Gerber-Shiu metrics (i.e., the time of ruin $\tau(u)$, the value of the reserve process *immediately before* ruin $X(\tau(u)-) = u - Y(\tau(u)-)$, and the value of the reserve process *at* ruin $X(\tau(u)) = u - Y(\tau(u))$). This concretely means that in this univariate setting we analyze, with $\boldsymbol{\gamma} := (\gamma_1,\gamma_2,\gamma_3)$, the object

$$p(u,t,\boldsymbol{\gamma}) := \mathbb{E}\big(e^{-\gamma_1\tau(u)-\gamma_2X(\tau(u)-)-\gamma_3X(\tau(u))}1\{\tau(u)\leqslant t\}\big);$$

as $\tau(u)$ and $X(\tau(u)-)$ are non-negative whereas $X(\tau(u))$ is non-positive, we let $\gamma_1,\gamma_2 \geqslant 0$ and $\gamma_3 \leqslant 0$. The objective of this section is to compute the transform with respect to u and t: with T_β an exponentially distributed time with parameter β sampled independently from everything else, we wish to find a closed-form expression for

$$\pi(\alpha,\beta,\boldsymbol{\gamma}) := \int_0^\infty\int_0^\infty e^{-\alpha u}\beta e^{-\beta t}p(u,t,\boldsymbol{\gamma})\,\mathrm{d}t\,\mathrm{d}u = \int_0^\infty e^{-\alpha u}p(u,T_\beta,\boldsymbol{\gamma})\,\mathrm{d}u.$$

For conciseness, in the remainder of this section we write $p(u) \equiv p(u,T_\beta,\boldsymbol{\gamma})$ and $\pi(\alpha) \equiv \pi(\alpha,\beta,\boldsymbol{\gamma})$. Observe that the level u can be first exceeded either at a claim arrival, thus leading to positive undershoot and overshoot, or between claim arrivals (i.e., due to the Brownian component), thus leading to the undershoot and overshoot both being equal to zero. We refer to Figure 1 for a pictorial illustration of both cases. In the remainder of this section, we denote the Laplace exponent of the process $Y(t)$ by

$$\varphi(\alpha) := \log\mathbb{E}\,e^{-\alpha Y(1)} = \varphi_W(\alpha) - \lambda(1-b(\alpha)) = \frac{\sigma^2}{2}\alpha^2 + r\alpha - \lambda(1-b(\alpha)),$$

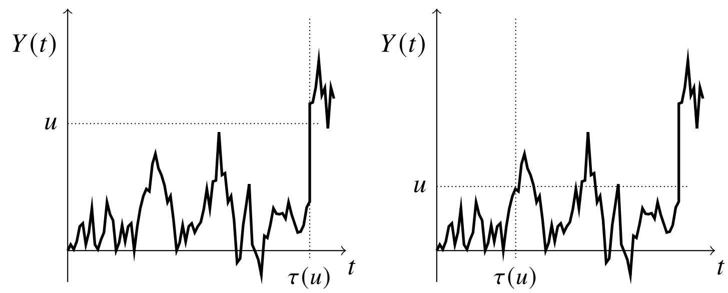

Figure 1. Net cumulative claim process $Y(t)$. Left panel: level u is exceeded due to a jump of the compound Poisson component. Right panel: level u is exceeded due to the Brownian component, i.e., with zero undershoot and overshoot.

and let $\psi(\cdot)$ be its right-inverse. In the unperturbed model, i.e., the model in which $\sigma^2 = 0$, various known methods can be used to identify $\pi(\alpha)$. As can be found in e.g. [20, Exercise 1.2] and, for a more general model, [7],

$$\pi(\alpha) = \frac{\lambda}{\varphi(\alpha) - \gamma_1 - \beta}\left(\frac{b(-\gamma_3) - b(\psi(\gamma_1+\beta)+\gamma_2)}{\psi(\gamma_1+\beta)+\gamma_2+\gamma_3} - \frac{b(-\gamma_3) - b(\alpha+\gamma_2)}{\alpha+\gamma_2+\gamma_3}\right).$$

A ‘density version’ of this result is given in [17, Theorem 5.5]: there the so-called Gerber-Shiu measure is expressed in terms of scale functions pertaining to the underlying compound Poisson process.

We proceed by an analysis of the univariate perturbed model. As will become clear in the analysis, the fact that we assume σ^2 to be positive is a crucial element in the derivation. It is noted that this univariate model has already been studied by Li et al. [18], but it seems difficult to extend their approach to the bivariate case. In view of this, and also because the univariate model is very natural and important in its own right, we present two methods via which it can be analyzed in detail: a method that is based on conditioning on the first event (Section 2.1 and Appendix A), and a method based on Kolmogorov-type equations that seems particularly suitable for the bivariate model (Section 2.2). In Sections 3-5 we extensively build upon the ideas and results of the present section.

2.1 Solution by conditioning on the first event

Define $p_-(u)$ as the counterpart of $p(u)$, but with a claim arriving at time 0. By conditioning on the maximum of the Brownian component until the first claim arrival, we obtain that, with $\tau_W(u)$ denoting the ruin time corresponding to the Brownian motion $Y_W(t)$,

$$p(u) = \int_0^\infty \mathbb{P}(\tau_W(u) \in \mathrm{d}v)\, \mathbb{P}(T_{\lambda+\beta} \geqslant v)\, e^{-\gamma_1 v} +$$
$$\int_{t=0}^\infty \int_{w=0}^\infty \int_{v=0}^u \lambda e^{-(\lambda+\beta)t}\, \mathbb{P}(\bar{Y}_W(t) \in \mathrm{d}v)\, \mathbb{P}(-\underline{Y}_W(t) \in \mathrm{d}w)\, e^{-\gamma_1 t}\, p_-(u - v + w)\, \mathrm{d}t.$$

This decomposition can be understood as follows. The first term in the right-hand side of the previous display corresponds to the scenario that before the first event (i.e., either expiration of the 'exponential clock' T_β or a claim arrival) the level u has already been exceeded, such that the undershoot and overshoot are both equal to zero. The second term corresponds to the scenario that $Y_W(t)$ has remained below level u until the first event (where it is observed that the densities of $\bar{Y}_W(t)$ and $\underline{Y}_W(t)$ factorize by the Wiener-Hopf decomposition, noting that the expression is weighted by an exponential density). Relying on a similar argumentation, we also observe that

$$p_-(u) = \int_0^u \mathbb{P}(B \in \mathrm{d}v)\, p(u-v) + \int_u^\infty \mathbb{P}(B \in \mathrm{d}v)\, e^{-\gamma_2 u}\, e^{-\gamma_3(u-v)}. \tag{3}$$

From these relations for $p(u)$ and $p_-(u)$, the transform $\pi(\alpha)$ can be identified in a relatively straightforward manner, as pointed out in Appendix A. Define

$$\Pi(\alpha) := \frac{b(-\gamma_3) - b(\alpha+\gamma_2)}{\alpha+\gamma_2+\gamma_3}. \tag{4}$$

Theorem 1 *For any $\alpha \geqslant 0$,*

$$\pi(\alpha) = \frac{(\alpha - \psi(\gamma_1+\beta))\,\sigma^2/2 + \lambda\,(\Pi(\psi(\gamma_1+\beta)) - \Pi(\alpha))}{\varphi(\alpha) - \gamma_1 - \beta}.$$

This theorem has nice ramifications. If $\varphi'(0) = r - \lambda\,\mathbb{E}B < 0$, (i.e., ruin eventually happens), then the stationary overshoot can be analyzed by considering $\alpha\,\pi(\alpha)$ with $\beta = \gamma_1 = \gamma_2 = 0$ and letting $\alpha \downarrow 0$. To this end, we first verify that, for $\alpha \geqslant 0$ and $\gamma_3 \leqslant 0$,

$$\int_0^\infty \alpha e^{-\alpha u}\,\mathbb{E}\,e^{-\gamma_3 X(\tau(u))}\,\mathrm{d}u = \frac{\alpha}{\varphi(\alpha)}\left((\alpha - \psi(0))\frac{\sigma^2}{2} + \lambda\,(\Pi(\psi(0)) - \Pi(\alpha))\right).$$

Observe that $X(\tau(u))$ has an atom in zero, due to the fact that ruin can possibly be reached 'due to the Brownian component' (i.e., between two subsequent claims). By letting $\gamma_3 \to -\infty$, we readily obtain

$$\int_0^\infty \alpha e^{-\alpha u}\,\mathbb{P}(X(\tau(u)) = 0)\,\mathrm{d}u = \frac{\alpha}{\varphi(\alpha)}\left((\alpha - \psi(0))\frac{\sigma^2}{2}\right).$$

Considering this expression as $\alpha \downarrow 0$, by an application of L'Hôpital's rule, we have proven the following result.

Corollary 1 *If $\varphi'(0) < 0$, then*

$$\lim_{u\to\infty} \mathbb{P}(X(\tau(u)) = 0) = -\frac{\psi(0)\,\sigma^2}{2\varphi'(0)}. \tag{5}$$

The expression in the right-hand side of (5) is indeed between 0 and 1, as can be seen as follows. Because $b(\alpha) \geqslant 1 - \alpha\,\mathbb{E}B$,

$$0 = \varphi(\psi(0)) = \frac{\sigma^2}{2}\psi(0)^2 + r\psi(0) - \lambda\big(1 - b(\psi(0))\big) \geqslant \psi(0)\left(\frac{\sigma^2}{2}\psi(0) + r - \lambda\,\mathbb{E}B\right).$$

Observing that $\psi(0)$ is positive and that $\varphi'(0) = r - \lambda\,\mathbb{E}B$, we have the right-hand side of (5) is at most 1.

2.2 Solution using Kolmogorov-type equations

In this subsection we detail an alternative approach, which will be relied upon later in the paper. In this approach we compare the object of interest $p(u)$ at two points in time (located 'close to' each other). Indeed, as $\Delta t \downarrow 0$, owing to the memoryless property of the killing time T_β, the following Kolmogorov-type identity holds: up to $o(\Delta t)$-terms,

$$p(u) = e^{-\gamma_1 \Delta t}\Bigg(\lambda\,\Delta t \int_0^u \mathbb{P}(B \in \mathrm{d}v)\, p(u-v) + \\ \lambda\,\Delta t \int_u^\infty \mathbb{P}(B \in \mathrm{d}v)\, e^{-\gamma_2 u}\, e^{-\gamma_3(u-v)} + \\ (1-\lambda\,\Delta t - \beta\,\Delta t) \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2\,\Delta t}} \exp\left(-\frac{y^2}{2\sigma^2\,\Delta t}\right) p(u + r\,\Delta t - y)\,\mathrm{d}y\Bigg). \tag{6}$$

The first term between the brackets on the right-hand side of (6) corresponds to the scenario with an arrival of a claim of size smaller than u, the second to the scenario with an arrival of a claim larger than u (so that the overshoot and undershoot can be assigned a value), and the third to the scenario of neither a claim arrival nor killing. Performing the substitution $z := y/(\sigma\sqrt{\Delta t})$, we obtain, up to $o(\Delta t)$-terms,

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2\,\Delta t}} \exp\left(-\frac{y^2}{2\sigma^2\,\Delta t}\right) p(u + r\,\Delta t - y)\,\mathrm{d}y \\
&\quad = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, p(u + r\,\Delta t - z\sigma\,\sqrt{\Delta t})\,\mathrm{d}z \\
&\quad = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \left(p(u) + p'(u)(r\,\Delta t - z\sigma\,\sqrt{\Delta t}) + \frac{1}{2}p''(u)(r\,\Delta t - z\sigma\,\sqrt{\Delta t})^2\right)\mathrm{d}z \\
&\quad = p(u) + p'(u)\,r\,\Delta t + \frac{\sigma^2}{2}p''(u)\,\Delta t.
\end{aligned}$$

The next step is to convert this equation into an integro-differential equation. With $\omega := \gamma_1 + \lambda + \beta$ we obtain after dividing the entire equation by Δt that

$$-r\,p'(u) - \frac{\sigma^2}{2}p''(u) = \lambda \int_0^u \mathbb{P}(B \in \mathrm{d}v)\, p(u-v) + \\ \lambda \int_u^\infty \mathbb{P}(B \in \mathrm{d}v)\, e^{-\gamma_2 u}\, e^{-\gamma_3(u-v)} - \omega\, p(u).$$

We now multiply the entire equation by $e^{-\alpha u}$ and integrate over u. To this end, it can be verified that integration by parts yields

$$\begin{aligned}
-\int_0^\infty p'(u)\, e^{-\alpha u}\,\mathrm{d}u &= p(0) - \alpha\,\pi(\alpha), \\
-\int_0^\infty p''(u)\, e^{-\alpha u}\,\mathrm{d}u &= p'(0) + \alpha\big(p(0) - \alpha\,\pi(\alpha)\big).
\end{aligned}$$

Due to the local behavior of Brownian motion, $p(0) = 1$ (for any $\sigma^2 > 0$), so that

$$r - r\alpha\,\pi(\alpha) + \frac{\sigma^2}{2}\left(p'(0) + \alpha - \alpha^2\,\pi(\alpha)\right)$$
$$= \lambda b(\alpha)\pi(\alpha) + \lambda\,\frac{b(-\gamma_3) - b(\alpha+\gamma_2)}{\alpha+\gamma_2+\gamma_3} - \omega\,\pi(\alpha).$$

Solving $\pi(\alpha)$ from this relation, recognizing the Laplace exponent $\varphi(\alpha)$, we find

$$\pi(\alpha) = \frac{(\alpha + p'(0))\,\sigma^2/2 + r - \lambda\Pi(\alpha)}{\varphi(\alpha) - \gamma_1 - \beta}.$$

The unknown $p'(0)$ can be found using that any non-negative root of the denominator is a root of the denominator as well. This root being $\psi(\gamma_1+\beta)$, we obtain

$$p'(0) = \frac{2}{\sigma^2}\left(\lambda\,\Pi(\psi(\gamma_1+\beta)) - r - \frac{\sigma^2}{2}\psi(\gamma_1+\beta)\right).$$

Inserting this $p'(0)$, we recover Theorem 1.

3 Bivariate risk model: general results

So as to treat the bivariate case, in principle both methods that were demonstrated in the previous section can be used. As it turns out, in this bivariate setting the Kolmogorov-type approach introduced in Section 2.2 works particularly conveniently: we derive an integral equation for $p(\boldsymbol{u})$ in the bivariate case using this technique. In Sections 4 and 5 we successively solve this integral equation for the unperturbed ($\sigma^2 = 0$) and perturbed ($\sigma^2 > 0$) case. As the two cases have to be treated intrinsically differently, this division into two sections is natural.
Define, for $i = 1, 2$, the *univariate* counterparts of $p(\boldsymbol{u})$:

$$p_i(u) \equiv p(u, \beta, \gamma_{1i}, \gamma_{2i}, \gamma_{3i})$$
$$:= \mathbb{E}\big(e^{-\gamma_{1i}\tau_i(u) - \gamma_{2i}(u - Y_i(\tau_i(u)-)) - \gamma_{3i}(u - Y_i(\tau_i(u)))} 1\{\tau_i(u) \leqslant T_\beta\}\big),$$

describing the ruin times, undershoots and overshoots of the individual processes. In Section 2 we found the transform of $p_i(u)$, in both the unperturbed and the perturbed case, which in the sequel we denote by $\pi_i(\alpha)$, for $i = 1, 2$ and $\alpha \geqslant 0$. As $\Delta t \downarrow 0$, following the Kolmogorov-type reasoning, we find that the bivariate counterpart of (6) is, for any $\boldsymbol{u} \geqslant \boldsymbol{0}$ and with $\mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})$ denoting $\mathbb{P}(B^{(1)} \in \mathrm{d}v_1, B^{(2)} \in \mathrm{d}v_2)$, up to terms of order $o(\Delta t)$,

$$p(\boldsymbol{u}) = e^{-\boldsymbol{\gamma}_1{}^\top \boldsymbol{1}\,\Delta t}\Big(\lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=0}^{u_2} p(\boldsymbol{u}-\boldsymbol{v})\,\mathbb{P}(\boldsymbol{B}\in \mathrm{d}\boldsymbol{v}) +$$
$$\lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=u_2}^{\infty} p_1(u_1 - v_1)\,e^{-\gamma_{22}u_2 - \gamma_{32}(u_2 - v_2)}\,\mathbb{P}(\boldsymbol{B}\in \mathrm{d}\boldsymbol{v}) +$$
$$\lambda\,\Delta t \int_{v_1=u_1}^{\infty}\int_{v_2=0}^{u_2} p_2(u_2 - v_2)\,e^{-\gamma_{21}u_1 - \gamma_{31}(u_1 - v_1)}\,\mathbb{P}(\boldsymbol{B}\in \mathrm{d}\boldsymbol{v}) +$$

$$\lambda\,\Delta t \int_{v_1=u_1}^{\infty}\int_{v_2=u_2}^{\infty} e^{-\boldsymbol{\gamma}_2^\top \boldsymbol{u}-\boldsymbol{\gamma}_3^\top(\boldsymbol{u}-\boldsymbol{v})}\,\mathbb{P}(\boldsymbol{B}\in \mathrm{d}\boldsymbol{v}) +$$
$$\big(1-(\lambda+\beta)\Delta t\big)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\,\Delta t}}\exp\left(-\frac{y^2}{2\sigma^2\,\Delta t}\right)p(\boldsymbol{u}+r\,\mathbf{1}\Delta t-y\,\mathbf{1})\,\mathrm{d}y\Big), \quad (7)$$

in the case that $\sigma^2>0$; if $\sigma^2=0$, the last term between brackets on the right-hand side has to be replaced by

$$\big(1-(\lambda+\beta)\Delta t\big)\,p(\boldsymbol{u}+r\,\mathbf{1}\Delta t).$$

Using the same procedure as in the univariate case, we find that (7) leads to the following integro-differential equation: for $\boldsymbol{u}\geqslant \mathbf{0}$,

$$\begin{aligned}
-r\Big(\frac{\partial}{\partial u_1}p(\boldsymbol{u})+\frac{\partial}{\partial u_2}p(\boldsymbol{u})\Big)-\frac{\sigma^2}{2}\Big(\frac{\partial^2}{\partial u_1^2}p(\boldsymbol{u})+2\frac{\partial^2}{\partial u_1 u_2}p(\boldsymbol{u})+\frac{\partial^2}{\partial u_2^2}p(\boldsymbol{u})\Big)&\\
=\lambda\int_0^{u_1}\int_0^{u_2} p(\boldsymbol{u}-\boldsymbol{v})\,\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})+&\\
\lambda\int_0^{u_1}\int_{u_2}^{\infty} p_1(u_1-v_1)\,e^{-\gamma_{22}u_2-\gamma_{32}(u_2-v_2)}\,\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})+&\\
\lambda\int_{u_1}^{\infty}\int_0^{u_2} p_2(u_2-v_2)\,e^{-\gamma_{21}u_1-\gamma_{31}(u_1-v_1)}\,\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})+&\\
\lambda\int_{u_1}^{\infty}\int_{u_2}^{\infty} e^{-\boldsymbol{\gamma}_2^\top\boldsymbol{u}-\boldsymbol{\gamma}_3^\top(\boldsymbol{u}-\boldsymbol{v})}\,\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})-\big(\mathbf{1}^\top\boldsymbol{\gamma}_1+\lambda+\beta\big)\,p(\boldsymbol{u}).&\quad (8)
\end{aligned}$$

From this point on we have to distinguish between the cases $\sigma^2=0$, which will be handled in Section 4, and $\sigma^2>0$, which will be handled in Section 5.

We conclude this section by a useful auxiliary result, that is of crucial importance in Sections 4 and 5. Importantly, it applies irrespective of whether σ^2 is positive or zero. Observe that, for a given $\beta>0$, we can rewrite the equation $\varphi(\boldsymbol{\alpha})-\beta=0$ as

$$\lambda b(\boldsymbol{\alpha})=c(\boldsymbol{\alpha}):=\lambda+\beta-r\mathbf{1}^\top\boldsymbol{\alpha}-\frac{\sigma^2}{2}(\alpha_1+\alpha_2)^2. \quad (9)$$

Fixing α_1 with $\mathrm{Re}\,\alpha_1>0$ and β, due to the lemma below we can identify a unique α_2 such that $\varphi(\boldsymbol{\alpha})-\beta=0$ in the right part of the complex plane; we denote this α_2 by $\omega_2(\alpha_1,\beta)$.

Lemma 1 *For every α_1 with $\mathrm{Re}\,\alpha_1>0$ and $\beta>0$, there exists a unique $\alpha_2=\omega_2(\alpha_1,\beta)$ with $\mathrm{Re}\,\omega_2(\alpha_1,\beta)>\mathrm{Re}\,(-\alpha_1)$ that satisfies Equation (9). For any $\beta>0$, the function $\alpha_1\mapsto\omega_2(\alpha_1,\beta)$ is analytic in $\mathrm{Re}\,\alpha_1>0$.*

Proof. We prove this by applying Rouché's theorem. To this end, consider the contour $\mathscr{C}$ consisting of (i) the line segment

$$\mathscr{C}_\ell:=\{-\alpha_1+\mathrm{i}\omega\,|\,\omega\in[-R,R]\},$$

with $R>0$, and (ii) to its right the semicircle

$$\mathscr{C}_s:=\{-\alpha_1+Re^{\mathrm{i}\phi}\,|\,\phi\in[-\pi/2,\pi/2]\}.$$

The next step is to show that on this contour $\mathscr{C}$

$$|\lambda b(\boldsymbol{\alpha})| < |c(\boldsymbol{\alpha})|; \tag{10}$$

cf. Equation (9). First observe that we can write

$$\lambda b(\boldsymbol{\alpha}) = \lambda\,\mathbb{E}\exp\left(-\alpha_1(B^{(1)} - B^{(2)}) - (\alpha_1 + \alpha_2)B^{(2)}\right). \tag{11}$$

For all $\boldsymbol{\alpha}$ such that $\mathrm{Re}\,\alpha_2 > \mathrm{Re}\,(-\alpha_1)$ we have that

$$|\lambda b(\boldsymbol{\alpha})| \leqslant \lambda\,\mathbb{E}\exp\left(-\mathrm{Re}\,\alpha_1\big(B^{(1)} - B^{(2)}\big) - \mathrm{Re}(\alpha_1 + \alpha_2)B^{(2)}\right) \leqslant \lambda,$$

recalling that we have assumed in (1) that $B^{(1)} \geqslant B^{(2)}$ almost surely. To prove (10) it therefore suffices to show that $|c(\boldsymbol{\alpha})| > \lambda$. This is done as follows.

- Observe that on the line segment $\mathscr{C}_\ell$ we have that

  $$c(\boldsymbol{\alpha}) = \lambda + \beta - r\,\mathrm{i}\omega + \frac{\sigma^2}{2}\omega^2.$$

  This implies that $|c(\boldsymbol{\alpha})| > \lambda$ on $\mathscr{C}_\ell$.

- Observe that on the semicircle $\mathscr{C}_s$ we have that

  $$c(\boldsymbol{\alpha}) = \lambda + \beta - r\,Re^{\mathrm{i}\phi} - \frac{\sigma^2}{2}R^2e^{2\mathrm{i}\phi}.$$

  As a consequence, for R large enough we have that $|c(\boldsymbol{\alpha})| > \lambda$ on $\mathscr{C}_s$.

Now that we know that $|\lambda b(\boldsymbol{\alpha})| < |c(\boldsymbol{\alpha})|$ on $\mathscr{C}$, it follows by Rouché's theorem that the equation $\lambda b(\boldsymbol{\alpha}) = c(\boldsymbol{\alpha})$ has, for a given α_1 with $\mathrm{Re}\,\alpha_1 > 0$ and $\beta > 0$, a unique solution $\alpha_2 = \omega_2(\alpha_1,\beta)$ with $\mathrm{Re}\,\omega_2(\alpha_1,\beta) > \mathrm{Re}\,(-\alpha_1)$. We use here that the quadratic function $c(\boldsymbol{\alpha})$ has a unique zero inside $\mathscr{C}$, which can be seen as follows. With $\alpha_2 = -\alpha_1 + y$ the equation $c(\boldsymbol{\alpha}) = 0$ becomes $(\sigma^2/2)\,y^2 + ry - (\lambda + \beta) = 0$, which has precisely one positive root.
The first claim of the lemma now follows by sending R to ∞, and observing that $\lambda b(\boldsymbol{\alpha})$, as given by (11), and $c(\boldsymbol{\alpha})$ are analytic on and inside $\mathscr{C}$. The fact that $\omega_2(\alpha_1,\beta)$ is analytic in $\mathrm{Re}\,\alpha_1 > 0$ follows by using the implicit function theorem; cf. [11, p. 101]. □

4 Bivariate unperturbed risk model

This section treats the unperturbed case (i.e., $\sigma^2 = 0$); we shall see in Section 5 that having $\sigma^2 > 0$ will force us to make a few significant changes in the ensuing analysis. With the integro-differential equation (8) at our disposal, we proceed by performing the (double) transform with respect to $\boldsymbol{u}$: we multiply the full equation by $e^{-\boldsymbol{\alpha}^\top\boldsymbol{u}}$ (with $\boldsymbol{\alpha} \geqslant \mathbf{0}$) and integrate over the non-negative u_1 and u_2. After some elementary calculus it is found that the sum of the five terms on the right-hand side equals

$$\big(\lambda b(\boldsymbol{\alpha}) - \mathbf{1}^\top\boldsymbol{\gamma}_1 - \lambda - \beta\big)\,\pi(\boldsymbol{\alpha}) + \lambda\zeta(\boldsymbol{\alpha}),$$

where

$$\zeta(\boldsymbol{\alpha}) :=$$
$$\pi_1(\alpha_1)\,\frac{b(\alpha_1,-\gamma_{32})-b(\alpha_1,\alpha_2+\gamma_{22})}{\alpha_2+\gamma_{22}+\gamma_{32}}+\pi_2(\alpha_2)\,\frac{b(-\gamma_{31},\alpha_2)-b(\alpha_1+\gamma_{21},\alpha_2)}{\alpha_1+\gamma_{21}+\gamma_{31}}+$$
$$\frac{b(-\gamma_{31},-\gamma_{32})-b(-\gamma_{31},\alpha_2+\gamma_{22})-b(\alpha_1+\gamma_{21},-\gamma_{32})+b(\alpha_1+\gamma_{21},\alpha_2+\gamma_{22})}{(\alpha_1+\gamma_{21}+\gamma_{31})(\alpha_2+\gamma_{22}+\gamma_{32})}.$$

The left-hand side can be evaluated using integration by parts, so as to obtain

$$-r\,\mathbf{1}^\top\boldsymbol{\alpha}\,\pi(\boldsymbol{\alpha})+r\pi_1^\circ(\alpha_2)+r\pi_2^\circ(\alpha_1),$$

where

$$\pi_1^\circ(\alpha) := \int_0^\infty p(0,u)\,e^{-\alpha u}\,\mathrm{d}u,\qquad \pi_2^\circ(\alpha) := \int_0^\infty p(u,0)\,e^{-\alpha u}\,\mathrm{d}u.$$

Recalling that $\varphi(\boldsymbol{\alpha}) = r\,\mathbf{1}^\top\boldsymbol{\alpha}-\lambda(1-b(\boldsymbol{\alpha}))$, this leads to the following result.

Proposition 1 *For any $\boldsymbol{\alpha}\geqslant\mathbf{0}$, $\beta>0$, $\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\geqslant\mathbf{0}$, $\boldsymbol{\gamma}_3\leqslant\mathbf{0}$,*

$$\pi(\boldsymbol{\alpha}) = \frac{r(\pi_1^\circ(\alpha_2)+\pi_2^\circ(\alpha_1))-\lambda\zeta(\boldsymbol{\alpha})}{\varphi(\boldsymbol{\alpha})-\mathbf{1}^\top\boldsymbol{\gamma}_1-\beta}. \tag{12}$$

We have, however, not yet identified the functions $\pi_i^\circ(\alpha)$. The key idea is that the almost sure ordering (1), which enforces $Y_1(t)\geqslant Y_2(t)$, can be used to evaluate $\pi_1^\circ(\alpha)$, where a crucial role is played by the fact that $\tau_1(0)\leqslant\tau_2(u)$ for all $u\geqslant 0$. Then, by Lemma 1 with $\sigma^2=0$ (which we emphasize by writing $\omega_2^0(\cdot,\cdot)$ rather than just $\omega_2(\cdot,\cdot)$),

$$\pi_2^\circ(\alpha) = -\pi_1^\circ\big(\omega_2^0(\alpha,\mathbf{1}^\top\boldsymbol{\gamma}_1+\beta)\big)+\frac{\lambda}{r}\,\zeta\big(\alpha,\omega_2^0(\alpha,\mathbf{1}^\top\boldsymbol{\gamma}_1+\beta)\big). \tag{13}$$

We proceed by determining $\pi_1^\circ(\alpha)$ in two steps: we first express $\pi_1^\circ(\alpha)$ in terms of an auxiliary function, and then we evaluate that function.

In our analysis an important role is played by

$$\boldsymbol{Z}(u)\equiv(Z_1(u),Z_2(u))^\top := (Y_2(\tau_1(u)),B_2^\circ(u))^\top,$$

where $B_2^\circ(u)$ denotes the size of the claim in the net cumulative claim process $Y_2(t)$ at the ruin time $\tau_1(u)$ corresponding to the net cumulative claim process $Y_1(t)$. A key object is the transform of the ruin time, undershoot and overshoot related to the process $Y_1(t)$, jointly with this random vector $\boldsymbol{Z}(u)$:

$$\bar{p}_1(u,\mathrm{d}\boldsymbol{z}) := \mathbb{E}\big(e^{-\mathbf{1}^\top\boldsymbol{\gamma}_1\tau_1(u)-\gamma_{21}(u-Y_1(\tau_1(u)-))-\gamma_{31}(u-Y_1(\tau_1(u)))}\mathbb{I}(u,\mathrm{d}\boldsymbol{z})\big),$$

with $\mathbb{I}(u,\mathrm{d}\boldsymbol{z}) := 1\{\tau_1(u)\leqslant T_\beta,\boldsymbol{Z}(u)\in\mathrm{d}\boldsymbol{z}\}$. Then the 'master equation' is given by

$$p(0,u) = \int_{z_1=u}^\infty\int_{z_2=0}^\infty \bar{p}_1(0,\mathrm{d}\boldsymbol{z})\,e^{-\gamma_{22}(u-z_1+z_2)-\gamma_{32}(u-z_1)}+$$

$$\int_{z_1=-\infty}^{u}\int_{z_2=0}^{\infty}\bar{p}_1(0,\mathrm{d}z)\,p_2(u-z_1). \tag{14}$$

To parse this decomposition, first note that the first term on the right-hand side corresponds to the scenario that $Y_2(t)$ exceeds u for the first time at $\tau_1(0)$ (i.e., $\tau_1(0)=\tau_2(u)$), and the second term to the scenario that u is not yet exceeded by $Y_2(t)$ at time $\tau_1(0)$ (i.e., $\tau_1(0)<\tau_2(u)$). In this reasoning we use that $\tau_1(0)\leqslant\tau_2(u)$ for all $u\geqslant 0$, implying that $\tau_1(0)>\tau_2(u)$ cannot occur. We refer to Figure 2 for an illustration of both possible scenarios. Then the right-hand side of (14) follows by the following observations:

- In case $\tau_1(0)=\tau_2(u)$, the undershoot is

$$\begin{aligned} u-Y_2(\tau_2(u)-) &= u-Y_2(\tau_1(0)-)\\ &= u-Y_2(\tau_1(0))+B_2^{\circ}(0)=u-Z_1(0)+Z_2(0), \end{aligned}$$

  and the corresponding overshoot is

$$u-Y_2(\tau_2(u))=u-Y_2(\tau_1(0))=u-Z_1(0).$$

- In case $\tau_1(0)<\tau_2(u)$, at $\tau_1(0)$ the second component still needs to bridge a residual distance of $u-Y_2(\tau_1(0))=u-Z_1(0)$. In this case, $\tau_2(u)$ is decomposed into $\tau_1(0)$, in (14) taken care of by the contribution $e^{-\gamma_{12}\tau(0)}$ in $\bar{p}_1(0,\mathrm{d}z)$, plus $\tau_2(u)-\tau_1(0)$, in (14) taken care of by $p_2(y-z_1)$.

By appealing to (14), it is immediate that the object of our interest equals

$$\begin{aligned} \pi_1^{\circ}(\alpha)= &\int_0^{\infty}e^{-\alpha u}\int_{z_1=u}^{\infty}\int_{z_2=0}^{\infty}\bar{p}_1(0,\mathrm{d}z)\,e^{-\gamma_{22}(u-z_1+z_2)-\gamma_{32}(u-z_1)}\,\mathrm{d}u\,+\\ &\int_0^{\infty}e^{-\alpha u}\int_{z_1=-\infty}^{u}\int_{z_2=0}^{\infty}\bar{p}_1(0,\mathrm{d}z)\,p_2(u-z_1)\,\mathrm{d}u. \end{aligned} \tag{15}$$

The main idea of the rest of our analysis is that we uniquely characterize $\pi_1^{\circ}(\alpha)$ by identifying the transform of $\bar{p}_1(0,\mathrm{d}z)$. To this end, we define, for $\boldsymbol{\delta}=(\delta_1,\delta_2)^{\top}$,

$$\begin{aligned} \xi(\boldsymbol{\delta}) &:= \mathbb{E}\big(e^{-\mathbf{1}^{\top}\boldsymbol{\gamma}_1\tau_1(0)+\gamma_{21}Y_1(\tau_1(0)-)+\gamma_{31}Y_1(\tau(0))-\delta_1Y_2(\tau_1(0))-\delta_2B_2^{\circ}(0)}1\{\tau_1(0)\leqslant T_{\beta}\}\big)\\ &= \mathbb{E}\big(e^{-\mathbf{1}^{\top}\boldsymbol{\gamma}_1\tau_1(0)+\gamma_{21}Y_1(\tau_1(0)-)+\gamma_{31}Y_1(\tau(0))-\boldsymbol{\delta}^{\top}\boldsymbol{Z}(0)}1\{\tau_1(0)\leqslant T_{\beta}\}\big)\\ &= \int_{z_1=-\infty}^{\infty}\int_{z_2=0}^{\infty}\bar{p}_1(0,\mathrm{d}z)\,e^{-\boldsymbol{\delta}^{\top}\boldsymbol{z}}. \end{aligned}$$

Conclude that if we are able to compute $\xi(\boldsymbol{\delta})$ for $\boldsymbol{\delta}\geqslant\mathbf{0}$, we have (albeit implicitly) identified $\bar{p}_1(0,\mathrm{d}z)$, which enables us to compute $\pi_1^{\circ}(\alpha)$ by (15); recall that we know $p_2(u-z_1)$, also appearing in (15) (via its transform, that is).
To evaluate $\xi(\boldsymbol{\delta})$, we first study a more general object, from which later $\xi(\boldsymbol{\delta})$ can be found:

$$\check{p}_1(u,\boldsymbol{\delta}):=\mathbb{E}\big(e^{-\mathbf{1}^{\top}\boldsymbol{\gamma}_1\tau_1(u)-\gamma_{21}(u-Y_1(\tau_1(u)-))-\gamma_{31}(u-Y_1(\tau(u)))-\boldsymbol{\delta}^{\top}\boldsymbol{Z}(u)}1\{\tau_1(u)\leqslant T_{\beta}\}\big);$$

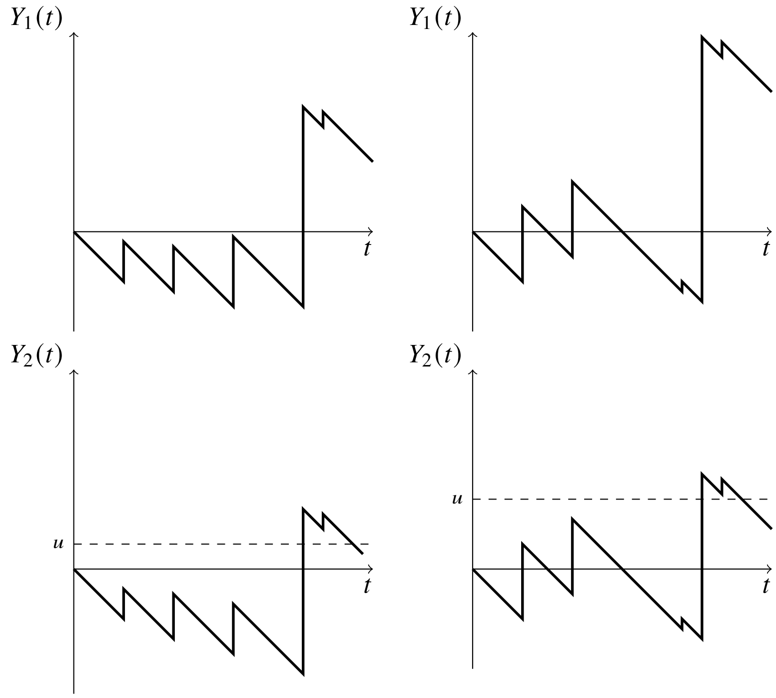

Figure 2. Net cumulative claim processes $Y_1(t)$ and $Y_2(t)$ such that $Y_1(t) \geqslant Y_2(t)$ for all $t \geqslant 0$. The left panels display a scenario of $(Y_1(t), Y_2(t))$ in which $\tau_1(0) = \tau_2(u)$, whereas the right panels display a scenario of $(Y_1(t), Y_2(t))$ in which $\tau_1(0) < \tau_2(u)$.

note that $\xi(\boldsymbol{\delta}) = \check{p}_1(0, \boldsymbol{\delta})$. Again relying on a Kolmogorov-type derivation, we can determine the transform of $\check{p}_1(u, \boldsymbol{\delta})$. Indeed, as $\Delta t \downarrow 0$,

$$\begin{aligned}\check{p}_1(u, \boldsymbol{\delta}) = e^{-\mathbf{1}^\top \boldsymbol{\gamma}_1 \Delta t + r\,\delta_1 \Delta t} \Big(&\lambda \,\Delta t \int_{v_1=0}^{u} \int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, \check{p}_1(u - v_1)\, e^{-\delta_1 v_2} + \\ &\lambda \,\Delta t \int_{v_1=u}^{\infty} \int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, e^{-\gamma_{21} u}\, e^{-\gamma_{31}(u - v_1)}\, e^{-\mathbf{1}^\top \boldsymbol{\delta}\, v_2} + \\ &(1 - \lambda \,\Delta t - \beta \,\Delta t)\, \check{p}_1(u + r\,\Delta t, \boldsymbol{\delta}) \Big), \end{aligned} \tag{16}$$

up to $o(\Delta t)$-terms. The first double integral in (16) is the contribution due to the scenario that there is a claim arrival but that, despite this claim, $Y_1(t)$ remains below u; the second double integral represents the contribution due to the scenario that there is a claim arrival by which $Y_1(t)$ exceeds u. Subtracting $\check{p}_1(u + r\,\Delta t \boldsymbol{\delta})$ from both sides of (16), dividing by Δt, and letting $\Delta t \downarrow 0$, we obtain an integro-differential equation. Multiplying this equation by $e^{-\alpha u}$ and integrating over u, using the same steps as in

Section 3, we eventually find that

$$\check{\pi}_1(\alpha,\boldsymbol{\delta}) := \int_0^\infty e^{-\alpha u}\,\check{p}_1(u,\boldsymbol{\delta})\,\mathrm{d}u$$

equals

$$\frac{1}{\varphi(\alpha,\delta_1)-\mathbf{1}^\top\boldsymbol{\gamma}_1-\beta}\left(r\,\check{p}_1(0,\boldsymbol{\delta})-\lambda\frac{b(-\gamma_{31},\mathbf{1}^\top\boldsymbol{\delta})-b(\alpha+\gamma_{21},\mathbf{1}^\top\boldsymbol{\delta})}{\alpha+\gamma_{21}+\gamma_{31}}\right). \tag{17}$$

Now (17) can be used to determine $\check{p}_1(0,\boldsymbol{\delta})$, as follows. Recall that for any α (with non-negative real part, that is) for which $\varphi(\alpha,\delta_1)-\mathbf{1}^\top\boldsymbol{\gamma}_1-\beta=0$, the term between brackets in (17) should vanish as well. Writing $\alpha^\circ \equiv \alpha^\circ(\beta,\boldsymbol{\gamma}_1,\delta_1) := \psi_1(\mathbf{1}^\top\boldsymbol{\gamma}_1+\beta)$, with $\beta\mapsto\psi_1(\beta)$ denoting the right-inverse of $\alpha\mapsto\varphi(\alpha,\delta_1)$, we conclude

$$\check{p}_1(0,\boldsymbol{\delta}) = \xi(\boldsymbol{\delta}) = \frac{\lambda}{r}\frac{b(-\gamma_{31},\mathbf{1}^\top\boldsymbol{\delta})-b(\alpha^\circ+\gamma_{21},\mathbf{1}^\top\boldsymbol{\delta})}{\alpha^\circ+\gamma_{21}+\gamma_{31}}. \tag{18}$$

We have now found all ingredients that allow the evaluation of $\pi_1^\circ(\alpha)$.

Proposition 2 *For any $\alpha \geqslant 0$, $\beta > 0$, $\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$, we have that $\pi_1^\circ(\alpha)$ is given by Equation* (15)*, where the transform $\xi(\boldsymbol{\delta})$ is given by Equation* (18)*.*

We have thus uniquely characterized the Gerber-Shiu metrics of the coupled risk system in the unperturbed case. The idea is that the transform $\xi(\boldsymbol{\delta})$ defines $\bar{p}_1(0,\mathrm{d}\boldsymbol{z})$, through which we can evaluate $\pi_1^\circ(\alpha)$ via Equation (15), which enables the calculation of transform $\pi_2^\circ(\alpha)$ via Equation (13), after which $\pi(\boldsymbol{\alpha})$ follows from (12).

Theorem 2 *For any $\boldsymbol{\alpha} \geqslant \mathbf{0}$, $\beta > 0$, $\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$, we have that $\pi(\boldsymbol{\alpha})$ is given by* (12)*. The transform $\pi_1^\circ(\alpha)$ follows from Proposition* 2*, and the transform $\pi_2^\circ(\alpha)$ from Equation* (13)*.*

5 Bivariate perturbed risk model

In this section we consider the perturbed case, i.e., from now on we have that $\sigma^2 > 0$. Like in the unperturbed case, the starting point is the integro-differential equation (8). Because of the Brownian term, we have that in this case both $Y_1(t)$ and $Y_2(t)$ attain positive values before T_β, with probability 1. In particular, we now have $\tau_1(0) = \tau_2(0) = 0$, and hence $p(\mathbf{0}) = 1$, $p(u_1,0) = p_1(u_1)$ and $p(0,u_2) = p_2(u_2)$. In the previous section, we needed to determine the 'boundary transforms' $\pi_1^\circ(\alpha)$ and $\pi_2^\circ(\alpha)$; in the current $\sigma^2 > 0$ case, however, $\pi_1^\circ(\alpha) = \pi_2(\alpha)$ and $\pi_2^\circ(\alpha) = \pi_1(\alpha)$ are known.
Defining

$$\partial\pi_1^\circ(\alpha) := \int_0^\infty \left.\frac{\partial}{\partial u_1}p(u_1,u_2)\right|_{u_1=0} e^{-\alpha u_2}\,\mathrm{d}u_2,$$
$$\partial\pi_2^\circ(\alpha) := \int_0^\infty \left.\frac{\partial}{\partial u_2}p(u_1,u_2)\right|_{u_2=0} e^{-\alpha u_1}\,\mathrm{d}u_1,$$

a lengthy but elementary computation reveals that,

$$\int_0^\infty \int_0^\infty e^{-\alpha_1 u_1 - \alpha_2 u_2}\Big(\frac{\partial^2}{\partial u_1^2} p(\boldsymbol{u}) + 2\frac{\partial^2}{\partial u_1 u_2} p(\boldsymbol{u}) + \frac{\partial^2}{\partial u_2^2} p(\boldsymbol{u})\Big) \mathrm{d}u_2\, \mathrm{d}u_1$$
$$= 2 + (\alpha_1+\alpha_2)^2 \pi(\boldsymbol{\alpha}) - \Sigma(\boldsymbol{\alpha}) - \partial\pi_1^\circ(\alpha_2) - \partial\pi_2^\circ(\alpha_1),$$

with the (known) function

$$\Sigma(\boldsymbol{\alpha}) := (\alpha_2 + 2\alpha_1)\pi_1(\alpha_1) + (\alpha_1 + 2\alpha_2)\pi_2(\alpha_2).$$

Define, for $\boldsymbol{\alpha} \geqslant \mathbf{0}$, $\varphi(\boldsymbol{\alpha}) := \frac{1}{2}\sigma^2\,(\alpha_1+\alpha_2)^2 + r\,\mathbf{1}^\top\boldsymbol{\alpha} - \lambda(1-b(\boldsymbol{\alpha}))$ as our bivariate Laplace exponent. As an intermediate result, we have obtained the following proposition.

Proposition 3 *For any $\boldsymbol{\alpha} \geqslant \mathbf{0}$, $\beta > 0$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$,*

$$\pi(\boldsymbol{\alpha}) = \frac{(\Sigma(\boldsymbol{\alpha}) + \partial\pi_1^\circ(\alpha_2) + \partial\pi_2^\circ(\alpha_1) - 2)\sigma^2/2 + r(\pi_1(\alpha_1) + \pi_2(\alpha_2)) - \lambda\zeta(\boldsymbol{\alpha})}{\varphi(\boldsymbol{\alpha}) - \mathbf{1}^\top\boldsymbol{\gamma}_1 - \beta}. \tag{19}$$

Observe that, reassuringly, Proposition 3 reduces to Proposition 1 when taking $\sigma^2 = 0$ (despite that we could not follow the same argumentation). By Lemma 1, and using that any root of the denominator is a root of the numerator as well, we obtain that

$$\partial\pi_2^\circ(\alpha) = -\partial\pi_1^\circ\big(\omega_2(\alpha, \mathbf{1}^\top\boldsymbol{\gamma}_1 + \beta)\big) - \Sigma\big(\alpha, \omega_2(\alpha, \mathbf{1}^\top\boldsymbol{\gamma}_1 + \beta)\big) + 2$$
$$+ \frac{\lambda}{r}\,\frac{2}{\sigma^2}\,\zeta\big(\alpha, \omega_2(\alpha, \mathbf{1}^\top\boldsymbol{\gamma}_1 + \beta)\big) - \frac{2r}{\sigma^2}\,(\pi_1(\alpha) + \pi_2(\omega_2(\alpha, \mathbf{1}^\top\boldsymbol{\gamma}_1 + \beta))); \tag{20}$$

cf. (13). We are thus left with identifying $\partial\pi_1^\circ(\alpha)$, which can be done relying on our ordering, mimicking the procedure followed in Section 4.
Denote, with $\bar{p}_1(u, \mathrm{d}z)$ as defined in Section 4,

$$\bar{p}_1'(u, \mathrm{d}z) := \frac{\partial}{\partial u}\bar{p}_1(u, \mathrm{d}z)$$

(which, in this perturbed case, does not have any irregular behavior at $u = 0$). The new master equation becomes

$$\frac{\partial}{\partial u_1} p(u_1, u_2)\Big|_{u_1=0} = \int_{z_1=u_2}^\infty \int_{z_2=0}^\infty \bar{p}_1'(0, \mathrm{d}z)\, e^{-\gamma_{22}(u_2 - z_1 + z_2) - \gamma_{32}(u_2 - z_1)} + \int_{z_1=-\infty}^{u_2} \int_{z_2=0}^\infty \bar{p}_1'(0, \mathrm{d}z)\, p_2(u_2 - z_1), \tag{21}$$

so that

$$\partial\pi_1^\circ(\alpha) = \int_0^\infty e^{-\alpha u} \int_{z_1=u}^\infty \int_{z_2=0}^\infty \bar{p}_1'(0, \mathrm{d}z)\, e^{-\gamma_{22}(u - z_1 + z_2) - \gamma_{32}(u - z_1)}\, \mathrm{d}u + \int_0^\infty e^{-\alpha u} \int_{z_1=-\infty}^{u} \int_{z_2=0}^\infty \bar{p}_1'(0, \mathrm{d}z)\, p_2(u - z_1)\, \mathrm{d}u. \tag{22}$$

The object of our interest, $\partial\pi_1^\circ(\alpha)$, is the transform of (21) with respect to u_2. It therefore suffices to identify $\bar{p}_1'(0, \mathrm{d}z)$, which we do via its transform; bear in mind that we found the transform of $p_2(u)$ in Section 2. Similar to what we have done before, we define $\xi(\boldsymbol{\delta})$ now by

$$\xi(\boldsymbol{\delta}) := \int_{z_1=-\infty}^{\infty}\int_{z_2=0}^{\infty} \bar{p}_1'(0, \mathrm{d}\boldsymbol{z})\, e^{-\boldsymbol{\delta}^\top \boldsymbol{z}}.$$

Let $\check{p}_1(u, \boldsymbol{\delta})$ be as in Section 4, so that $\xi(\boldsymbol{\delta}) = \check{p}_1'(0, \boldsymbol{\delta})$. As before, we find $\check{p}_1'(0, \boldsymbol{\delta})$ by first identifying the transform of $\check{p}_1'(u, \boldsymbol{\delta})$. In our perturbed system, we have the following Kolmogorov-type equation: as $\Delta t \downarrow 0$,

$$\begin{aligned}
\check{p}_1(u, \boldsymbol{\delta}) =\ & e^{-\mathbf{1}^\top \boldsymbol{\gamma}_1 \Delta t + r\delta_1 \Delta t}\Bigg(\lambda\, \Delta t \int_{v_1=0}^{u}\int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, \check{p}_1(u - v_1, \boldsymbol{\delta})\, e^{-\delta_1 v_2} + \\
& \lambda\, \Delta t \int_{v_1=u}^{\infty}\int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, e^{-\gamma_{21} u}\, e^{-\gamma_{31}(u - v_1)}\, e^{-\mathbf{1}^\top \boldsymbol{\delta}\, v_2} + \\
& (1 - \lambda\, \Delta t - \beta\, \Delta t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\, \Delta t}} \exp\left(-\frac{y^2}{2\sigma^2\, \Delta t}\right) \check{p}_1(u + r\, \Delta t - y, \boldsymbol{\delta})\, e^{-\delta_1 y}\, \mathrm{d}y \Bigg),
\end{aligned}$$

up to $o(\Delta t)$-terms. From this we obtain, as usual, an integro-differential equation: by the substitution $z := y/(\sigma\sqrt{\Delta t})$ and working out the Taylor expansions of the various functions involved, we find, with $\omega := \lambda + \mathbf{1}^\top \boldsymbol{\gamma}_1 + \beta - r\delta_1$,

$$\begin{aligned}
-\frac{\sigma^2}{2}\delta_1^2\, \check{p}_1(u, \boldsymbol{\delta}) &- (r + \delta_1\sigma^2)\, \check{p}_1'(u, \boldsymbol{\delta})\ -\ \frac{\sigma^2}{2}\check{p}_1''(u, \boldsymbol{\delta}) \\
&= \lambda \int_{v_1=0}^{u}\int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, \check{p}_1(u - v_1, \boldsymbol{\delta})\, e^{-\delta_1 v_2} + \\
&\quad \lambda \int_{v_1=u}^{\infty}\int_{v_2=0}^{\infty} \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\, e^{-\gamma_{21} u}\, e^{-\gamma_{31}(u - v_1)}\, e^{-\mathbf{1}^\top \boldsymbol{\delta}\, v_2} - \omega\, \check{p}_1(u, \boldsymbol{\delta});
\end{aligned}$$

here $\check{p}_1'(u, \boldsymbol{\delta})$ and $\check{p}_1''(u, \boldsymbol{\delta})$ are the first and second order derivatives of $\check{p}_1(u, \boldsymbol{\delta})$ with respect to u, and we have used that

$$\begin{aligned}
&\check{p}_1(u + r\, \Delta t - z\sigma\sqrt{\Delta t}, \boldsymbol{\delta})\, e^{-\delta_1 z\sigma\sqrt{\Delta t}} \\
&\quad = \check{p}_1(u, \boldsymbol{\delta})\left(1 + \frac{\sigma^2}{2}\delta_1^2\, \Delta t\right) + \check{p}_1'(u, \boldsymbol{\delta})(r + \delta_1\sigma^2)\, \Delta t + \check{p}_1''(u, \boldsymbol{\delta})\frac{\sigma^2}{2}\, \Delta t + o(\Delta t).
\end{aligned}$$

Now transform with respect to u, so as to obtain the transform $\check{\pi}_1(\alpha)$. Multiplying by $e^{-\alpha u}$ and integrating over u, and using that $\check{p}_1(0) = 1$ (bearing in mind that $\tau_1(0) = 0$ in this perturbed case), after considerable calculus we obtain that $\check{\pi}_1(\alpha)$ is given by

$$\frac{1}{\varphi(\alpha, \delta_1) - \omega'}\left((\alpha + \check{p}_1'(0, \boldsymbol{\delta}))\frac{\sigma^2}{2} + r + \delta_1\sigma^2 - \lambda\frac{b(-\gamma_{31}, \mathbf{1}^\top\boldsymbol{\delta}) - b(\alpha + \gamma_{21}, \mathbf{1}^\top\boldsymbol{\delta})}{\alpha + \gamma_{21} + \gamma_{31}}\right),$$

with $\omega' := \mathbf{1}^\top \boldsymbol{\gamma}_1 + \beta$. Writing $\alpha^\circ \equiv \alpha^\circ(\beta, \boldsymbol{\gamma}_1, \delta_1) := \psi_1(\omega')$, with $\beta \mapsto \psi_1(\beta)$ denoting the right-inverse of $\alpha \mapsto \varphi(\alpha, \delta_1)$, this implies that $\xi(\boldsymbol{\delta}) = \check{p}_1'(0, \boldsymbol{\delta})$ is given by

$$\xi(\boldsymbol{\delta}) = \frac{2}{\sigma^2}\left(\lambda\frac{b(-\gamma_{31}, \mathbf{1}^\top\boldsymbol{\delta}) - b(\alpha^\circ + \gamma_{21}, \mathbf{1}^\top\boldsymbol{\delta})}{\alpha^\circ + \gamma_{21} + \gamma_{31}} - r - \delta_1\sigma^2 - \frac{\sigma^2}{2}\alpha^\circ\right). \tag{23}$$

Upon combining the above, we have identified $\partial\pi_1^\circ(\alpha)$.

Proposition 4 *For any $\alpha \geqslant 0$, $\beta > 0$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$, we have that $\partial\pi_1^\circ(\alpha)$ is given by Equation* (22)*, with $\xi(\boldsymbol{\delta})$ given by Equation* (23)*.*

We have thus found the Gerber-Shiu metrics in the coupled perturbed risk system, providing the joint distribution of the ruin times, undershoots and overshoots pertaining to $Y_1(t)$ and $Y_2(t)$.

Theorem 3 *For any $\boldsymbol{\alpha} \geqslant \mathbf{0}$, $\beta > 0$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$, we have that $\pi(\boldsymbol{\alpha})$ is given by* (12)*. The transform $\partial\pi_1^\circ(\alpha)$ follows from Proposition 4, and the transform $\partial\pi_2^\circ(\alpha)$ from Equation* (20)*.*

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Appendix A: proof of Theorem 1.

Define $\pi_-(\alpha)$ analogously to $\pi(\alpha)$, but with $p(u)$ replaced by $p_-(u)$. By a few standard computations, it follows from (3) that

$$\pi_-(\alpha) = b(\alpha)\,\pi(\alpha) + \Pi(\alpha), \tag{24}$$

with $\Pi(\alpha)$ as given in (4). It is immediate that, with $\theta := \lambda + \gamma_1 + \beta$,

$$\int_0^\infty \mathbb{P}(\tau_W(u) \in \mathrm{d}v)\,\mathbb{P}(T_{\lambda+\beta} \geqslant v)\,e^{-\gamma_1 v} = \mathbb{P}\left(\bar{Y}_W(T_\theta) \geqslant u\right),$$

so that, with x_θ^+ as defined above,

$$\int_0^\infty e^{-\alpha u} \int_0^\infty \mathbb{P}(\tau_W(u) \in \mathrm{d}v)\,\mathbb{P}(T_{\lambda+\beta} \geqslant v)\,e^{-\gamma_1 v}\,\mathrm{d}u = \frac{1}{\alpha + x_\theta^+}.$$

Also,

$$\begin{aligned}\int_{t=0}^\infty \int_{w=0}^\infty \int_{v=0}^u \lambda e^{-(\lambda+\beta)t}\,\mathbb{P}(\bar{Y}_W(t) \in \mathrm{d}v)\,\mathbb{P}(-\underline{Y}_W(t) \in \mathrm{d}w)\,e^{-\gamma_1 t}\,p_-(u - v + w)\,\mathrm{d}t \\ = \frac{\lambda}{\theta} \int_{w=0}^\infty \int_{v=0}^u \mathbb{P}(\bar{Y}_W(T_\theta) \in \mathrm{d}v)\,\mathbb{P}(-\underline{Y}_W(T_\theta) \in \mathrm{d}w)\,p_-(u - v + w).\end{aligned}$$

Upon combining the above, we obtain after some straightforward calculus, recalling that $-\underline{Y}_W(T_\theta)$ is exponentially distributed with parameter x_θ^-,

$$\pi(\alpha) = \frac{1}{\alpha + x_\theta^+} + \frac{\lambda}{\theta}\,x_\theta^-\,\mathbb{E}\,e^{-\alpha \bar{Y}_W(T_\theta)}\,\frac{\pi_-(x_\theta^-) - \pi_-(\alpha)}{\alpha - x_\theta^-}$$

$$
\begin{aligned}
&= \frac{1}{\alpha + x_\theta^+} + \frac{\lambda}{\varphi_W(\alpha) - \theta}\big(\pi_-(x_\theta^-) - \pi_-(\alpha)\big) \\
&= \frac{(\alpha - x_\theta^-)\,\sigma^2/2\, + \lambda\,(\pi_-(x_\theta^-) - \pi_-(\alpha))}{\varphi_W(\alpha) - \theta} \\
&= \frac{(\alpha - x_\theta^-)\,\sigma^2/2\, + \lambda\,(\pi_-(x_\theta^-) - b(\alpha)\,\pi(\alpha) - \Pi(\alpha))}{\varphi_W(\alpha) - \theta},
\end{aligned}
$$

where the last equality is due to (24). Now $\pi(\alpha)$ can be solved:

$$
\pi(\alpha) = \frac{(\alpha - x_\theta^-)\,\sigma^2/2\, + \lambda\,(\pi_-(x_\theta^-) - \Pi(\alpha))}{\varphi(\alpha) - \gamma_1 - \beta}.
$$

The last step is to determine the constant $\pi_-(x_\theta^-)$. Using the well-known principle that any root (in the right-half of the complex plane, that is) of the denominator should be a root of the numerator as well, we conclude that, with $\psi(\cdot)$ denoting the right inverse of $\varphi(\cdot)$,

$$
\big(\psi(\gamma_1 + \beta) - x_\theta^-\big)\,\sigma^2/2\, + \lambda\,\big(\pi_-(x_\theta^-) - \Pi(\gamma_1 + \beta)\big) = 0,
$$

from which $\pi_-(x_\theta^-)$ can be found. We have thus established Theorem 1.