Perishable inventory models with restrictions

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Abstract

We extend the basic perishable inventory model with random input stream by adding two realistic new features. In the basic model the items enter the inventory at Poisson times, the demand arrival times are Poisson as well, and each item has a finite deterministic lifespan in the inventory. The new features considered here are (1) batch demands of random size, and (2) finite storage space in the inventory. In Model 1 demands are only accepted if they can be fully satisfied, in Model 2 if a fresh item arrives at a full system it replaces the oldest item present. In both models we analytically determine the steady-state, long-run behavior and the important performance measures in closed form.

1 Introduction

In perishable inventory systems (PIS), items of a certain type arrive at a collecting point from where they are removed by incoming demands. The arrival times of the fresh items and the demands are assumed to be random. When arriving at a non-empty shelf, a demand always takes away the oldest items. Items are perishable: they may deteriorate or exceed an expiration time before being issued, and then they become outdated and are scrapped. This paper is devoted to the study of two PIS, each with a restriction. In the first system, the restriction is on the demand, and in the second on the supply.

Examples of PIS abound; they include storage of commodities, blood banks, spot markets for special goods, distribution sites for transplantation

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organs, and peer-to-peer lending agencies in the internet (where many individuals deposit money for a limited period, which can be borrowed by debtors; cf. [2]). Below we shall argue that the restrictions we impose arise naturally in several such PIS.

The literature on the probabilistic analysis of PIS is extensive. We refer to the monograph of Nahmias [11] and his earlier review [10] for studies on PIS *optimization* problems, focusing on optimal ordering policies. Our paper fits in another line of research, that focuses on the probabilistic analysis of PIS; the input as well as the output is random and not ordered by a controller. The survey of Karaesmen et al. [8] contains a comprehensive survey of such PIS studies until 2011. For more recent overviews, also containing various extensions, we refer to Krishnamoorthy [9] and to [3].

We consider PIS under the quite common assumptions of independent Poisson arrival processes of items and of demands for items, and a fixed maximum shelf life of items – which is without loss of generality set equal to one. In the sequel we focus on two PIS with an important restriction. In our first model, we allow demands for *batches*. The demand type is *all or nothing*: if, at the moment of a demand arrival, the size of the demand batch does not exceed the number of items on the shelf, then the demand is fully satisfied; but otherwise, none of the items on the shelf is removed and the whole batch demand leaves unsatisfied.

In our second model we consider another restriction, this time on the *supply* instead of on the demand. The number of items on the shelf is restricted to being at most n_0 (integer). If the shelf contains precisely n_0 items and a new item arrives, then this new item is not rejected, but replaces the *oldest item* on the shelf. The advantage of this policy is that it rejuvenates the item population; in many applications, young items are worth more than older ones.

Each of these two models arises very naturally in various settings. First consider batch demand, focusing on blood transfusions. Blood is fluid, but blood as a medical product in a blood bank is packaged and stored on the shelf in the refrigerator in special plastic bags that weigh approximately 0.6 kg. Each such dose is considered to be one unit (item). It is accepted that a donation of one such unit causes no medical risk to the donor. From the demand side, though, one unit may not be sufficient. In blood banks it is hence natural to consider batch demands. In this same application area, batch demand of the *all or nothing* type may also be natural: if a patient requires more units than there are available, he/she may leave unsatisfied (and consider other options). Often several blood portions are simultaneously required in the case of, e.g., major traffic incidents; for such a pile-up the assumption of geometrically distributed batch demand sizes seems reasonable.

Blood banks also provide a natural motivation for studying our second PIS variant, with restrictions on the number of items stored on the shelf. Issuing 'young' doses to a patient clearly has priority over issuing 'old' doses. Hence replacing the oldest item by a newly arriving item has several advantages: it stochastically reduces the number of perishable blood units and enhances the rejuvenation of the population of blood items on the shelf. The capacity of the shelf (the maximum number of items) may be dictated by practical constraints, and should be chosen such that the risk of unsatisfied demand is small. It should also be mentioned that blood is a primary product in a supply chain, from which it is possible to produce related secondary products (plasma), so that the loss caused by the removal of the oldest item is limited.

Main results. In both models, we focus on the steady-state *age of the oldest item* A and the so-called *virtual outdating time* $V = 1 - A$. The reason for this is that the probability density $f(\cdot)$ of V turns out to provide the key to determining all relevant key performance measures of the two PIS models under consideration: outdating rate, unsatisfied demand rate, number of items on the shelf. For both models, we derive an integral equation for density $f(\cdot)$, solve it, and use the result to determine the above-mentioned performance measures.

Organization of the paper. Section 2 contains a description of the basic PIS model and some preliminary results, including a discussion of the age process A and the virtual outdating time process V, and a general conservation law for PIS. Section 3 considers Model 1, viz., the PIS with batch demands, with an all-or-nothing restriction on the demands. Section 4 is devoted to Model 2, viz., the PIS model with a restriction on the total number of items on the shelf. Some conclusions and suggestions for further research are presented in Section 5.

2 Preliminaries

In this section we first describe the PIS model that forms the basis for Models 1 and 2. The specific restrictions which are imposed in those models will be introduced and discussed in more detail in Sections 3 and 4. In the present section we also introduce the concepts of age and virtual outdating time (VOT) in some detail, as these two concepts play a key role in Sections 3 and 4.

Description of the basic PIS model. Items arrive according to a Poisson process with rate λ. Each arriving item is stored on a shelf, and has a unit life time. Demands for items arrive according to a Poisson process with rate μ, independent of the item arrival process. Upon arrival, a demand removes the oldest item from the shelf, or leaves unsatisfied if the shelf is empty. An item that has not been taken within one time unit becomes outdated and must be scrapped.

Number of items on the shelf and conservation law. Let $K(t)$ be the number of items on the shelf at time t, with $K(0) = 0$. $K(t)$ equals the number of items that have arrived up to t minus the number of items that have left until then. The latter term equals the sum of the number of outdatings and that of satisfied demands. Denoting the rate of outdating by λ^* and the rate of unsatisfied demands by μ^*, we clearly have the following *satisfied demand conservation law*:

$$\lambda - \lambda^* \; = \; \mu - \mu^*. \tag{1}$$

Age of the oldest item and virtual outdating time. It is important to observe that, even under the above exponentiality assumptions, $\mathbf{K} = \{K(t), t \geq 0\}$ is not a Markov process, because at any given time t_0 the distribution of $(K(t))_{t \geq t_0}$ depends on the evolution of the process before t_0: the ages of the items on the shelf are important. Let us therefore focus on the age $A(t)$ of the oldest item on the shelf. If the shelf is empty at time t, we define $A(t)$ to be minus the time until the next item arrival (a "negative age"). Subsequently set $V(t) = 1 - A(t)$; cf. Figure 1. A little reflection shows that $V(t)$ is the time that would pass from t until the next outdating if no new demands arrived after t. This "virtual" process $\mathbf{V} = \{V(t), t \geq 0\}$, the so-called Virtual Outdating Process (VOT), is closely related to $\mathbf{K}$ provided that any newly arriving demand is always satisfied by the oldest item present, if at all. Indeed, in the batch demand model the shelf is empty if and only if the age of the oldest item is negative ($A(t) < 0$, so $V(t) > 1$), and the number of items on the shelf equals n if and only if $n - 1$ items have arrived during the age time interval of the oldest item. For the model with finite shelf size this relation between $\mathbf{K}$ and $\mathbf{V}$ needs to be slightly adapted.

It is good to pay special attention to what happens when $\mathbf{V}$ reaches zero. The oldest item now reaches its maximal shelf life time of 1 and is scrapped. Hence $\mathbf{V}$ instantaneously jumps to 1 minus the shelf age of the item that until then was the second oldest item.

For both PIS models in this paper $\mathbf{V}$ turns out to be a key process. It is

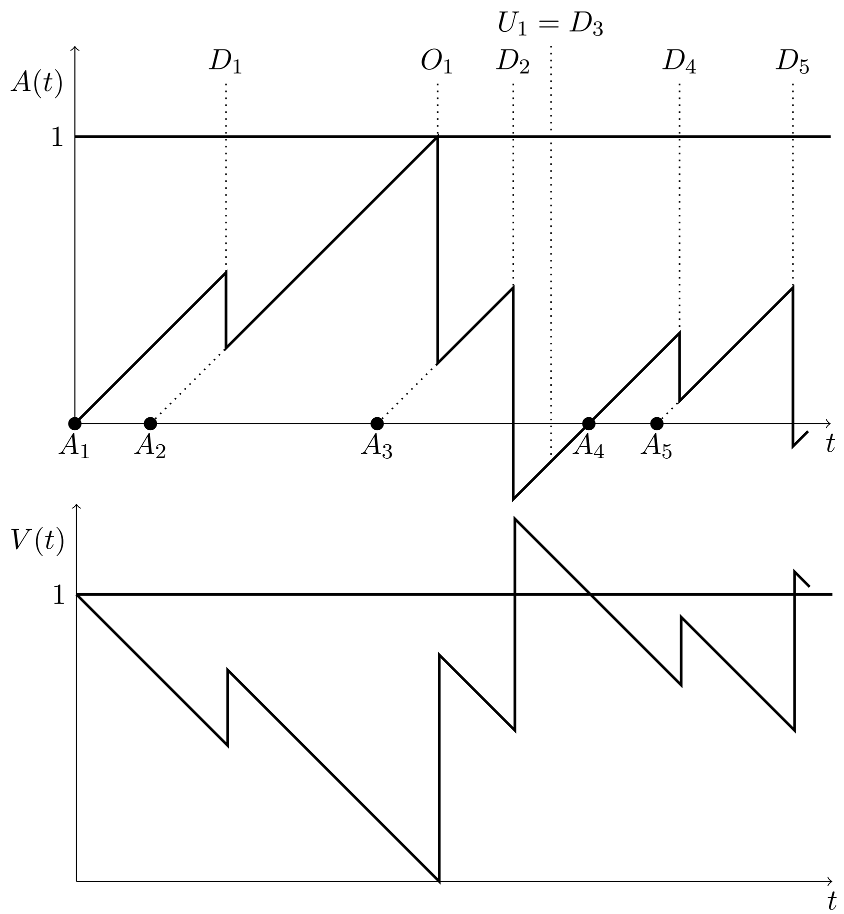

Figure 1: A typical sample path of the age process $\mathbf{A}$ (top panel) and of the VOT process $\mathbf{V}$ (bottom panel). A_n (D_n) denotes the arrival time of the nth item (demand); O_1 denotes the first outdating, and U_1 the first unsatisfied demand; it coincides with D_3.

not difficult to see that under our Poisson-type assumptions $\mathbf{V}$ is a Markov process and a regenerative process with finite expected cycle lengths, where cycles are defined as the intervals between successive times at which the shelf is emptied. Therefore $\mathbf{V}$ is stable and has a stationary distribution (see also [3]).

The main goal of this paper is to derive the steady-state density f of the VOT for both models. The main tool is level crossing theory, which is based on the observation that $f(x)$ is equal to the long-run rate of downcrossings of level x by the VOT. Once f is known, the performance measures of the PIS can be determined as follows. Define K and V as the steady-state number of items on the shelf and of the VOT, respectively.
(a) Outdatings occur precisely at the times when the VOT reaches zero (of course from above). Thus their long-run rate is given by $\lambda^* = f(0)$.
(b) All demands that find the shelf empty (i.e., the VOT above 1) depart unsatisfied. Therefore, the long-run rate of unsatisfied demands equals

$$\mu^* = \mu\mathbb{P}(V > 1) = \mu \int_1^\infty f(x)\mathrm{d}x.$$

We next derive the generating function of the steady-state number of items on the shelf. The formulas for the distribution of K slightly differ for the batch demand model and the finite shelf size model.

For the batch demand model a little reflection shows that

$$\begin{aligned} K(t) &= 0, \quad V(t) > 1, \\ K(t) &= n, \quad V(t) \le 1 \ \text{ and } \ n-1 \text{ arrivals during age of oldest item at } t. \end{aligned} \tag{2}$$

As the arrival times form a Poisson process, we have

$$\mathbb{P}(K(t) = n \mid V(t) = w) = \mathrm{e}^{-\lambda(1-w)}(\lambda(1-w))^{n-1}/(n-1)!, \ \ 0 \le w \le 1, \ \ n = 1, 2, \ldots.$$

Clearly, $\lim_{t\to\infty} \mathbb{P}(V(t) > 1) = \mathbb{P}(V > 1)$ and (by dominated convergence) $\lim_{t\to\infty} \mathbb{E}z^{K(t)} = \mathbb{E}z^K, \ \ |z| < 1$, so that a quick calculation yields

$$\begin{aligned} \mathbb{E}z^K &= \mathbb{P}(V > 1) + \int_0^1 \sum_{n=1}^{\infty} z^n \mathbb{P}(K(t) = n \mid V(t) = w) f(w)\mathrm{d}w \\ &= \int_1^\infty f(w)\mathrm{d}w + z\int_0^1 \mathrm{e}^{-\lambda(1-w)(1-z)} f(w)\mathrm{d}w. \end{aligned} \tag{3}$$

For the model with finite shelf size (2) needs to be modified as follows:

$$\begin{aligned} K(t) &= 0, \quad V(t) > 1, \\ K(t) &= n, \quad V(t) \leq 1 \text{ and } n-1 \text{ arrivals during age of oldest item at } t, \\ &\qquad\qquad n = 1, \dots, n_0 - 1, \\ K(t) &= n_0, \quad V(t) \leq 1 \text{ and } \geq n_0 - 1 \text{ arrivals during age of oldest item at } t. \end{aligned} \tag{4}$$

For this model we have, for $0 \leq w \leq 1$,

$$\mathbb{P}(K(t) = n_0 \mid V(t) = w) = 1 - \mathrm{e}^{-\lambda(1-w)} \sum_{k=0}^{n_0-2} (\lambda(1-w))^k / k!,$$

whereas

$$\mathbb{P}(K(t) = j \mid V(t) = w) = \mathrm{e}^{-\lambda(1-w)((\lambda(1-w))^{j-1}/(j-1)!}, \quad j = 1, \dots, n_0 - 1.$$

The probabilities $\mathbb{P}(K = j)$, $j = 1, \dots, n_0$, now follow by integrating the above expressions over $(0, 1)$ after multiplication by $f(w)$; and $\mathbb{P}(K = 0) = \int_1^\infty f(w)\, \mathrm{d}w$.

3 Model 1: Batch demand of the all or nothing type

In the PIS model that will be studied in this section we extend the basic PIS model described in the beginning of Section 2 in the following way. Demands arrive *in batches* according to a Poisson process with rate μ. A generic batch size is denoted by B with distribution $\mathbb{P}(B = n) = \theta_n$, $n = 1, 2, \dots$. All batch sizes are independent of each other and of all item and demand interarrival times. If, at the arrival instant of a batch, the size of the demand does not exceed the number of items on the shelf, then the demand is immediately and fully satisfied; otherwise, the whole demand leaves unsatisfied ('all or nothing'). In Figure 2 we display the VOT process, highlighting a satisfied demand (left), an unsatisfied demand (middle) and an outdating (right).

We first formulate and prove, in Subsection 3.1, an integral equation for the density $f(\cdot)$ of the VOT $\mathbf{V}$. We explicitly solve this integral equation in Subsection 3.2 for geometrically distributed batch demand (which also includes the case $B \equiv 1$) and in Subsection 3.3 for batches that either have size $B = 1$ or $B = 2$. A formal solution for the case of general batch size distribution is provided in Subsection 3.4. In practice the latter solution

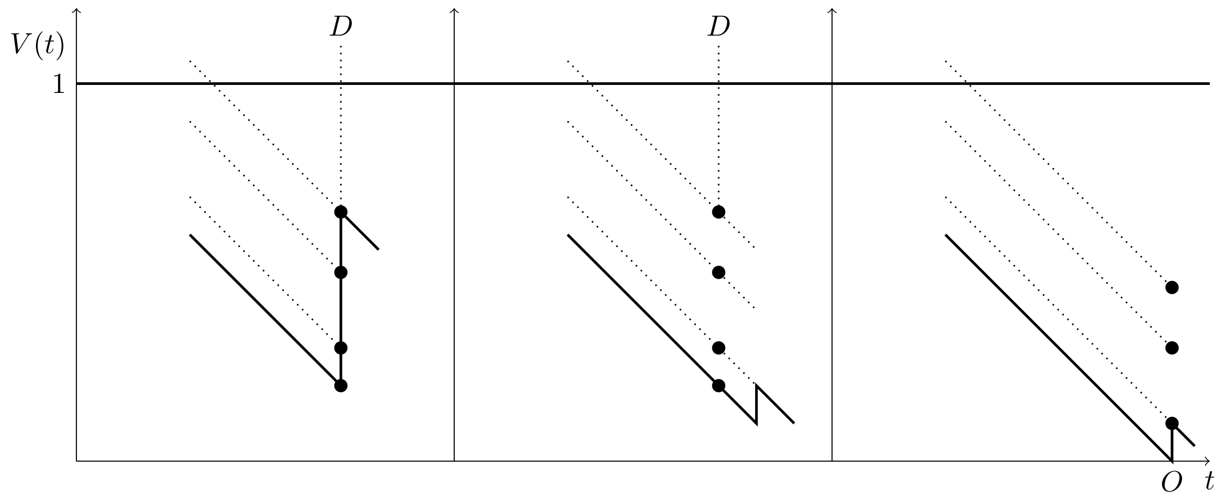

Figure 2: A typical sample path of the VOT process $\mathbf{V}$ for all-or-nothing demand. There are 4 items on the shelf. Left panel: Demand (at D) for 3 items; all are satisfied (notice that a demand for 4 items would also be satisfied, and would lead to a jump above 1 of $\exp(\lambda)$ distributed size). Middle panel: Demand (at D) for 5 items; the demand leaves unsatisfied. Right panel: Outdating (at O); the second oldest item becomes the oldest.

might be less easy to implement, but the solutions for the very natural cases of geometric batch demand and for batches of size 1 or 2 could in practice be used to provide an approximation for a more general batch demand.

3.1 The integral equation

We consider the above-described PIS, with Poisson(λ) item arrival process and fixed (unit) shelf life time in steady state The density $f(\cdot)$ of $\mathbf{V}$ satisfies the following Pollaczek-Khinchine type integral equation.

Theorem 1 *For* $0 \leq x \leq 1$,

$$\begin{aligned} f(x) &= \mu \int_{w=0}^{x} \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} \mathrm{e}^{-\lambda(x-w)} \frac{[\lambda(x-w)]^k}{k!} \\ &\quad \times \sum_{j=n-1-k}^{\infty} \mathrm{e}^{-\lambda(1-x)} \frac{[\lambda(1-x)]^j}{j!} \; f(w) \; \mathrm{d}w + f(0)\mathrm{e}^{-\lambda x}, \end{aligned} \tag{5}$$

and for $x > 1$:

$$f(x) = \mu \mathrm{e}^{-\lambda(x-1)} \int_{w=0}^{1} \sum_{n=1}^{\infty} \theta_n \mathrm{e}^{-\lambda(1-w)} \frac{[\lambda(1-w)]^{n-1}}{(n-1)!} \ f(w) \ \mathrm{d}w \ + \ f(0)\mathrm{e}^{-\lambda x}. \tag{6}$$

Proof. As mentioned above, we employ the *Level Crossing Technique* (LCT), cf. [4, 5, 6]: in steady state, the rate of downcrossings of any level x should equal the rate of upcrossings of that level. Alternatively, one could obtain these equations by first deriving the Kolmogorov forward equation for the Markov process $\mathbf{V}$. We distinguish $0 \le x \le 1$ and $x > 1$.

The case $0 \le x \le 1$. The lefthand side of (5) gives the rate of downcrossings of level x. We now show that the righthand side of that equation gives the corresponding upcrossing rate. An upcrossing of x occurs when there is a demand (rate μ) while V is at some level below x and jumps above x because of that demand. If the shelf is empty ($V = 0$, the $f(0)$ term), then the jump exceeds x iff the time since the last item arrival exceeds x – which has probability $\mathrm{e}^{-\lambda x}$. If $V = w \in (0, x)$ and the size of the batch demand is n, then the jump exceeds x in each of the following disjoint events: k item arrivals in an interval of length $x - w$, and at least $n - 1 - k$ item arrivals in an interval of length $1 - x$, for $k = 0, 1, \dots, n - 1$. The independence of the numbers of arrivals in these two intervals allows multiplication of the corresponding Poisson probabilities. Observe that the 'all or nothing' policy implies that, in case of a batch of size n, no jump occurs if the number of items on the shelf is less than n.

The case $x > 1$. Starting from 0, again the jump exceeds x iff the time since the last arrival exceeds x. Starting from $w \in (0, 1)$, an upcrossing of level $x > 1$ ($V > 1$, so $A < 0$; the shelf becomes empty) only occurs if the size of the batch demand equals the number of items on the shelf. That is, such an upcrossing occurs, starting from w, when two independent events both occur: (i) the batch size is n and the number of item arrivals during $1 - w$ is $n - 1$ ($n = 1, 2, \dots$), and (ii) no item arrivals during an interval of length $x - 1$. ■

3.2 Geometric batch sizes

In this subsection we assume that batch sizes are geometrically distributed, i.e., $\theta_n = (1 - \theta)\theta^{n-1}$ for $n = 1, 2, \dots$. Our goal is to solve (5)-(6) for this case. Interchange the three summations in the righthand side of (5) in the following way, introducing $r = n-1-k$ in the second step:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{j=n-1-k}^{\infty} = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\sum_{j=n-1-k}^{\infty} = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=r}^{\infty} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{j}.$$

In this way, the factor $(1-\theta)\theta^{n-1}$ becomes $(1-\theta)\theta^{k+r}$, which can be summed over r. Subsequently performing the summation over j, and finally over k, we obtain the following integral equation for $f(x)$, for $0 \le x \le 1$:

$$f(x) = \mu \int_0^x \mathrm{e}^{-\lambda(1-\theta)(x-w)}[1-\theta\mathrm{e}^{-\lambda(1-\theta)(1-x)}]\ f(w)\ \mathrm{d}w + f(0)\mathrm{e}^{-\lambda x}. \quad (7)$$

Multiplying both sides of (7) by $\mathrm{e}^{-\lambda(1-\theta)(1-x)}$, and introducing $g(x) := f(x)\mathrm{e}^{-\lambda(1-\theta)(1-x)}$, we can rewrite (7) into

$$g(x) = \mu[1-\theta\mathrm{e}^{-\lambda(1-\theta)(1-x)}]\int_0^x g(w)\ \mathrm{d}w\ +\ g(0)\mathrm{e}^{-\lambda\theta x}. \quad (8)$$

Introducing $h(x) := \int_0^x g(w)\ \mathrm{d}w$ we obtain

$$h'(x)\ =\ \mu[1-\theta\mathrm{e}^{-\lambda(1-\theta)(1-x)}]h(x)\ +\ g(0)\mathrm{e}^{-\lambda\theta x}. \quad (9)$$

The solution of the first-order inhomogeneous differential equation (9) is readily seen to be

$$\begin{aligned} h(x) &= \exp[\mu x - \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-x)}] \times \quad (10)\\ &\times\ [C + g(0)\int_0^x \mathrm{e}^{-\lambda\theta y}\exp[-\mu y + \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-y)}]\ \mathrm{d}y]. \end{aligned}$$

The constant C must be zero, because $h(0) = 0$. Since $g(x) = h'(x)$ it follow from (9) and (10) that

$$\begin{aligned} g(x) &= \mu[1-\theta\mathrm{e}^{-\lambda(1-\theta)(1-x)}]g(0)\exp[\mu x - \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-x)}]\\ &\times \int_0^x \mathrm{e}^{-\lambda\theta y}\exp[-\mu y + \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-y)}]\ \mathrm{d}y + g(0)\mathrm{e}^{-\lambda\theta x} \quad (11) \end{aligned}$$

and hence

$$\begin{aligned} f(x) &= \mu f(0)[\mathrm{e}^{-\lambda(1-\theta)x} - \theta\mathrm{e}^{-\lambda(1-\theta)}]\exp[\mu x - \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-x)}]\\ &\times \int_0^x \mathrm{e}^{-\lambda\theta y}\exp[-\mu y + \frac{\mu\theta}{\lambda(1-\theta)}\mathrm{e}^{-\lambda(1-\theta)(1-y)}]\ \mathrm{d}y + f(0)\mathrm{e}^{-\lambda x}. \quad (12) \end{aligned}$$

Now turn to Equation (6), for $x > 1$. For $\theta_n = (1-\theta)\theta^{n-1}$ it becomes:

$$f(x) = \mu(1-\theta)\mathrm{e}^{-\lambda(x-1)} \int_0^1 \mathrm{e}^{-\lambda(1-\theta)(1-w)}\ f(w)\ \mathrm{d}w\ +\ f(0)\mathrm{e}^{-\lambda x}, \qquad (13)$$

and the integral equals $\int_0^1 g(w)\ \mathrm{d}w = h(1)$, which is by now known up to $f(0)$. Finally, the one remaining unknown $f(0)$ follows via normalization: $\int_0^\infty f(x)\ \mathrm{d}x = 1$.

Remark 1 Taking $\theta = 0$ corresponds to having $\theta_1 = 1$ and $\theta_n = 0$ for $n \geq 2$: all batches have size 1. It can be checked either from (12) or from (5) that now $f(x) = f(0)\mathrm{e}^{(\mu-\lambda)x}$ for $0 \leq x \leq 1$.

Remark 2 We could also have derived the integral equation (7) from first principles, instead as a special case of (5). We show this for the first term in the righthand side of (7). At a moment of demand arrival, a jump from $w > 0$ to a level $\in (x, 1]$ occurs iff one out of two disjoint events occurs:
(i) The number of items on the shelf equals the batch size. Since the shelf becomes empty, the jump upcrosses level 1 and hence also level $x \leq 1$. The probability that the number of items on the shelf is n equals the probability that $n-1$ items have arrived during the lifetime (age) of the oldest item on the shelf, i.e., during $1-w$; that probability equals $P(n-1) := \mathrm{e}^{-\lambda(1-w)}(\lambda(1-w))^{n-1}/(n-1)!$. The probability that the number of items on the shelf equals the batch size is

$$Z_1(w) := \sum_{n=1}^{\infty} \mathbb{P}(B = n)P_n = (1-\theta)\mathrm{e}^{-\lambda(1-\theta)(1-w)},$$

where in the last step we have used that the batch size equals n with probability $\mathbb{P}(B = n) = (1-\theta)\theta^{n-1}$, $n = 1, 2, \dots$.
(ii) The number of items K on the shelf is strictly greater than the batch size B, but B is larger than the number of items Y_1 on the shelf with ages between $1-x$ and $1-w$ (excluding the oldest item on the shelf, with age $1-w$). To determine the probability of this event, viz., $\mathbb{P}(Y_1 < B < K)$, observe that Y_1 is Poisson($\lambda(x-w)$) distributed, and hence $\mathbb{P}(B > Y_1) = \mathrm{e}^{-\lambda(1-\theta)(x-w)}$. By the memoryless property, the conditional distribution of $B - Y_1$, given that it is positive, is also geometrically distributed with parameter θ. Hence, realizing that $K - 1 - Y_1$ equals the Poisson($\lambda(1-x)$) distributed number of items on the shelf with age in $(0, 1-x)$,

$$Z_2(x, w) := \mathbb{P}(Y_1 < B < K) = \mathbb{P}(Y_1 < B) - \mathbb{P}(Y_1 < B)\mathbb{P}(K - 1 < B)$$

$$= \mathrm{e}^{-\lambda(1-\theta)(x-w)} - \mathrm{e}^{-\lambda(1-\theta)(x-w)}\mathrm{e}^{-\lambda(1-\theta)(1-x)}.$$

Adding $Z_1(w)$ and $Z_2(x,w)$, the level crossing argument yields the first expression in the righthand side of (7).

Remark 3 In blood transfusions, one might distinguish between individuals and groups of random size (the latter due to some accident or incident). If we assume that a person typically needs one bag of blood (one unit), and that the size of a group is geometrically distributed, then we arrive at nearly geometric batch demand, with

$$\begin{aligned} \mathbb{P}(B=1) &= 1-\theta+p\theta, \\ \mathbb{P}(B=n) &= (1-p)(1-\theta)\theta^{n-1}, \quad n=2,3,\dots\,. \end{aligned} \tag{14}$$

It is straightforward to handle this case, using the above results. Indeed, assume that the second equation of (14) *also* holds for $n=1$; that would give rise to a multiplicative factor $1-p$ in the first term in the righthand side of (9). Subsequently for the $n=1$ term, subtract $(1-p)(1-\theta)$ from $1-\theta+p\theta$ to get p. The contribution of $n=1$ in (5) then just is one term, viz., $\mu p \int_{w=0}^{x} \mathrm{e}^{-\lambda(x-w)} f(w)\,\mathrm{d}w$, yielding $\mu p\, h(x)$ in (9). Altogether, (9) becomes an equation of nearly the same form, that can be solved in exactly the same way:

$$h'(x) = \mu[p+(1-p)(1-\theta\mathrm{e}^{-\lambda(1-\theta)(1-x)})]h(x) + g(0)\mathrm{e}^{-\lambda\theta x}. \tag{15}$$

3.3 The case of one or two units

In blood transfusions, the demand of a patient typically ranges from 0.3 to 1.2 kg. As blood is stored in bags of approximately 0.6 kg, demands will typically ask for one or for 2 units. In this section we therefore consider the case that a batch either has size 1 or size 2, with probabilities θ_1 and θ_2. Equation (5) now reduces as follows: for $0 \le x \le 1$,

$$\begin{aligned} f(x) &= \mu\theta_1 \int_0^x \mathrm{e}^{-\lambda(x-w)}\, f(w)\,\mathrm{d}w \\ &+ \mu\theta_2(1-\mathrm{e}^{-\lambda(1-x)}) \int_0^x \mathrm{e}^{-\lambda(x-w)}\, f(w)\,\mathrm{d}w \\ &+ \mu\theta_2 \int_0^x \lambda(x-w)\mathrm{e}^{-\lambda(x-w)}\, f(w)\,\mathrm{d}w + f(0)\mathrm{e}^{-\lambda x}. \end{aligned} \tag{16}$$

Introducing $g(x) := \mathrm{e}^{\lambda x} f(x)$ and multiplying both sides of (16) by $\mathrm{e}^{\lambda x}$ we obtain

$$\begin{aligned} g(x) &= [\mu\theta_1 + \mu\theta_2(1 - \mathrm{e}^{-\lambda(1-x)})] \int_0^x g(w)\ \mathrm{d}w \\ &+ \mu\theta_2 \int_0^x \lambda(x-w)\ g(w)\ \mathrm{d}w + g(0). \end{aligned} \quad (17)$$

Introducing $h(x) := \int_0^x g(w)\ \mathrm{d}w$, we can rewrite (17), after one more differentiation, into the following ordinary second-order differential equation:

$$h"(x) - [\mu\theta_1 + \mu\theta_2(1 - \mathrm{e}^{-\lambda(1-x)})]h'(x) - \mu\theta_2\lambda(1 - \mathrm{e}^{-\lambda(1-x)})h(x) = 0, \quad (18)$$

with initial conditions $h(0) = 0$, $h'(0) = f(0)$, where $f(0)$ eventually can be derived from the normalization condition.
Now let $H(z) := h(x)$, with $z := \mathrm{e}^{\lambda x}$. Equation (18) transforms into another second-order differential equation, but now with coefficients that are polynomial in z:

$$\lambda^2 z^2 H"(z) + \lambda^2 z H'(z) - [\mu\theta_1 + \mu\theta_2(1 - \mathrm{e}^{-\lambda} z)]\lambda z H'(z) - \mu\theta_2\lambda(1 - \mathrm{e}^{-\lambda} z)H(z) = 0, \quad (19)$$

so

$$H"(z) + [\frac{1}{z} - \frac{\mu\theta_1}{\lambda z} - \frac{\mu\theta_2}{\lambda z}(1 - \mathrm{e}^{-\lambda} z)]H'(z) + \frac{\mu\theta_2}{\lambda z^2}(\mathrm{e}^{-\lambda} z - 1)H(z) = 0. \quad (20)$$

This equation has the form $H"(z) + p(z)H'(z) + q(z)H(z) = 0$, with $p(z)$ and $q(z)$ quotients of simple polynomials. Let $I(z) := q(z) - \frac{1}{2}p'(z) - \frac{1}{4}(p(z))^2$, and let

$$y(z) := \exp(\frac{1}{2}\int^z p(u)\mathrm{d}u)H(z) = z^{\frac{1}{2}(1 - \frac{\mu\theta_1}{\lambda} - \frac{\mu\theta_2}{\lambda})} \mathrm{e}^{\frac{\mu\theta_2}{2\lambda}\mathrm{e}^{-\lambda} z} H(z). \quad (21)$$

Then (20) becomes $y"(z) + I(z)y(z) = 0$. We can write $I(z)$ in the form

$$I(z) = -\frac{a^2 z^2 + bz + c}{4z^2}, \quad (22)$$

with

$$a := \frac{\mu\theta_2}{\lambda}\mathrm{e}^{-\lambda}, \quad (23)$$

$$b := -2a(1 + \mu\theta_1/\lambda + \mu\theta - 2/\lambda), \quad (24)$$

$$c := -1 + 4\mu\theta_2/\lambda + (\mu\theta_1/\lambda + \mu\theta_2/\lambda)^2. \quad (25)$$

According to Equation (25) of [7], $y(z)$ is then given by a weighted sum (determined by the boundary conditions) of the following two expressions:

$$y_{\pm}(z) = (az)^{\frac{1}{2}(1 \pm \sqrt{1+c})} \mathrm{e}^{-az/2} M(\frac{b}{4a} + \frac{1}{2}(1 \pm \sqrt{1+c}); 1 \pm \sqrt{1+c}; az). \quad (26)$$

$M(\alpha; \beta; z)$ is Kummer's confluent hypergeometric function, solution of the differential equation $zw"(z) + (\beta - z)w'(z) + \alpha w(z) = 0$, of which many properties are known; cf. [7] or [1]. Now one can go back from $y(z)$ to $H(z)$, to $h(x)$, to $g(x)$ and finally to $f(x)$. We thus get, using (21) in the second line:

$$\begin{aligned} f(x) &= \mathrm{e}^{-\lambda x} g(x) = \mathrm{e}^{-\lambda x} h'(x) = \lambda \frac{\mathrm{d}}{\mathrm{dz}} H(z)|_{z=\mathrm{e}^{\lambda x}} \\ &= \lambda z^{-\frac{1}{2}(1-\frac{\mu\theta_1}{\lambda}-\frac{\mu\theta_2}{\lambda})} \mathrm{e}^{-\frac{\mu\theta_2}{2\lambda}\mathrm{e}^{-\lambda z}} [y'(z) - \frac{1}{2} p y(z)]|_{z=\mathrm{e}^{\lambda x}}. \quad (27) \end{aligned}$$

Alternatively, a formal solution procedure is to apply Picard iteration to (17), which is a Volterra integral equation of the second kind. Introducing $K(x,w) := \theta_1 + \theta_2(1 - \mathrm{e}^{-\lambda(1-x)}) + \theta_2 \lambda (x - w)$, we can rewrite (17) into

$$g(x) = g(0) + \mu \int_{w=0}^{x} K(x,w)\, g(w)\, \mathrm{d}w. \quad (28)$$

One iteration results in

$$g(x) = g(0) + \mu g(0) \int_{w=0}^{x} K(x,w)\, \mathrm{d}w + \mu^2 \int_{w=0}^{x} \int_{y=0}^{w} K(x,w) K(w,y) g(y)\, \mathrm{d}y \mathrm{d}w. \quad (29)$$

Introducing $L_n(x) := \int_{w=0}^{x} K(x,w) L_{n-1}(w)\, \mathrm{d}w$, $n = 1, 2, \ldots$, with $L_0(x) := 1$, we arrive at

$$g(x) = g(0) \sum_{n=0}^{\infty} \mu^n L_n(x), \quad 0 \le x \le 1. \quad (30)$$

The convergence of the series in (30) is easily seen; observe that $K(x,w)$ is for $x \in [0,1]$ nonnegative and bounded, say, by C, yielding $0 \le L_n(x) \le C^n/n!$.

3.4 The general case

For the case of a general batch size distribution, we return to Theorem 1, Equations (5) and (6). Equation (6) poses no problems; it just states that $f(x) = C\mathrm{e}^{-\lambda x}$ for $x > 1$, where C has to follow from continuity of $f(x)$ at

$x = 1$ and the normalization condition. Multiplying (5) on both sides by $\mathrm{e}^{\lambda x}$ and again defining $g(x) := \mathrm{e}^{\lambda x} f(x)$, that equation becomes for $0 \le x \le 1$:

$$g(x) = \mu \mathrm{e}^{-\lambda} \int_{w=0}^{x} K(x, w) g(w) \; \mathrm{d}w + g(0), \tag{31}$$

where now

$$K(x, w) := \sum_{n=1}^{\infty} \theta_n \sum_{k=0}^{n-1} \frac{[\lambda(x - w)]^k}{k!} \sum_{j=n-1-k}^{\infty} \frac{[\lambda(1 - x)]^{n-1}}{(n-1)!}. \tag{32}$$

Just as at the end of the previous subsection, we formally have a relation that is suitable for Picard iteration. However, we shall not pursue this further here.

4 Model 2: A restriction on the number of items on the shelf

The PIS model under consideration in this section is a variant of the basic model described in the beginning of Section 2. We assume that the shelf has a *finite* capacity of n_0 items. If a new item arrives when there are already n_0 items on the shelf, then the oldest item is scrapped and replaced by the new arrival.

We first formulate and prove, in Subsection 3.1, an integral equation for the density $f(\cdot)$ of the VOT $\mathbf{V}$. We explicitly solve this integral equation in Subsection 3.2 for geometrically distributed batch demand (which also includes the case $B \equiv 1$) and in Subsection 3.3 for batches that either have size $B = 1$ or $B = 2$. A formal solution for the case of general batch size distribution is provided in Subsection 3.4. In practice the latter solution might be less easy to implement, but the solutions for the very natural cases of geometric batch demand and for batches of size 1 or 2 could in practice be used to provide an approximation for a more general batch demand.

We first formulate and prove, in Subsection 4.1, an integral equation for the density $f(\cdot)$ of the VOT $\mathbf{V}$. We explicitly solve this integral equation in Subsection 4.2.

4.1 The integral equation

Just like for Model 1, the finiteness of the life times of items guarantees that $\mathbf{V}$ has a steady-state distribution. Its steady-state density $f(\cdot)$ satisfies the following integral equation.

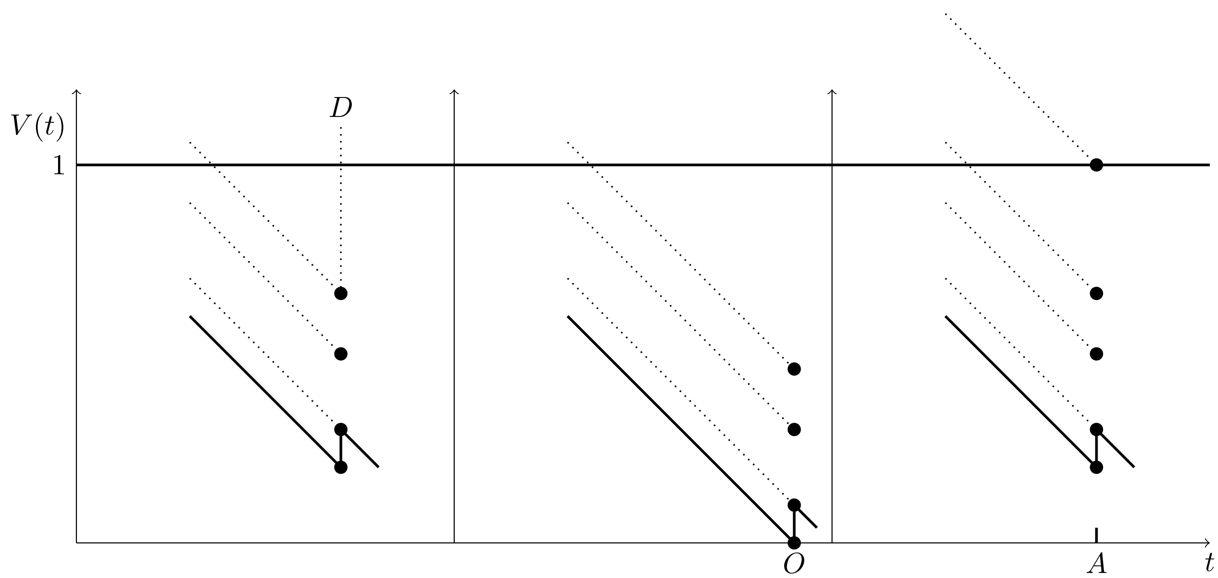

Figure 3: A typical sample path of the VOT process $\mathbf{V}$ with a finite shelf ($n_0 = 5$). Left panel: A satisfied demand (at D; there were 4 items on the shelf). Middle panel: An outdating (at O); the second oldest item becomes the oldest. Right panel: An arriving item (at A) finds a full shelf; it replaces the oldest item, and the second oldest item becomes the oldest.

Theorem 2 *For $0 \leq x \leq 1$,*

$$f(x) = \int_{w=0}^{x} \mathrm{e}^{-\lambda(x-w)} \tag{33}$$
$$\times \quad \left[\mu \sum_{j=0}^{n_0-1} \mathrm{e}^{-\lambda(1-x)} \frac{(\lambda(1-x))^j}{j!} + \lambda \mathrm{e}^{-\lambda(1-x)} \frac{(\lambda(1-x))^{n_0-1}}{(n_0-1)!}\right] f(w)\ \mathrm{d}w + f(0)\mathrm{e}^{-\lambda x}$$

and for $x > 1$, with k some positive constant,

$$f(x) = k\mathrm{e}^{-\lambda x}. \tag{34}$$

Proof. We again employ LCT, equaling the rate level x is downcrossed (i.e., $f(x)$) and upcrossed. Equation (34) immediately follows from the $\exp(\lambda)$ item interarrival times, and the same holds for the $f(0)\mathrm{e}^{-\lambda x}$ term in (33) that reflects an upcrossing of x by a jump from zero (an outdating has occurred, and the time until the next item arrival exceeds x). Let us now focus on the upcrossing rate from some level $w \in (0, x)$ in (33). Such a jump occurs when the age of the oldest item is $1 - w$, in one of the following two scenarios.
(i) Next to the oldest item, there are $j \in \{0, 1, \ldots, n_0 - 1\}$ items on the shelf when a demand arrives, and all those items arrived in the last $1 - x$ part (so that the age of the one-but-oldest item is at most $1 - x$). The probability of j arrivals during the age $1 - w$ equals $\mathrm{e}^{-\lambda(1-w)}(\lambda(1-w))^j/j!$ Since the Poisson arrival epochs in $(0, 1 - w)$ are independent and uniformly distributed on that interval, the probability that all j arrived in the last $1 - x$ part equals $((1 - x)/(1 - w))^j$. That gives the first term in the righthand side of (33). (Alternatively, one might argue that there must be no arrival in an interval of length $x - w$, so that the one-but-oldest item causes a jump of at least $x - w$; and j arrivals in the last $1 - x$ part of $1 - w$.)
(ii) A new item arrives who finds the oldest item and $n_0 - 1$ other items on the shelf, while the one-but-oldest item – that now becomes the oldest item – has an age less than $1 - x$. The new item arrives at rate λ; the probability of having $n_0 - 1$ arrivals in $(0, 1 - w)$ equals $\mathrm{e}^{-\lambda(1-w)}(\lambda(1-w))^{n_0-1}/(n_0-1)!$; and the probability of all of them arriving in the last $1 - x$ part of $1 - w$ equals $((1 - x)/(1 - w))^{n_0-1}$. ∎

4.2 Solution of the integral equation

To determine $f(x)$ from (33) and (34), we introduce, just like in Subsection 3.3,

$$g(x) := \mathrm{e}^{\lambda x} \quad \text{and} \quad h(x) := \int_{w=0}^{x} g(w)\ \mathrm{d}w. \tag{35}$$

Further introducing

$$R(x) := \mu \sum_{j=0}^{n_0-1} \mathrm{e}^{-\lambda(1-x)} \frac{(\lambda(1-x))^j}{j!} \ + \ \lambda \mathrm{e}^{-\lambda(1-x))} \frac{(\lambda(1-x))^{n_0-1}}{(n_0-1)!}, \qquad (36)$$

(33) gives rise to the following first-order inhomogeneous differential equation:

$$h'(x) = R(x)h(x) \ + \ f(0), \quad 0 \leq x \leq 1. \qquad (37)$$

This equation is routinely solved, finally resulting in:

Theorem 3 *The density $f(\cdot)$ of $\mathbf{V}$ in the PIS with restricted accessibility is given by*

$$f(x) \ = \ f(0)\mathrm{e}^{-\lambda x} \left[1 + R(x) \int_{y=0}^{x} \mathrm{e}^{\int_{u=y}^{x} R(u) \ \mathrm{d}u} \ \mathrm{d}y \right], \quad 0 \leq x \leq 1, \qquad (38)$$

and, for $x > 1$,

$$f(x) \ = \ f(0)\mathrm{e}^{-\lambda x} \ \left[1 + (\mu + \lambda I(n_0 = 1)) \int_{y=0}^{1} \mathrm{e}^{\int_{u=y}^{1} R(u) \ \mathrm{d}u} \ \mathrm{d}y \right], \qquad (39)$$

where $f(0)$ follows from $\int_{x=0}^{\infty} f(x) \ \mathrm{d}x = 1$ and where $I(\cdot)$ denotes an indicator function.

Proof. The solution of (37), with boundary condition $h(0) = 0$, is:

$$h(x) \ = \ f(0) \int_{y=0}^{x} \mathrm{e}^{\int_{u=y}^{x} R(u) \ \mathrm{d}u} \ \mathrm{d}y, \quad 0 \leq x \leq 1. \qquad (40)$$

Observing that $g(x) = h'(x) = R(x)h(x) + f(0)$, we obtain (38). Equation (39) follows from (34), the continuity of $f(x)$ at $x = 1$, and the fact that $R(1) = \mu + \lambda I(n_0 = 1)$. ■

5 Conclusions and suggestions for further research

We have shown that two important extensions of the basic PIS are analytically tractable: (i) batch demands of random size, and (ii) finite storage space in the inventory. We derived integral equations for the steady-state densities of the underlying virtual outdating processes of a PIS with batch demands and for the PIS with finite storage space, and from this we obtained closed-form solutions. The relevant performance measures can then be determined in terms of these densities. Finally, we would like to emphasize once more that the Virtual Outdating Time **V** appears to be particularly useful in studying PIS with random input. Several other examples of this, as well as a list of open PIS problems, are presented in [3].

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