The friendship paradox for sparse random graphs

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Abstract

Let G_n be an undirected finite graph on $n \in \mathbb{N}$ vertices labelled by $[n] = \{1, \ldots, n\}$. For $i \in [n]$, let $\Delta_{i,n}$ be the *friendship bias* of vertex i, defined as the difference between the average degree of the neighbours of vertex i and the degree of vertex i itself when i is not isolated, and zero when i is isolated. Let μ_n denote the *friendship-bias empirical distribution*, i.e., the measure that puts mass $\frac{1}{n}$ at each $\Delta_{i,n}$, $i \in [n]$. The friendship paradox says that if G_n has no self-loops, then $\int_{\mathbb{R}} x\mu_n(\mathrm{d}x) \geq 0$, with equality if and only if in each connected component of G_n all the degrees are the same.

We show that if $(G_n)_{n\in\mathbb{N}}$ is a sequence of sparse random graphs that converges to a rooted random tree in the sense of convergence locally in probability, then μ_n converges weakly to a limiting measure μ that is expressible in terms of the law of the rooted random tree. We study μ for four classes of sparse random graphs: the homogeneous Erdős-Rényi random graph, the inhomogeneous Erdős-Rényi random graph, the configuration model and the preferential attachment model. In particular, we compute the first two moments of μ, identify the right tail of μ, and argue that $\mu([0,\infty)) \geq \frac{1}{2}$, a property we refer to as *friendship-paradox significance*.

Keywords: Sparse random graphs, local convergence, friendship bias.

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1 Introduction and outline

1.1 Background and motivation

In 1991, the American sociologist Scott Feld discovered the paradoxical phenomenon that 'your friends are more popular than you' [6]. This statement means the following. Consider a group of individuals who form a connected friendship network. For each individual in the group, compute the difference between the average number of friends of friends and the number of friends (all friendships in the group are mutual). Average these numbers over all the individuals in the group. It turns out that the latter average is always non-negative, and is strictly positive as soon as not all individuals have exactly the same number of friends. This bias, which at first glance seems counterintuitive, goes under the name of *friendship paradox*, even though it is a hard fact. An equivalent, and possibly more soothing, version of the paradox reads 'your enemies have more enemies than you' [13].

Implications. Apart from being interesting in itself, the friendship paradox has useful implications. For instance, it can be used to slow down the spread of an infectious disease. Suppose that there is a group of individuals whose friendship network we do not know explicitly. Suppose that an infectious disease breaks out and we only have one vaccine at our disposal, which we want to use as effectively as possible. In other words, we want to vaccinate the individual who has the most friends. One way we could do this is by choosing an individual at random and giving the vaccine to him or her. Another way could be to select an individual at random, and let him or her choose a friend at random to give the vaccine to. Since the more popular individuals are more likely to be chosen, the second approach is more effective in combating the disease.

The friendship bias can be viewed as a *centrality measure*, akin to PageRank [15] centrality and degree centrality. In [7], numerous intriguing features of PageRank were explored in the context of sparse graphs with the help of local weak convergence. Similar interesting features emerge in the analysis of the friendship bias. In [2] and [16], the behaviour of the friendship paradox for random graphs like the Erdős-Rényi random graph was studied. Through an empirical analysis of the limit of large Erdős-Rényi random graphs conditioned not to have isolated points, [2] concluded that "for small degrees no meaningful friendship paradox applies" (a statement that we will refute in the present paper). Moreover, [2] examined a special case of the configuration model and provided an empirical analysis by using kernel density estimation. Furthermore, a random network model was proposed with degree correlation and, with the help of the notion of Shannon entropy, an attempt was made to investigate the impact of the assortativity coefficient on the friendship paradox.

There have also been studies on what is called the *generalised friendship paradox*, in which attributes other than popularity are considered that produce a similar paradox [5]. For instance, an analysis of two co-authorship networks of Physical Review journals and Google Scholar profiles reveals that on average the co-authors of a person have more collaborations, publications and citations than that person [5]. On Twitter, for most users on average their friends share and tweet more viral content, and over 98 percent of the users have fewer followers than their followers [9]. In [2], it is shown in an informal way that the generalised friendship paradox holds when the attribute correlates positively with popularity. Other works have examined the implications of the friendship paradox for individual biases in perception and thought contagion, noting that our social norms are influenced by our perceptions of others, which are strongly shaped by the people around us [12]. For instance, individuals whose acquaintances smoke are more likely to smoke themselves [4]. The impact of the friendship paradox on behaviour explains why we should expect more-connected people to behave systematically differently from less-connected people, which in turn biases overall behaviour in society [12]. In [14], a new strategy based on the friendship paradox is imposed to improve poll predictions of elections, instead of randomly sampling individuals and asking such questions as ‘Who are you voting for? Who do you think will win?’ In [3], the friendship paradox is looked at from a probabilistic point of view, and it is shown that a randomly chosen friend of a randomly chosen individual stochastically has more friends than that individual.

Mathematical modelling. The friendship paradox raises many questions. ‘How large is the bias? How does it depend on the architecture of the friendship network? Are there universal features beyond the fact that the bias is always non-negative?’ Mathematically, the friendship structure is modelled by a graph, where the vertices represent individuals and the edges represent friendships. The graph is typically random and its size is typically large. In the present paper we take a *quantitative* look at the friendship paradox for *sparse random graphs*. We focus on the *friendship-bias empirical distribution*, i.e., the distribution of the biases of all the individuals in the network. We show that if a sequence of sparse random graphs converges to a rooted random tree in the sense of convergence locally in probability, then the friendship-bias empirical distribution converges weakly to a limiting measure that is expressible in terms of the law of the rooted random tree. We study this limiting measure for four classes of sparse random graphs. In particular, we compute its first two moments, identify its right tail, and argue that it puts at least one half of its mass on non-negative biases, a property we refer to as *friendship-paradox significance*.

1.2 Friendship paradox

Random graphs. Throughout the sequel, $G = (V(G), E(G))$ is an undirected simple graph or multi-graph. The vertices $V(G)$ represent individuals, the edges $E(G)$ represent mutual friendships. Denote by

$$(d_i^{(G)})_{i\in V(G)}, \qquad A^{(G)} = (A_{i,j}^{(G)})_{i,j\in V(G)},$$

the *degree sequence*, respectively, the *adjacency matrix* of G, where $A_{i,j}^{(G)}$ is the number of edges from i to j (each self-loop adds 2 to the degree).

Let G_n be a finite graph on $n \in \mathbb{N}$ vertices labeled by $[n] = \{1, \dots, n\}$. For $i \in [n]$, define the *friendship bias* as

$$\Delta_{i,n} = \left[\frac{\sum_{j\in[n]} A_{i,j}^{(G_n)} d_j^{(G_n)}}{d_i^{(G_n)}} - d_i^{(G_n)} \right] \mathbb{1}_{\{d_i^{(G_n)} \neq 0\}}.$$

Let $\mu_n \colon \mathcal{B}(\mathbb{R}) \to [0,1]$ be the *quenched friendship-bias empirical distribution*

$$\mu_n = \frac{1}{n} \sum_{i\in[n]} \delta_{\Delta_{i,n}},$$

where δ_x denotes the Dirac measure concentrated at x, and $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$. Let $\tilde{\mu}_n \colon \mathcal{B}(\mathbb{R}) \to [0,1]$ be the *annealed friendship-bias empirical distribution* defined by

$$\tilde{\mu}_n(\cdot) = \mathbb{E}_n[\mu_n(\cdot)],$$

where $\mathbb{E}_n$ denotes expectation with respect to the random graph G_n.

The *average friendship bias* is defined as

$$\Delta_{[n]} = \frac{1}{n} \sum_{i \in [n]} \Delta_{i,n} = \int_{\mathbb{R}} x \mu_n(\mathrm{d}x).$$

If G_n has no self-loop, then

$$\begin{aligned}\Delta_{[n]} &= \frac{1}{n} \sum_{\substack{i \in [n] \\ d_i^{(G_n)} \neq 0}} \sum_{\substack{j \in [n] \\ d_j^{(G_n)} \neq 0}} A_{i,j}^{(G_n)} \left(\frac{d_j^{(G_n)}}{d_i^{(G_n)}} - 1 \right) \\ &= \frac{1}{2n} \sum_{\substack{i \in [n] \\ d_i^{(G_n)} \neq 0}} \sum_{\substack{j \in [n] \\ d_j^{(G_n)} \neq 0}} A_{i,j}^{(G_n)} \left(\sqrt{\frac{d_j^{(G_n)}}{d_i^{(G_n)}}} - \sqrt{\frac{d_i^{(G_n)}}{d_j^{(G_n)}}} \right)^2 \geq 0,\end{aligned}$$

with equality if and only if all connected components of G_n are regular. This property is known as the *friendship paradox*, because if the edges of the graph represent mutual friendships, then $\Delta_{[n]} > 0$ means that in a community with n individuals on average the friends of an individual have more friends than the individual itself.

Note that when G_n contains self-loops, it may happen that $\Delta_{[n]} < 0$ (see Figure 1). Also note that

$$\mathbb{E}_n[\Delta_{[n]}] = \frac{1}{n} \sum_{i \in [n]} \mathbb{E}_n[\Delta_{i,n}] = \int_{\mathbb{R}} x \tilde{\mu}_n(\mathrm{d}x).$$

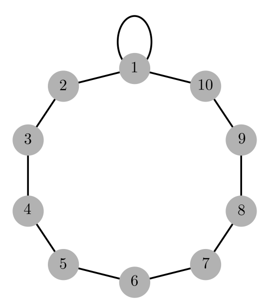

Figure 1: *A graph with a self-loop:* $n = 10$ *and* $\Delta_{[n]} = -\frac{1}{n}$.

Rooted random graphs. A *rooted* graph G with root vertex o is denoted by the pair (G, o). Consider an almost surely locally finite rooted random tree (G_∞, ϕ), let d_ϕ be the degree of ϕ, and let d_j be the size of the offspring of neighbour $j \in [d_\phi] = \{1, \dots, d_\phi\}$ of ϕ (labelled in an arbitrary ordering). Define

$$\Delta_\phi = \left[\frac{1}{d_\phi} \sum_{j=1}^{d_\phi} (d_j + 1) - d_\phi \right] \mathbb{1}_{\{d_\phi \neq 0\}}, \tag{1.1}$$

and let $\mu\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ be the law of Δ_ϕ. Write $\bar{\mu}$ to denote the law of (G_∞, ϕ).

For the special case where (G_∞, ϕ) is a *rooted Galton-Watson tree* in which each individual in the population independently gives birth to a random number of children in $\mathbb{N}_0$ according to a common law ν, the measure μ equals

$$\mu(\cdot) = \nu(0)\delta_0(\cdot) + \sum_{k \in \mathbb{N}} \nu(k) \nu^{\circledast k}\big(k(\cdot + k - 1)\big), \tag{1.2}$$

where $\nu(k) = \nu(\{k\})$ is the probability that an individual has k children, and $\nu^{\circledast k}$ is the k-fold convolution of ν.

Outline. In Section 2 we use the theory of local convergence of sparse random graphs to show that the friendship-bias empirical distribution μ_n converges to a limit μ as $n \to \infty$. In Section 3 we study μ for four classes of sparse random graphs : the homogeneous Erdős-Rényi random graph, the inhomogeneous Erdős-Rényi random graph, the configuration model and the preferential attachment model. In particular, we compute the first two moments of μ, identify the right tail of μ, and argue that $\mu([0,\infty)) > \frac{1}{2}$, a property we refer to as *friendship paradox significance*. Proofs are given in Section 4.

Notation. For sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of real numbers we write $a_n \sim b_n$ when $\lim_{n\to\infty} a_n/b_n = 1$, $a_n \asymp b_n$ when $0 < \liminf_{n\to\infty} a_n/b_n \le \limsup_{n\to\infty} a_n/b_n < \infty$, $a_n \lesssim b_n$ when $\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$, and $a_n \gtrsim b_n$ when $\liminf_{n\to\infty} \frac{a_n}{b_n} > 0$.

2 Local convergence: main theorems

The asymptotic behaviour of the empirical distribution μ_n as $n \to \infty$ provides information on the friendship paradox for large complex networks. Theorems 2.3–2.4 below show that μ_n and $\tilde{\mu}_n$ converge weakly to μ for all locally tree-like random graphs, and quantify the friendship paradox.

We start by setting up the notion of local convergence for random graphs from [11, Chapter 2]. Let $B_r^{(G)}(o)$ denote the rooted subgraph of (G, o) in which all vertices have graph distance at most r to o, i.e., if dist_G denotes the graph distance in G, then

$$B_r^{(G)}(o) = \big((V(B_r^{(G)}(o)), E(B_r^{(G)}(o))), o\big)$$

with

$$V(B_r^{(G)}(o)) = \{i \in V(G)\colon\ \mathrm{dist}_G(o,i) \le r\},$$
$$E(B_r^{(G)}(o)) = \{e \in E(G)\colon\ e = \{i,j\},\ \max\{\mathrm{dist}_G(o,i), \mathrm{dist}_G(o,j)\} \le r\},$$

where for representing an edge we allow multisets. Write $G_1 \simeq G_2$ when G_1 and G_2 are *isomorphic*. Let $\mathscr{G}$ be the set of all connected locally finite rooted graphs equipped with the metric

$$\mathrm{d}_{\mathscr{G}}\big((G_1,o_1),(G_2,o_2)\big) = \Big(1+\sup\big\{r \geq 0\colon\ B_r^{(G_1)}(o_1) \simeq B_r^{(G_2)}(o_2)\big\}\Big)^{-1},$$

with the convention that two connected locally finite rooted graphs (G_1, o_1) and (G_2, o_2) are identified when $(G_1, o_1) \simeq (G_2, o_2)$.

Definition 2.1. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite random graphs. For $n \in \mathbb{N}$, let $\mathscr{C}(G_n, U_n)$ denote the connected component of a uniformly chosen vertex $U_n \in V(G_n)$ in G_n, viewed as a rooted graph with root vertex U_n.

(a) G_n converges *locally weakly* to $(G, o) \in \mathscr{G}$ with (deterministic) law $\tilde{\nu}$ if, for every bounded and continuous function $h\colon \mathscr{G} \to \mathbb{R}$,

$$\mathbb{E}_{\tilde{\nu}_n}\big[h(\mathscr{C}(G_n,U_n))\big] \to \mathbb{E}_{\tilde{\nu}}\big[h((G,o))\big],$$

where $\mathbb{E}_{\tilde{\nu}_n}$ is the expectation with respect to the random vertex U_n and the random graph G_n with joint law $\tilde{\nu}_n$, while $\mathbb{E}_{\tilde{\nu}}$ is the expectation with respect to (G, o) with law $\tilde{\nu}$.

(b) G_n converges *locally in probability* to $(G, o) \in \mathscr{G}$ with (possibly random) law $\tilde{\nu}$ if, for every bounded and continuous function $h\colon \mathscr{G} \to \mathbb{R}$,

$$\mathbb{E}_{\tilde{\nu}_n}\big[h(\mathscr{C}(G_n,U_n)) \mid G_n\big] \xrightarrow{\mathbb{P}} \mathbb{E}_{\tilde{\nu}}\big[h((G,o))\big].$$

♠

Remark 2.2. Note that if G_n converges locally in probability to $(G, o) \in \mathscr{G}$ with law $\tilde{\nu}$, then G_n converges locally weakly to $(\bar{G}, \bar{o}) \in \mathscr{G}$ with law $\bar{\nu}(\cdot) = \mathbb{E}[\tilde{\nu}(\cdot)]$ defined, for every bounded and continuous function $h\colon \mathscr{G} \to \mathbb{R}$, by

$$\mathbb{E}_{\bar{\nu}}\big[h((\bar{G},\bar{o}))\big] = \mathbb{E}\Big[\mathbb{E}_{\tilde{\nu}}\big[h((G,o))\big]\Big].$$

♠

Theorem 2.3. *If $(G_n)_{n\in\mathbb{N}}$ converges locally in probability to the almost surely locally finite rooted random tree (G_∞, ϕ), then $\mu_n \Longrightarrow \mu$ as $n \to \infty$ in probability.*

Theorem 2.4. *If $(G_n)_{n\in\mathbb{N}}$ converges locally weakly to the almost surely locally finite rooted random tree (G_∞, ϕ), then $\tilde{\mu}_n \Longrightarrow \mu$ as $n \to \infty$.*

Remark 2.5. Theorems 2.3-2.4 also apply when, instead of a tree, the limit is any almost surely locally finite rooted graph (G, o) and μ is defined as the law of the random variable

$$\Delta_o^{(G)} = \left[\frac{\sum_{j\in V(G)} A_{o,j}^{(G)} d_j^{(G)}}{d_o^{(G)}} - d_o^{(G)}\right] \mathbb{1}_{\left\{d_o^{(G)}\neq 0\right\}}.$$

♠

We already know from Section 1.2 that $\int_{\mathbb{R}} x\mu_n(\mathrm{d}x) \geq 0$, and we expect from Theorem 2.3 that

$$\int_{\mathbb{R}} x\mu_n(\mathrm{d}x) \to \int_{\mathbb{R}} x\mu(\mathrm{d}x).$$

We therefore divide the friendship paradox into two classes:

Definition 2.6. We say that the friendship paradox is *significant* when $\mu([0,\infty)) \geq \frac{1}{2}$ and *insignificant* when $\mu([0,\infty)) < \frac{1}{2}$. We say that the friendship paradox is *marginally significant* when $\mu([0,\infty)) = \frac{1}{2}$. ♠

In Section 3 we analyse μ for four classes of sparse random graphs: the homogeneous Erdős-Rényi random graph (Section 3.1), the inhomogeneous Erdős-Rényi random graph (Section 3.2), the configuration model (Section 3.3), and the preferential attachment model (Section 3.4). We prove that the friendship paradox is significant for specific one-parameter classes of configuration models and preferential attachment models. Furthermore, we demonstrate numerically that the friendship paradox is significant also for the homogeneous Erdős-Rényi random graph and the inhomogeneous Erdős-Rényi random graph.

As part of our analysis of significance we propose the following conjecture:

Conjecture 2.7. *Let $(X_i)_{i\in\mathbb{N}_0}$ be i.i.d. Binomial, Geometric or Poisson random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then*

$$\mathbb{P}\left\{ \sum_{i=1}^{X_0} X_i \geq X_0(X_0-1) \right\} \geq \frac{1}{2}$$

with the empty sum equal to zero.

Conjecture 2.7 says that if $d_\phi, d_1, d_2, \dots$ are i.i.d. Binomial, Geometric or Poisson random variables, then the friendship paradox is significant (recall (1.1)). We believe that this conjecture also holds for many other distributions with support in $\mathbb{N}_0$ (and that the i.i.d. assumption can even be relaxed), but it is not true for general discrete distributions with support in $\mathbb{N}_0$. For example, if $(X_i)_{i\in\mathbb{N}_0}$ are i.i.d. copies of

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{5}, \\ 20, & \text{with probability } \frac{4}{5}, \end{cases}$$

then $\mathbb{P}\{\sum_{i=1}^{X_0} X_i \geq X_0(X_0-1)\} = \frac{1}{5} + 6\left(\frac{4}{5}\right)^{21} < \frac{1}{2}$.

Remark 2.8. Theorem 2.3 shows that, for all locally tree-like random graphs,

$$\mu_n([0,\infty)) \xrightarrow{\mathbb{P}} \mu([0,\infty)) \text{ whenever } \mu(\{0\}) = 0.$$

We will see that, from our four classes, the configuration model and the preferential attachment model have $\mu(\{0\}) = 0$, while the homogeneous and the inhomogeneous Erdős-Rényi random graph have $\mu(\{0\}) > 0$. Still, we expect the same convergence to hold for the latter two. ♠

3 Four classes of sparse random graphs: main theorems

In this section we focus on the computation of $\mu([0,\infty))$, $\mathbb{E}_{\bar{\mu}}[\Delta_\phi]$ and $\mathbb{E}_{\bar{\mu}}[\Delta_\phi^2]$. We write $\mathbb{P}_{\bar{\mu}}$ to denote the probability measure of the probability space on which (G_∞, ϕ) is defined. $\mathbb{E}_{\bar{\mu}}$ will be the expectation with respect to (G_∞, ϕ) with law $\bar{\mu}$.

3.1 Homogeneous Erdős-Rényi random graph

Fix $\lambda \in (0,\infty)$. For $n \in \mathbb{N}$, let $\mathrm{HER}_n(\frac{\lambda}{n} \wedge 1)$ be the random graph on $[n]$ where each pair of vertices is independently connected by an edge with probability $\frac{\lambda}{n} \wedge 1$. This random graph converges locally in probability to a *Galton-Watson tree with a Poisson offspring distribution with mean* λ [11, Theorem 2.18].

Theorem 3.1. *For every* $\lambda \in (0,\infty)$*,*

$$\mu([0,\infty)) = \sum_{k\in\mathbb{N}_0} \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \sum_{l\geq k(k-1)} \frac{\mathrm{e}^{-\lambda k}(\lambda k)^l}{l!}.$$

In particular, $\lim_{\lambda\downarrow 0}\mu([0,\infty)) = 1$ *and* $\lim_{\lambda\to\infty}\mu([0,\infty)) = \frac{1}{2}$*, i.e., the friendship paradox is significant for both small and large* λ*, and becomes marginally significant in the limit as* $\lambda \to \infty$*.*

Theorem 3.2. *For every* $\lambda \in (0,\infty)$*,*

$$\begin{aligned}
\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] &= 1 - (\lambda+1)\,\mathrm{e}^{-\lambda},\\
\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi^2\big] &= \lambda\Big(\sum_{k\in\mathbb{N}} k^{-1}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} + 1\Big) - (\lambda+1)^2\,\mathrm{e}^{-\lambda} + 1.
\end{aligned}$$

In particular,

$$\begin{aligned}
\lim_{\lambda\downarrow 0}\mathbb{E}_{\bar{\mu}}[\Delta_\phi] = 0, &\quad \lim_{\lambda\downarrow 0}\mathbb{E}_{\bar{\mu}}[\Delta_\phi^2] = 0,\\
\lim_{\lambda\to\infty}\mathbb{E}_{\bar{\mu}}[\Delta_\phi] = 1, &\quad \lim_{\lambda\to\infty}\mathbb{E}_{\bar{\mu}}[\Delta_\phi^2] = \infty.
\end{aligned}$$

Theorem 3.3. *For every* $\lambda \in (0,\infty)$*,*

$$\mathbb{P}_{\bar{\mu}}\{\Delta_\phi \geq x\} \sim \mathbb{P}_{\bar{\mu}}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar{\mu}}\big\{d_1 \geq x\big\}, \qquad x \to \infty.$$

In particular,

$$\mathbb{P}_{\bar{\mu}}\{\Delta_\phi \geq x\} = \mu([x,\infty)) \sim \frac{1}{\sqrt{2\pi x}}\,\lambda\,\mathrm{e}^{-2\lambda}\exp\Big\{-x\log\Big(\frac{x}{\lambda\mathrm{e}}\Big)\Big\}, \qquad x \to \infty.$$

Theorems 3.1–3.3 show that for the homogeneous Erdős-Rényi random graph, which is a simple graph, the properties of μ are explicitly computable. In addition, as seen in Figure 2, numerical computations indicate that $\mu([0,\infty))$ is a strictly decreasing function of λ, which leads us to *conjecture* that μ is significant for all $\lambda \in (0,\infty)$ (in support of Conjecture 2.7).

3.2 Inhomogeneous Erdős-Rényi random graph

Let $\mathcal{F}$ be the class of *non-constant* Riemann integrable functions $f\colon [0,1] \to (0,\infty)$ satisfying

$$M_- = M_-(f) = \inf_{x\in[0,1]} f(x) > 0, \qquad M_+ = M_+(f) = \sup_{x\in[0,1]} f(x) < \infty.$$

Fix $\lambda \in (0,\infty)$. For $n \in \mathbb{N}$, let $\mathrm{IER}_n(\lambda f)$ be the random graph on $[n]$ where each pair of vertices $i,j \in [n]$ is independently connected by an edge with probability $\frac{\lambda}{n}f(\frac{i}{n})f(\frac{j}{n}) \wedge 1$. This random graph converges locally in probability to a *unimodular multi-type marked Galton-Watson tree* with the following properties [11, Theorem 3.14]:

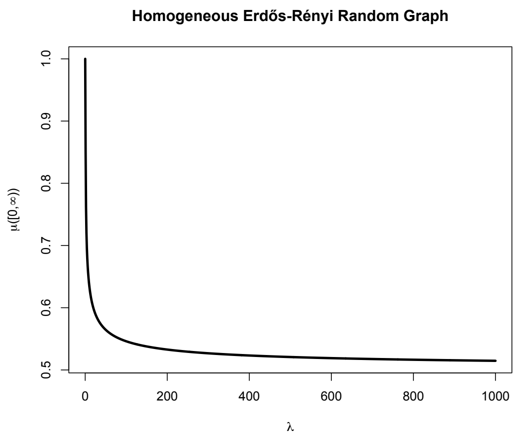

Figure 2: *Numerical plot of the map* $\lambda \mapsto \sum_{k=0}^{10^4} \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \sum_{l \geq k(k-1)} \frac{\mathrm{e}^{-\lambda k}(\lambda k)^l}{l!}$ *for* $10^{-8} \leq \lambda \leq 10^3$.

(1) The root has type Q with

$$\rho\{Q \in A\} = \ell_1(A) \qquad \forall\, A \subseteq [0,1],$$

where ρ is the probability distribution of the types of the vertices, and ℓ_1 is the Lebesgue measure on $[0,1]$.

(2) Let $\beta_m = \int_{[0,1]} \mathrm{d}y\, f^m(y)$. Any vertex other than the root takes an independent type Q' with

$$\rho\{Q' \in A\} = \beta_1^{-1} \int_A \mathrm{d}y\, f(y) \qquad \forall\, A \subseteq [0,1].$$

(3) A vertex of type x independently has offspring distribution $\mathrm{Poisson}(\lambda\beta_1 f(x))$.

Property (1) says that Q has a uniform $[0,1]$ distribution, while property (2) says that

$$h_{Q'}(y) = \beta_1^{-1} f(y) \mathbb{1}_{\{y \in [0,1]\}}$$

is the density function of Q'.

Theorem 3.4. *Let* $c_k(z)$ *be the probability that a random variable with a* $\mathrm{Poisson}(\lambda\beta_1 z)$*-distribution takes the value* $k \in \mathbb{N}_0$*.*

(a) For every $f \in \mathcal{F}$ *and* $\lambda \in (0,\infty)$*,*

$$\mu([0,\infty)) = \sum_{k \in \mathbb{N}_0} \left(\int_{[0,1]} \mathrm{d}x\, c_k(f(x)) \right) \left[\beta_1^{-k} \prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}x_j f(x_j) \sum_{l \geq k(k-1)} c_l \left(\sum_{j=1}^{k} f(x_j) \right) \right],$$

where $\prod_{j=1}^{0} = 0$*.*

(b) *For every* $f \in \mathcal{F}$,

$$\lim_{\lambda \downarrow 0} \mu([0,\infty)) = 1,$$
$$\lim_{\lambda \to \infty} \mu([0,\infty)) = \mathbb{P}_{\bar{\mu}}\{(Q,Q') \in A_f\} + \frac{1}{2}\mathbb{P}_{\bar{\mu}}\{(Q,Q') \in B_f\},$$

where

$$A_f = \big\{(x,y) \in [0,1] \times [0,1] \colon\ f(x) < f(y)\big\},$$
$$B_f = \big\{(x,y) \in [0,1] \times [0,1] \colon\ f(x) = f(y)\big\}.$$

(c) *For every* $f \in \mathcal{F}$, $\lim_{\lambda\to\infty} \mu([0,\infty)) \geq \frac{1}{2}$, *which together with (a) implies that the friendship paradox is significant for both small and large* λ.

(d) *If* ℓ_2 *is the Lebesgue measure on* $[0,1]\times[0,1]$ *and* $\ell_2(A_f) > 0$, *then* $\lim_{\lambda\to\infty} \mu([0,\infty)) > \frac{1}{2}$. *Moreover, if* $\ell_2(A_f) = 0$, *then* $\lim_{\lambda\to\infty} \mu([0,\infty)) = \frac{1}{2}$, *i.e., the friendship paradox is marginally significant in the limit as* $\lambda \to \infty$.

(e) *If* $f \in \mathcal{F}$ *is injective, then* $\lim_{\lambda\to\infty} \mu([0,\infty)) = \mathbb{P}_{\bar{\mu}}\{(Q,Q') \in A_f\}$.

Theorem 3.5. *For every* $f \in \mathcal{F}$ *and* $\lambda \in (0,\infty)$,

$$\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] = 1 - (\lambda\beta_2 + 1) \int_{[0,1]} \mathrm{d}x\, \mathrm{e}^{-\lambda\beta_1 f(x)} + \lambda(\beta_2 - \beta_1^2) \tag{3.1}$$

and

$$\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi^2\big] = \lambda\big(\beta_2 + \lambda(\beta_1\beta_3 - \beta_2^2)\big) \int_{[0,1]} \mathrm{d}x \sum_{k\in\mathbb{N}} k^{-1} \frac{\mathrm{e}^{-\lambda\beta_1 f(x)}(\lambda\beta_1 f(x))^k}{k!}$$
$$- \big(\lambda^2\beta_2^2 + 2\lambda\beta_2 + 1\big) \int_{[0,1]} \mathrm{d}x\, \mathrm{e}^{-\lambda\beta_1 f(x)} + \lambda^2(\beta_2^2 - \beta_2\beta_1^2) + \lambda(2\beta_2 - \beta_1^2) + 1.$$

In particular,

$$\lim_{\lambda\downarrow 0} \mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] = 0, \quad \lim_{\lambda\downarrow 0} \mathbb{E}_{\bar{\mu}}\big[\Delta_\phi^2\big] = 0,$$
$$\lim_{\lambda\to\infty} \mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] = \infty, \quad \lim_{\lambda\to\infty} \mathbb{E}_{\bar{\mu}}\big[\Delta_\phi^2\big] = \infty.$$

Theorem 3.6. *For every* $f \in \mathcal{F}$ *and* $\lambda \in (0,\infty)$,

$$\mathbb{P}_{\bar{\mu}}\{\Delta_\phi \geq x\} \sim \mathbb{P}_{\bar{\mu}}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar{\mu}}\big\{d_1 \geq x\big\}, \qquad x \to \infty.$$

In particular,

$$\mu([x,\infty)) \asymp \frac{1}{\sqrt{x}}\, \beta_x\, \mathrm{e}^{-x\log(\frac{x}{\lambda\beta_1 \mathrm{e}})}, \qquad x \to \infty,$$

where $\beta_x = \int_{[0,1]} \mathrm{d}y f^x(y)$. *The asymptotic behaviour of* β_x *can be derived in the following two cases:*

(a) *If* ℓ_1 *is the Lebesgue measure on* $[0,1]$ *and* $\ell_1(\{y \in [0,1] \colon\ f(y) = M_+\}) > 0$, *then* $\beta_x \asymp M_+^x$ *as* $x \to \infty$.

(b) If $f(y_\star) = M_+$ and $f(y_\star) - f(y) \asymp |y - y_\star|^\alpha$, $y \to y_\star$, for some $y_\star \in [0,1]$ and $\alpha \in (0,\infty)$, and f is bounded away from M_+ outside any neighbourhood of $y_\star$, then $\beta_x \asymp x^{-1/\alpha}\, M_+^x$ as $x \to \infty$.

Theorems 3.4–3.6 show that for the inhomogeneous Erdős-Rényi random graph, which is a simple graph, the properties of μ are again explicitly computable. In addition, as seen in Figure 3, numerical computations indicate that $\mu([0,\infty))$ is a strictly decreasing function of λ, regardless which $f \in \mathcal{F}$ is used, which leads us to *conjecture* that μ is significant for all $f \in \mathcal{F}$ and $\lambda \in (0,\infty)$ (in support of a possible extension of Conjecture 2.7 to a case where X_0 has a different distribution than the other random variables).

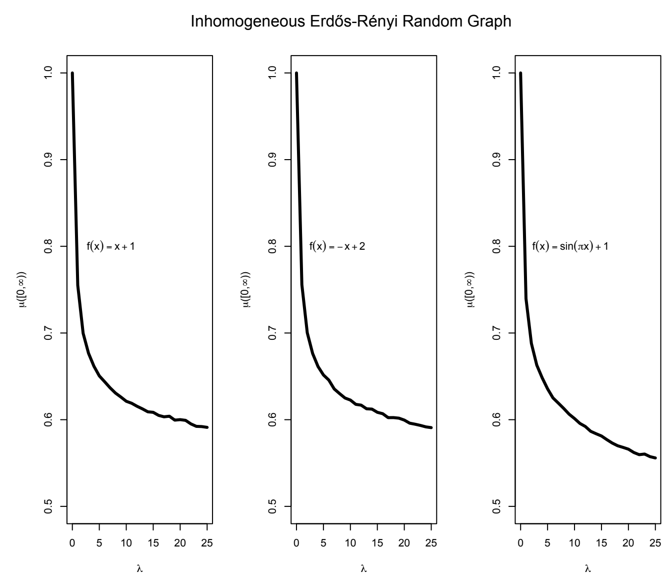

Figure 3: *Numerical plot of the map $\lambda \mapsto \mu([0,\infty))$ for $\mathrm{IER}_n(\lambda f)$ with an increasing, a decreasing and a non-monotonic function f, estimated with the help of Monte Carlo integration.*

In Theorem 3.5, the term $\beta_2 - \beta_1^2$ is the variance of $f(U)$ when U is a uniform $[0,1]$ random variable. If f would be a constant function, then the third term in (3.1) would vanish, which would yield $\mathbb{E}_{\bar{\mu}}[\Delta_\phi] \to 1$ as $\lambda \to \infty$. However, because f is non-constant, the inhomogeneity may cause an individual to have friends who have significantly more friends than the individual does. For example, if $f(x) = a\mathbb{1}_A(x) + b\mathbb{1}_B(x)$ with $a < b$, $A \cap B = \emptyset$ and $A \cup B = [0,1]$, then f divides the individuals into *two communities* $A_n = nA \cap [n]$ and $B_n = nB \cap [n]$. Two individuals inside community A_n are friends with probability $a^2\lambda/n$, while two individuals inside community B_n are friends with probability $b^2\lambda/n$. The friends of individuals in community A_n can live in community B_n, where individuals are more likely to have friends, which explains why $\mathbb{E}_{\bar{\mu}}[\Delta_\phi] \to \infty$ as $\lambda \to \infty$.

3.3 Configuration model

For $n \in \mathbb{N}$ and a sequence of deterministic non-negative integers $\mathbf{d}_n = (d_{1,n}, \ldots, d_{n,n})$, let $\mathrm{CM}_n(\mathbf{d}_n)$ be the random graph on $[n]$ drawn uniformly at random from the set of all graphs with the same degree sequence $\mathbf{d}_n$. Let D_n be the degree of a uniformly chosen vertex in $[n]$. If D_n converges in distribution to a random variable D with $\mathbb{P}\{D > 0\} = 1$ such that

$\mathbb{E}[D_n] = \frac{1}{n}\sum_{i\in[n]} d_{i,n} \to \mathbb{E}[D] < \infty$, then this random graph converges locally in probability to a *unimodular branching process tree* with *root offspring distribution* $p = (p_k)_{k\geq 0}$ given by $p_k = \mathbb{P}\{D = k\}$ [11, Theorem 4.1] and with *biased offspring distribution* $p^\star = (p^\star_k)_{k\geq 0}$ given by $p^\star_k = (k+1)p_{k+1}/\mathbb{E}[D]$ for all the other vertices [11, Definition 1.26].

Note that $\mathrm{CM}_n(\mathbf{d}_n)$ is a multi-graph, i.e., it possibly has self-loops and multiple edges. We saw in Section 1 that if a graph contains self-loops, then the friendship paradox may not hold (see Figure 1). In spite of this, since $\mathrm{CM}_n(\mathbf{d}_n)$ is locally tree-like we may assume that for large n it obeys the friendship paradox. There is a version of the configuration model without self-loops that also converges locally in probability to the same limit (see [11, Theorem 4.1]). In many real-life applications, D often exhibits a power-law distribution. Our first result shows that assuming D to have a power-law distribution with a finite first moment implies the significance of the friendship paradox.

Theorem 3.7. *Consider the special case where $p_k = k^{-\tau}/\zeta(\tau)$, $\tau > 2$, with ζ the Riemann function. Then $\mu([0,\infty)) > \frac{1}{2}$ for all $\tau > 2$. In addition, $\lim_{\tau\downarrow 2}\mu([0,\infty)) = 1$ and $\lim_{\tau\to\infty}\mu([0,\infty)) = 1$.*

The next result describes the first and second moment of Δ_ϕ and for this we do not need to assume that D has a power law distribution.

Theorem 3.8. *Let D be such that $\mathbb{P}(D > 0) = 1$ and $\mathbb{E}[D] < \infty$. Then*

$$\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] = \frac{\mathbb{V}ar(D)}{\mathbb{E}[D]}.$$

For every D satisfying the above conditions and $\mathbb{E}[D^2] < \infty$,

$$\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi^2\big] = \frac{\left(\mathbb{E}[D^3]\mathbb{E}[D] - (\mathbb{E}[D^2])^2\right)\mathbb{E}[D^{-1}] + \mathbb{E}[D^2]\mathbb{V}ar(D)}{(\mathbb{E}[D])^2}.$$

In particular, $\mathbb{V}ar_{\bar{\mu}}(\Delta_\phi) \geq \mathbb{V}ar(D)$.

The next result derives the tail of the measure μ when D follows a power law distribution.

Theorem 3.9. *Consider the special case where $p_k = k^{-\tau}/\zeta(\tau)$, $\tau > 2$. Then*

$$\mu([x,\infty)) \asymp x^{-(\tau-2)}, \qquad x \to \infty.$$

Theorems 3.7–3.9 show that for the configuration model, which is a multi-graph, the properties of μ are somewhat harder to come by than for the homogeneous or inhomogeneous Erdős-Rényi Random Graph. Fig. 4 shows numerical upper and lower bound on $\mu([0,\infty))$ for the special case where $p_k = k^{-\tau}/\zeta(\tau)$, $\tau > 2$. The black curves are given by the following maps:

$$\tau \mapsto \textstyle\sum_{k=1}^{200} p_k\mathbb{P}_{\bar{\mu}}\big\{d_1 \geq k(k-1)\big\},$$
$$\tau \mapsto 1 \wedge \textstyle\sum_{k=1}^{200} kp_k\mathbb{P}_{\bar{\mu}}\big\{d_1 \geq k-1\big\}.$$

Fig. 4 shows that $\mu([0,\infty))$ is not a monotone function of τ, and is always larger than $\frac{4}{5}$. It achieves its minimum value inside the interval $[2,4]$.

The following example shows that for the configuration model with a bimodal degree distribution the friendship paradox is significant as soon as one of the two degrees is 1.

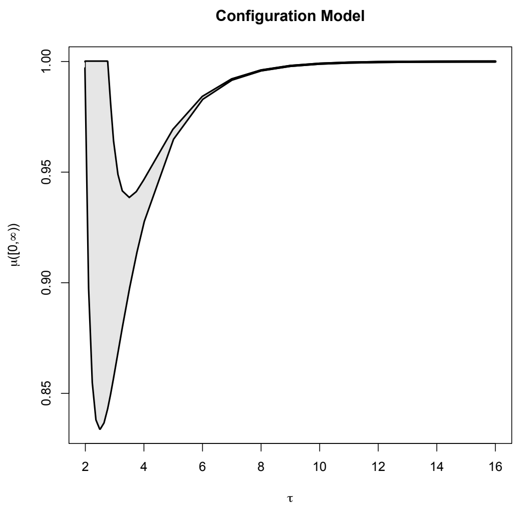

Figure 4: *Numerical plot of the region containing the map* $\tau \mapsto \mu([0,\infty))$ *for* $\mathrm{CM}_n(\mathbf{d}_n)$ *with* $p_k = k^{-\tau}/\zeta(\tau)$.

Example 3.10. Consider the case where the limit is a tree with $p_k = p\mathbb{1}_{\{M_1\}}(k) + (1-p)\mathbb{1}_{\{M_2\}}(k)$ for some $M_1, M_2 \in \mathbb{N}$, $M_1 \leq M_2$ and $p \in (0,1)$. Then

$$p_k^\star = \frac{M_1 p}{M_1 p + M_2(1-p)} \mathbb{1}_{\{M_1-1\}}(k) + \frac{M_2(1-p)}{M_1 p + M_2(1-p)} \mathbb{1}_{\{M_2-1\}}(k).$$

If $M_1 = M_2$, then $p_k = \mathbb{1}_{\{M_1\}}(k)$, $p_k^\star = \mathbb{1}_{\{M_1-1\}}(k)$ and $\mu([0,\infty)) = 1$. If $M_1 < M_2$, then

$$\mu([0,\infty)) = p + (1-p)\left(\frac{M_2(1-p)}{M_1 p + M_2(1-p)}\right)^{M_2}, \tag{3.2}$$

which is $> \frac{1}{2}$ when $p \in [\frac{1}{2}, 1)$. If $M_1 = 1 < M_2$, $p \in (0, \frac{1}{2})$. Suppose $\mu([0,\infty)) \leq \frac{1}{2}$, then (3.2) implies

$$M_2 \log\Big(1 + \frac{p}{M_2(1-p)}\Big) \geq \log\Big(\frac{1-p}{\frac{1}{2}-p}\Big),$$

which (because $\log(1+x) \leq x$, $x \geq 0$) in turn implies that

$$\log\Big(\frac{1-p}{\frac{1}{2}-p}\Big) - \frac{p}{1-p} \leq 0,$$

which is a contradiction. Consequently, if $M_1 = 1 < M_2$ and $p \in (0,1)$, then $\mu([0,\infty)) > \frac{1}{2}$. ♠

The following example shows that for the configuration model the friendship paradox is not always significant.

Example 3.11. Let $\mathbf{d}_n$, $n \in \mathbb{N}$, be such that $\lfloor \frac{n}{10} \rfloor$ degrees are 9 and the other degrees are 10. It is easily checked that $\mathrm{CM}_n(\mathbf{d}_n)$ converges locally in probability to a tree with root offspring distribution $p_k = \frac{1}{10}\mathbb{1}_{\{9\}}(k) + \frac{9}{10}\mathbb{1}_{\{10\}}(k)$, and that $\mu([0,\infty)) = \frac{1}{10} + \frac{9}{10}\big(\frac{10}{11}\big)^{10} < \frac{1}{2}$. The latter follows from (3.2) with $p = \frac{1}{10}$, $M_1 = 9$ and $M_2 = 10$. ♠

3.4 Preferential attachment model

Imagine a social network with new individuals arriving one by one, expanding the social network by one vertex at each arrival. The new individual makes connections with the other individuals by becoming acquainted with them. A realistic assumption is that the new individual is more likely to become acquainted with individuals who already have a large number of acquaintances, which is sometimes called the *rich-get-richer* phenomenon ([11, Chapter 1]). The preferential attachment model describes networks that grow over time, where newly added vertices are more likely to connect to old vertices with higher degrees than to old vertices with lower degrees. It is a graph sequence denoted by $(\mathrm{PAM}_n^{(m,\delta)})_{n\in\mathbb{N}}$, depending on two parameters $m \in \mathbb{N}$ and $\delta \geq -m$, such that at each time n there are n vertices and mn edges, i.e., the total degree is $2mn$. For the friendship paradox, it is natural to assume $m = 1$. For $m > 1$: at time 1 there is one vertex that establishes m friendship relations with itself, while at time 2 the new vertex either establishes m friendship relations with itself or m friendship relations with the previous vertex. So the friendship is higher when $m > 1$. Henceforth we restrict ourselves to the more natural case where $m = 1$, so that one vertex with one edge is added per unit time. The following brief overview is taken from [10, Chapter 8].

Fix $\delta \geq -1$. In this case, $\mathrm{PAM}_1^{(1,\delta)}$ consists of a single vertex v_1 with a single self-loop. For $n \in \mathbb{N}$, suppose that $v_1, \ldots, v_n$ are the vertices of $\mathrm{PAM}_n^{(1,\delta)}$ with degrees $d_{1,n}, \ldots, d_{n,n}$, respectively. Then, given $\mathrm{PAM}_n^{(1,\delta)}$, the rule for obtaining $\mathrm{PAM}_{n+1}^{(1,\delta)}$ is such that a single vertex v_{n+1} with a single edge attached to a vertex in $\{v_1, \ldots, v_{n+1}\}$ is added according to the following probabilities:

$$\mathbb{P}\{v_{n+1} \to v_i \mid \mathrm{PAM}_n^{(1,\delta)}\} = \begin{cases} \dfrac{d_{i,n} + \delta}{n(2+\delta) + (1+\delta)}, & \text{if } i \in [n], \\ \dfrac{1+\delta}{n(2+\delta) + (1+\delta)}, & \text{if } i = n+1. \end{cases}$$

The role of the parameter δ is that of a shift: for $\delta = 0$ the attachment probabilities are linear in the degrees, for $\delta \neq 0$ affine in the degrees. The δ also allows some flexibility in terms of the degree distribution. The power law exponent of the degree distribution is given by $3 + \delta/m$.

For $\delta > -1$, $\mathrm{PAM}_n^{(1,\delta)}$ converges locally in probability to a multi-type discrete-time branching process called the *Pólya point tree* [11, Theorem 5.26] (see also [1]).[1] In this tree, each vertex u is labeled by finite words as $u = u^1u^2 \cdots u^l$, as in the Ulam-Harris way of labelling trees and ϕ represents the empty word or the root of the tree. Then, $u = \phi$ carries a type A_u and a vertex $u \neq \phi$ carries a type (A_u, L_u), where $A_u \in [0, 1]$ specifies the age of u and L_u a label in $\{\mathsf{y}, \mathsf{o}\}$ that indicates whether u is younger than its parent by denoting $L_u = \mathsf{y}$ or older than its parent by denoting $L_u = \mathsf{o}$. An old child is actually the older neighbour to which the initial edge of the parent is connected, whereas younger children are younger vertices that use an edge to connect to their parent. The number of children with label o of a vertex is either 0 or 1 and defined deterministically, but the number of children with label y of a vertex is random. This rooted tree is defined inductively as follows.

First, the root ϕ takes an age A_ϕ that is chosen uniformly at random from $[0, 1]$, but it does not take any label in $\{\mathsf{y}, \mathsf{o}\}$ because it has no parent. Then, by recursion, the remainder of the tree is constructed in such a way that if the vertex u has the Ulam-Harris word $u^1u^2 \cdots u^l$ and the type (A_u, L_u) (or A_u in the case that $u = \phi$), then $u_j = uj = u^1u^2 \cdots u^l j$ is defined as follows:

[1] It is worth noting here that $(\mathrm{PAM}_n^{(1,\delta)})_{n\in\mathbb{N}}$ is denoted by $(\mathrm{PAM}_n^{(1,\delta)}(a))_{n\in\mathbb{N}}$ in [11].

(1) If either $L_u = \mathsf{o}$ or $u = \phi$, then u has $N^{(\mathsf{old})}(u) = 1$ offspring with label o. Otherwise it has $N^{(\mathsf{old})}(u) = 0$ offspring with label o. In the case $N^{(\mathsf{old})}(u) = 1$, set u_1 as the offspring of u with label o. This offspring has an age A_{u_1} given by

$$A_{u_1} = U_{u_1}^{\frac{2+\delta}{1+\delta}} A_u,$$

where U_{u_1} is a uniform $[0,1]$ random variable that is independent of everything else.

(2) Let $A_{u_{N^{(\mathsf{old})}(u)+1}}, \dots, A_{u_{N^{(\mathsf{old})}(u)+N^{(\mathsf{young})}(u)}}$ be the (ordered) points of a Poisson point process on $[A_u, 1]$ with intensity

$$\rho_u(x) = \frac{\Gamma_u}{2+\delta} \frac{x^{-(1+\delta)/(2+\delta)}}{A_u^{1/(2+\delta)}},$$

where Γ_u is a random variable selected independently of everything else as

$$\Gamma_u \overset{d}{=} \begin{cases} \text{Gamma}(1+\delta, 1), & \text{if } u \text{ is the root or of label } \mathsf{y}, \\ \text{Gamma}(2+\delta, 1), & \text{if } u \text{ is of label } \mathsf{o}. \end{cases}$$

Then the vertices $u_{N^{(\mathsf{old})}(u)+1}, \dots, u_{N^{(\mathsf{old})}(u)+N^{(\mathsf{young})}(u)}$ are defined as the children of u with label y.

Note that $(\mathrm{PAM}_n^{(1,\delta)})_{n\in\mathbb{N}}$ is a multi-graph, i.e., it possibly has self-loops and multiple edges. Since $(\mathrm{PAM}_n^{(1,\delta)})_{n\in\mathbb{N}}$ is locally tree-like we may assume that for large n it obeys the friendship paradox. There is a version of the preferential attachment model without self-loops that also converges locally in probability to the same limit (see [11, Theorem 5.8]).

Theorem 3.12. $\mu([0,\infty)) \geq \frac{1}{2}$. *In addition,* $\lim_{\delta\downarrow -1} \mu([0,\infty)) = 1$.

Theorem 3.13. *Abbreviate* $p_\delta = \mathbb{E}_{\bar{\mu}}[d_\phi^{-1}] = \sum_{k\in\mathbb{N}} \frac{(2+\delta)\Gamma(3+2\delta)\Gamma(k+\delta)}{k\Gamma(1+\delta)\Gamma(k+3+2\delta)}$. *Then*

$$\mathbb{E}_{\bar{\mu}}\big[\Delta_\phi\big] \begin{cases} \in \frac{2+\delta}{\delta}(\frac{1}{2} + p_\delta) + \big[-(1-p_\delta), 0\big], & \text{if } \delta \in (0,\infty), \\ = \infty, & \text{if } \delta \in (-1, 0], \end{cases}$$

and $\mathbb{E}_{\bar{\mu}}[\Delta_\phi^2] < \infty$ *if and only if* $\delta \in (1,\infty)$.

Theorem 3.14. *Abbreviate* $\tau = \tau(\delta) = 3 + \delta$. *Then, as* $x \to \infty$,

$$\mu([x,\infty)) \gtrsim \begin{cases} x^{-(3\tau-4)}, & \text{if } \delta \in (-1,0), \\ x^{-(2\tau-1)}, & \text{if } \delta \in [0,\infty), \end{cases}$$

and

$$\mu([x,\infty)) \lesssim \begin{cases} x^{-(3-\tau)}, & \text{if } \delta \in [-\frac{1}{2}, 0), \\ x^{-1}, & \text{if } \delta = 0, \\ x^{-(\tau-3)}, & \text{if } \delta \in (0,\infty). \end{cases}$$

Remark 3.15. Theorem 3.13 suggests that if $\delta \in (-1, 0]$, then $\mu([x,\infty)) \asymp x^{-1+\varepsilon}$ for some $\varepsilon \in [0,\infty)$, but we are unable to establish this scaling in the lower bound. Note that in the upper bound we do not cover the case $\delta \in (-1, -\frac{1}{2})$, which is due to technical hurdles. ♠

Figure 5 shows a numerical plot of a lower bound on $\delta \mapsto \mu([0,\infty))$ for $(\mathrm{PAM}_n^{(1,\delta)})_{n\in\mathbb{N}}$, given by

$$\sum_{k=1}^{50} \mathbb{P}_{\bar{\mu}}\big\{d_\phi = k\big\}\mathbb{P}_{\bar{\mu}}\big\{d_1 \geq k(k-1) \mid d_\phi = k\big\},$$

in which we have used Monte Carlo integration to estimate $\mathbb{P}_{\bar{\mu}}\{d_1 \geq k(k-1) \mid d_\phi = k\}$. Fig. 5 shows that $\mu([0,\infty))$ may not be a monotone function of δ, and is always greater than 0.99. In addition to confirming that $\lim_{\delta\downarrow -1}\mu([0,\infty)) = 1$, the figure leads us to *conjecture* that $\lim_{\delta\to\infty}\mu([0,\infty)) = 1$. Since the preferential attachment model captures self-reinforcement in friendship networks, these strong forms of the friendship paradox are perhaps plausible.

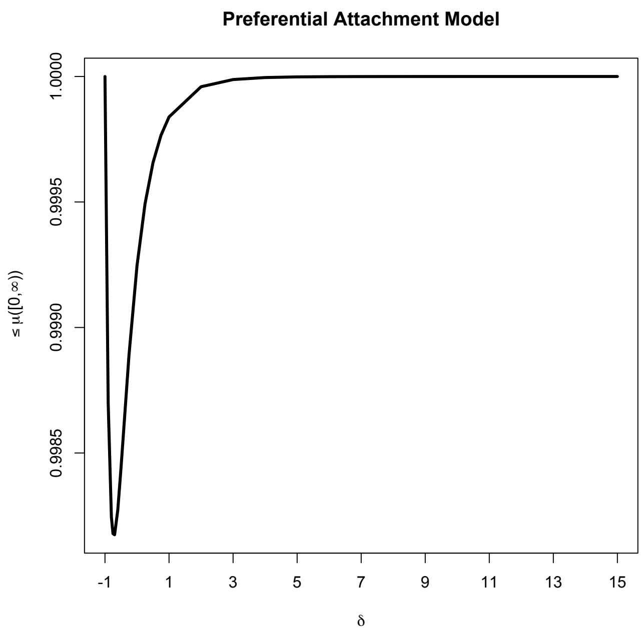

Figure 5: *Numerical plot of a lower bound on* $\delta \mapsto \mu([0,\infty))$ *for* $(\mathrm{PAM}_n^{(1,\delta)})_{n\in\mathbb{N}}$.

4 Proof of the main theorems

In this section we provide the proofs of all the results discussed before.

4.1 Local convergence

4.1.1 Convergence locally in probability

Proof of Theorem 2.3. We replace (G_∞, ϕ) by a connected locally-finite random graph (G'_∞, ϕ) rooted at ϕ defined by

$$(G'_\infty(\omega), \phi) = \begin{cases} (G_\infty(\omega), \phi), & \text{if } (G_\infty(\omega), \phi) \text{ is locally finite,} \\ (H, \phi), & \text{otherwise,} \end{cases}$$

where (H,ϕ) is an arbitrary connected locally-finite deterministic graph rooted at ϕ. Without loss of generality, we simplify the notation by assuming that $(G_\infty,\phi) \equiv (G'_\infty,\phi)$. Since G_n converges locally in probability to (G_∞,ϕ), for every bounded and continuous function $h\colon (\mathscr{G},\mathrm{d}_{\mathscr{G}}) \to (\mathbb{R},|\cdot|)$ we have

$$\mathbb{E}_{\bar{\mu}_n}\big[h(\mathscr{C}(G_n,U_n)) \mid G_n\big] \xrightarrow{\mathbb{P}} \mathbb{E}_{\bar{\mu}}\big[h((G_\infty,\phi))\big], \tag{4.1}$$

where the expectation in the left-hand side is the conditional expected value of $h(\mathscr{C}(G_n,U_n))$ given G_n when (G_n,U_n) has the joint law $\bar{\mu}_n$. Since

$$\mathbb{E}_{\bar{\mu}_n}\big[h(\mathscr{C}(G_n,U_n)) \mid G_n\big] = \frac{1}{n}\sum_{i\in[n]} h(\mathscr{C}(G_n,i)),$$

where $\mathscr{C}(G_n,i)$ is the connected component of i in G_n viewed as a rooted graph with root vertex i, it follows from (4.1) that

$$\frac{1}{n}\sum_{i\in[n]} h(\mathscr{C}(G_n,i)) \xrightarrow{\mathbb{P}} \mathbb{E}_{\bar{\mu}}\big[h((G_\infty,\phi))\big] \tag{4.2}$$

for every bounded and continuous function $h\colon (\mathscr{G},\mathrm{d}_{\mathscr{G}}) \to (\mathbb{R},|\cdot|)$.

Now let $f\colon (\mathbb{R},|\cdot|) \to (\mathbb{R},|\cdot|)$ be an arbitrary bounded and continuous function. Define the function $h\colon (\mathscr{G},\mathrm{d}_{\mathscr{G}}) \to (\mathbb{R},|\cdot|)$ by setting

$$h((G,o)) = f\left(\left[\frac{\sum_{j\in V(G)} A^{(G)}_{o,j} d^{(G)}_j}{d^{(G)}_o} - d^{(G)}_o\right] \mathbb{1}_{\{d^{(G)}_o \neq 0\}}\right). \tag{4.3}$$

Note that the boundedness of f implies the boundedness of h. Also, if (G,o) and $((G'_1,o'_1))_{n\in\mathbb{N}}$ are connected locally-finite deterministic rooted graphs in $\mathscr{G}$ such that $\mathrm{d}_{\mathscr{G}}((G'_n,o'_n),(G,o)) \to 0$, then by the definition of $\mathrm{d}_{\mathscr{G}}$ we have

$$\sup\big\{r \geq 0\colon\ B^{(G'_n)}_r(o'_n) \simeq B^{(G)}_r(o)\big\} \to \infty,$$

and so (see [11, Lemma A.10]),

$$\mathbb{1}_{\big\{(G'_n,o'_n)\ \simeq\ (G,o)\big\}} = \mathbb{1}_{\big\{B^{(G'_n)}_r(o'_n)\ \simeq\ B^{(G)}_r(o)\ \forall\, r\geq 0\big\}} \to 1.$$

In this case, for sufficiently large n, there is a bijection $g_n\colon V(G) \to V(G'_n)$ such that $g_n(o) = o'_n$ and $A^{(G)}_{i,j} = A^{(G'_n)}_{g_n(i),g_n(j)}$ for every $i,j \in V(G)$. Consequently, for every $i \in V(G)$,

$$\begin{aligned} d^{(G)}_i = \sum_{j\in V(G)\setminus\{i\}} A^{(G)}_{i,j} + 2A^{(G)}_{i,i} &= \sum_{j\in V(G)\setminus\{i\}} A^{(G'_n)}_{g_n(i),g_n(j)} + 2A^{(G'_n)}_{g_n(i),g_n(i)} \\ &= \sum_{j\in V(G'_n)\setminus\{g_n(i)\}} A^{(G'_n)}_{g_n(i),j} + 2A^{(G'_n)}_{g_n(i),g_n(i)} = d^{(G'_n)}_{g_n(i)} \end{aligned}$$

and

$$\sum_{j\in V(G)} A^{(G)}_{o,j} d^{(G)}_j = \sum_{j\in V(G)} A^{(G'_n)}_{o'_n,g_n(j)} d^{(G'_n)}_{g_n(j)} = \sum_{j\in V(G'_n)} A^{(G'_n)}_{o'_n,j} d^{(G'_n)}_j.$$

Hence

$$\left[\frac{\sum_{j\in V(G'_n)} A^{(G'_n)}_{o'_n,j} d^{(G'_n)}_j}{d^{(G'_n)}_{o'_n}} - d^{(G'_n)}_{o'_n}\right] \mathbb{1}_{\left\{d^{(G'_n)}_{o'_n}\neq 0\right\}} \to \left[\frac{\sum_{j\in V(G)} A^{(G)}_{o,j} d^{(G)}_j}{d^{(G)}_o} - d^{(G)}_o\right] \mathbb{1}_{\left\{d^{(G)}_o\neq 0\right\}},$$

and so, by the continuity of f,

$$h((G'_n, o'_n)) \to h((G, o)).$$

Hence also h is continuous. But $h(\mathscr{C}(G_n, i)) = f(\Delta_{i,n})$, and

$$\begin{aligned} h((G_\infty, \phi)) &= f\left(\left[\frac{\sum_{j\in V(G_\infty)} A^{(G_\infty)}_{\phi,j} d^{(G_\infty)}_j}{d^{(G_\infty)}_\phi} - d^{(G_\infty)}_\phi\right] \mathbb{1}_{\{d^{(G_\infty)}_\phi\neq 0\}}\right) \\ &= f\left(\left[\frac{\sum_{j=1}^{d_\phi}(d_j+1)}{d_\phi} - d_\phi\right] \mathbb{1}_{\{d_\phi\neq 0\}}\right) = f(\Delta_\phi) \qquad a.s. \end{aligned}$$

Inserting this into (4.2), we get

$$\frac{1}{n}\sum_{i\in[n]} f(\Delta_{i,n}) \xrightarrow{\mathbb{P}} \mathbb{E}_{\bar{\mu}}\big[f(\Delta_\phi)\big]. \tag{4.4}$$

On the other hand, $\int_{\mathbb{R}} f \mathrm{d}\mu_n = \frac{1}{n}\sum_{i\in[n]} f(\Delta_{i,n})$ and $\int_{\mathbb{R}} f \mathrm{d}\mu = \mathbb{E}_{\bar{\mu}}[f(\Delta_\phi)]$. Hence the convergence in (4.4) implies that

$$\int_{\mathbb{R}} f \mathrm{d}\mu_n \xrightarrow{\mathbb{P}} \int_{\mathbb{R}} f \mathrm{d}\mu,$$

which settles the claim. □

4.1.2 Convergence locally weakly

Proof of Theorem 2.4. We use the same notation as in the proof of Theorem 2.3. For $n \in \mathbb{N}$,

$$\tilde{\mu}_n(\cdot) = \frac{1}{n}\sum_{i\in[n]} \mathbb{E}_n[\delta_{\Delta_{i,n}}(\cdot)] = \frac{1}{n}\sum_{i\in[n]} \mu_{i,n}(\cdot),$$

where $\mu_{i,n}(\cdot)$ is the law of $\Delta_{i,n}$. Let $f\colon (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ be an arbitrary bounded and continuous function. We have

$$\int_{\mathbb{R}} f \mathrm{d}\tilde{\mu}_n = \frac{1}{n}\sum_{i\in[n]} \int_{\mathbb{R}} f \mathrm{d}\mu_{i,n} = \frac{1}{n}\sum_{i\in[n]} \mathbb{E}_n[f(\Delta_{i,n})] = \mathbb{E}_n\left[\frac{1}{n}\sum_{i\in[n]} f(\Delta_{i,n})\right].$$

Since G_n converges locally weakly to (G_∞, ϕ), for every bounded and continuous function $h\colon (\mathscr{G}, \mathrm{d}_{\mathscr{G}}) \to (\mathbb{R}, |\cdot|)$ we have

$$\mathbb{E}_{\bar{\mu}_n}\big[h(\mathscr{C}(G_n, U_n))\big] \to \mathbb{E}_{\bar{\mu}}\big[h((G_\infty, \phi))\big],$$

where the expectation in the left-hand side is with respect to (G_n, U_n) with law $\bar{\mu}_n$. Using the fact that

$$\mathbb{E}_{\bar{\mu}_n}\big[h(\mathscr{C}(G_n,U_n))\big] = \mathbb{E}_n\left[\mathbb{E}_{\bar{\mu}_n}\big[h(\mathscr{C}(G_n,U_n)) \mid G_n\big]\right] = \mathbb{E}_n\left[\frac{1}{n}\sum_{i\in[n]} h(\mathscr{C}(G_n,i))\right]$$

and taking h as in (4.3), we get

$$\int_{\mathbb{R}} f\,\mathrm{d}\tilde{\mu}_n = \mathbb{E}_n\left[\frac{1}{n}\sum_{i\in[n]} f(\Delta_{i,n})\right] \to \mathbb{E}_{\bar{\mu}}\big[f(\Delta_\phi)\big] = \int_{\mathbb{R}} f\,\mathrm{d}\mu,$$

which settles the claim. □

4.2 Four classes of sparse random graphs

4.2.1 Homogeneous Erdős-Rényi random graph

Proof of Theorem 3.1. Since $\nu = \text{Poisson}(\lambda)$, it follows from (1.2) that

$$\begin{aligned}\mu([0,\infty)) &= \nu(0) + \sum_{k\in\mathbb{N}} \nu(k)\nu^{\circledast k}\big([k(k-1),\infty)\big)\\ &= \mathrm{e}^{-\lambda} + \sum_{k\in\mathbb{N}} \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \sum_{l\geq k(k-1)} \frac{\mathrm{e}^{-\lambda k}(\lambda k)^l}{l!} = \sum_{k\in\mathbb{N}_0} \frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \sum_{l\geq k(k-1)} \frac{\mathrm{e}^{-\lambda k}(\lambda k)^l}{l!}.\end{aligned}$$

Since

$$\mu([0,\infty)) \geq \mathrm{e}^{-\lambda} + \lambda\mathrm{e}^{-\lambda},$$

it follows that $\lim_{\lambda\downarrow 0}\mu([0,\infty)) = 1$.

We now show that $\lim_{\lambda\to\infty}\mu([0,\infty)) = 1/2$. Fix $\lambda > 8$. Define

$$Z_\lambda = \frac{d_\phi - \lambda}{\sqrt{\lambda}}, \qquad Z'_\lambda = \frac{\sum_{j=1}^{d_\phi} d_j - \lambda d_\phi}{\sqrt{\lambda d_\phi}}.$$

Rewriting Δ_ϕ in terms of Z_λ and Z'_λ we have

$$\begin{aligned}\mu([0,\infty)) &= \mathrm{e}^{-\lambda} + \mathbb{P}_{\bar{\mu}}\left\{\sum_{j=1}^{d_\phi} d_j \geq d_\phi(d_\phi-1), d_\phi \geq 1\right\}\\ &= \mathrm{e}^{-\lambda} + \mathbb{P}_{\bar{\mu}}\left\{Z'_\lambda\sqrt{1+\tfrac{1}{\sqrt{\lambda}}Z_\lambda} \geq \sqrt{\lambda}Z_\lambda\big(1-\tfrac{1}{\lambda}\big) - 1 + Z_\lambda^2,\ d_\phi \geq 1\right\}.\end{aligned} \tag{4.5}$$

We note that $\mathbb{P}_{\bar{\mu}}\{Z_\lambda \geq -\sqrt{\lambda}\} = 1$ because d_ϕ is a non-negative random variable. Define the event $A_\lambda = \{|Z_\lambda| \leq \log\lambda, d_\phi \geq 1\}$. Since, by the continuous mapping theorem and central limit theorem, $\frac{|Z_\lambda|}{\log\lambda} \to 0$ in probability as $\lambda\to\infty$, we have

$$\mathbb{P}_{\bar{\mu}}\big\{A_\lambda^c\big\} \to 0, \qquad \lambda\to\infty. \tag{4.6}$$

On the other hand,

$$\mathbb{P}_{\bar\mu}\Big\{\Big\{Z'_\lambda\left(1+\tfrac{1}{\sqrt\lambda}Z_\lambda\right)^{1/2}\ge\sqrt\lambda Z_\lambda\big(1-\tfrac{1}{\lambda}\big)-1+Z_\lambda^2\Big\}\cap A_\lambda\Big\}$$
$$=\mathbb{P}_{\bar\mu}\Big\{\Big\{Z'_\lambda\left(1+O(\tfrac{\log\lambda}{\sqrt\lambda})\right)^{1/2}\ge\sqrt\lambda Z_\lambda\big(1-\tfrac{1}{\lambda}\big)-1+O(\log^2\lambda)\Big\}\cap A_\lambda\Big\}. \qquad (4.7)$$

Now define

$$B_{1,\lambda}=\Big\{Z_\lambda<-\frac{1}{\log\lambda}\Big\},\quad B_{2,\lambda}=\Big\{Z_\lambda>\frac{1}{\log\lambda}\Big\},\quad B_{3,\lambda}=\Big\{|Z_\lambda|\le\frac{1}{\log\lambda}\Big\}.$$

By the continuous mapping theorem and the central limit theorem, there exists a standard Gaussian random variable N defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that $Z_\lambda+\frac{1}{\log\lambda}\Rightarrow N$, $Z_\lambda-\frac{1}{\log\lambda}\Rightarrow N$ and $|Z_\lambda|-\frac{1}{\log\lambda}\Rightarrow|N|$ as $\lambda\to\infty$. Therefore as $\lambda\to\infty$

$$\mathbb{P}_{\bar\mu}\big\{B_{1,\lambda}\big\}\to\mathbb{P}\big\{N<0\big\}=\tfrac12, \qquad (4.8)$$
$$\mathbb{P}_{\bar\mu}\big\{B_{2,\lambda}\big\}\to\mathbb{P}\big\{N>0\big\}=\tfrac12,$$
$$\mathbb{P}_{\bar\mu}\big\{B_{3,\lambda}\big\}\to\mathbb{P}\big\{|N|\le0\big\}=0. \qquad (4.9)$$

Again, using the continuous mapping theorem and the central limit theorem, we have

$$\tilde Z_\lambda\frac{\log\lambda}{\sqrt\lambda}+O\big(\tfrac{\log^3\lambda}{\sqrt\lambda}\big)\Longrightarrow0,\qquad\lambda\to\infty,$$

where we abbreviate $\tilde Z_\lambda=Z'_\lambda(1+O(\frac{\log\lambda}{\sqrt\lambda}))^{\frac12}$. Note that $g(\lambda)=\frac{\log\lambda}{\sqrt\lambda}$ is a strictly decreasing function on the interval $(8,\infty)$, which implies that $\frac{\log\lambda}{\sqrt\lambda}<\frac{\log8}{\sqrt8}<\frac78$. Hence

$$\mathbb{P}_{\bar\mu}\Big\{\Big\{\tilde Z_\lambda\ge\sqrt\lambda Z_\lambda\big(1-\tfrac1\lambda\big)-1+O(\log^2\lambda)\Big\}\cap B_{2,\lambda}\Big\}$$
$$\le\mathbb{P}_{\bar\mu}\Big\{\tilde Z_\lambda\frac{\log\lambda}{\sqrt\lambda}+O\big(\tfrac{\log^3\lambda}{\sqrt\lambda}\big)>\tfrac78-\tfrac{\log8}{\sqrt8}\Big\}\to0,\qquad\lambda\to\infty. \qquad (4.10)$$

Moreover,

$$\mathbb{P}_{\bar\mu}\Big\{\Big\{\tilde Z_\lambda\ge\sqrt\lambda Z_\lambda\big(1-\tfrac1\lambda\big)-1+O(\log^2\lambda)\Big\}\cap A_\lambda\cap B_{1,\lambda}\Big\}$$
$$=\mathbb{P}_{\bar\mu}\big\{A_\lambda\cap B_{1,\lambda}\big\}-\mathbb{P}_{\bar\mu}\Big\{\Big\{\tilde Z_\lambda<\sqrt\lambda Z_\lambda\big(1-\tfrac1\lambda\big)-1+O(\log^2\lambda)\Big\}\cap A_\lambda\cap B_{1,\lambda}\Big\}$$
$$\to\tfrac12,\qquad\lambda\to\infty, \qquad (4.11)$$

because it follows from (4.6) and (4.8) that

$$\mathbb{P}_{\bar\mu}\big\{A_\lambda\cap B_{1,\lambda}\big\}=\mathbb{P}_{\bar\mu}\big\{B_{1,\lambda}\big\}-\mathbb{P}_{\bar\mu}\big\{A_\lambda^c\cap B_{1,\lambda}\big\}\to\tfrac12,\qquad\lambda\to\infty,$$

and

$$\mathbb{P}_{\bar\mu}\Big\{\Big\{\tilde Z_\lambda<\sqrt\lambda Z_\lambda\big(1-\tfrac1\lambda\big)-1+O(\log^2\lambda)\Big\}\cap A_\lambda\cap B_{1,\lambda}\Big\}$$
$$\le\mathbb{P}_{\bar\mu}\Big\{\tilde Z_\lambda\tfrac{\log\lambda}{\sqrt\lambda}+O\big(\tfrac{\log^3\lambda}{\sqrt\lambda}\big)<-\tfrac78\Big\}\to0,\qquad\lambda\to\infty.$$

Equations (4.5)–(4.7) and (4.9)–(4.11) imply that $\lim_{\lambda\to\infty}\mu([0,\infty))=\frac12$.

Proof of Theorem 3.2. Fix $\lambda \in (0,\infty)$. Since $(d_j)_{j\in\mathbb{N}}$ are i.i.d. copies of d_ϕ,

$$\begin{aligned}\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] &= \sum_{k\in\mathbb{N}} \nu(k)\mathbb{E}_{\bar\mu}\big[\Delta_\phi \mid d_\phi = k\big] = \sum_{k\in\mathbb{N}} \nu(k)\mathbb{E}_{\bar\mu}\Big[\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big] \\ &= \mathbb{E}_{\bar\mu}\big[d_\phi\big]\big(1-\nu(0)\big) + 1 - \nu(0) - \mathbb{E}_{\bar\mu}\big[d_\phi\big] = 1 - (\lambda+1)\,\mathrm{e}^{-\lambda},\end{aligned}$$

which tends to 0 as $\lambda \downarrow 0$ and 1 as $\lambda \to \infty$. Similarly,

$$\begin{aligned}\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] &= \sum_{k\in\mathbb{N}} \nu(k)\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2 \mid d_\phi = k\big] = \sum_{k\in\mathbb{N}} \nu(k)\mathbb{E}_{\bar\mu}\Big[\Big(\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big)^2\Big] \\ &= \sum_{k\in\mathbb{N}} \nu(k)\mathbb{E}_{\bar\mu}\Big[\frac{1}{k^2}\sum_{j=1}^k d_j^2 + \frac{1}{k^2}\sum_{i=1}^k\sum_{\substack{j=1\\ j\neq i}}^k d_i d_j + 1 + k^2 + \frac{2}{k}\sum_{j=1}^k d_j - 2\sum_{j=1}^k d_j - 2k\Big] \\ &= \mathbb{V}ar_{\bar\mu}\big(d_\phi\big)\mathbb{E}_{\bar\mu}\Big[d_\phi^{-1}\mathbb{1}_{\{d_\phi\neq 0\}}\Big] + \big(\mathbb{E}_{\bar\mu}\big[d_\phi\big]\big)^2\big(1-\nu(0)\big) + 1 - \nu(0) \\ &\quad + \mathbb{E}_{\bar\mu}\big[d_\phi^2\big] + 2\mathbb{E}_{\bar\mu}\big[d_\phi\big]\big(1-\nu(0)\big) - 2\big(\mathbb{E}_{\bar\mu}\big[d_\phi\big]\big)^2 - 2\mathbb{E}_{\bar\mu}\big[d_\phi\big] \\ &= \lambda\Big(\sum_{k\in\mathbb{N}} k^{-1}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} + 1\Big) - (\lambda+1)^2\,\mathrm{e}^{-\lambda} + 1.\end{aligned}$$

Since

$$\sum_{k\in\mathbb{N}} k^{-1}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} = \mathbb{E}_{\bar\mu}\Big[d_\phi^{-1}\mathbb{1}_{\{d_\phi\neq 0\}}\Big] \in [0,1],$$

it follows that $\lim_{\lambda\downarrow 0}\mathbb{E}_{\bar\mu}[\Delta_\phi^2] = 0$ and $\lim_{\lambda\to\infty}\mathbb{E}_{\bar\mu}[\Delta_\phi^2] = \infty$.

Proof of Theorem 3.3. Fix $\lambda \in (0,\infty)$. First, we show that for each $k \in \mathbb{N}$,

$$\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^k d_j \geq kx + k(k-1)\Big\} \sim \mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^k d_j = kx + k(k-1)\Big\}, \qquad x \to \infty. \tag{4.12}$$

Indeed, without loss of generality we may assume that $x \in \mathbb{N}$. Since, for each $k, l \in \mathbb{N}$,

$$\begin{aligned}&\frac{\mathbb{P}_{\bar\mu}\big\{\sum_{j=1}^k d_j = kx + k(k-1) + l\big\}}{\mathbb{P}_{\bar\mu}\big\{\sum_{j=1}^k d_j = kx + k(k-1)\big\}} \\ &= \frac{(\lambda k)^l}{(kx + k(k-1)+1)\times\cdots\times(kx+k(k-1)+l)} \to 0, \qquad x \to \infty,\end{aligned}$$

and

$$\sum_{l\in\mathbb{N}} \frac{\mathbb{P}_{\bar\mu}\big\{\sum_{j=1}^k d_j = kx + k(k-1) + l\big\}}{\mathbb{P}_{\bar\mu}\big\{\sum_{j=1}^k d_j = kx + k(k-1)\big\}} \leq \sum_{l\in\mathbb{N}} \frac{(\lambda k)^l}{l!} \leq \mathrm{e}^{\lambda k} < \infty,$$

(4.12) follows by the dominated convergence theorem.

Next, we show that

$$\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^k d_j = kx + k(k-1)\Big\} = o\left(\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}\right), \qquad x \to \infty. \tag{4.13}$$

Indeed, for $k \in \mathbb{N} \setminus \{1\}$ we have, by Stirling's approximation,

$$\frac{\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j = kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}} = \frac{\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j = kx\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}} \frac{(\lambda k)^{k(k-1)}}{(kx+1)\times\cdots\times(kx+k(k-1))}$$
$$\sim \frac{\mathrm{e}^{-\lambda(k-1)}}{\sqrt{k}} \Big(\frac{x}{\lambda \mathrm{e}}\Big)^{-x(k-1)} \frac{(\lambda k)^{k(k-1)}}{(kx+1)\times\cdots\times(kx+k(k-1))} \to 0, \qquad x \to \infty.$$

Observe that for $x \geq 1$,

$$\sum_{k\in\mathbb{N}} \frac{\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j \geq kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\}} \leq \sum_{k\in\mathbb{N}} k\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\} = \mathbb{E}_{\bar\mu}\big[d_\phi\big] < \infty.$$

Using (4.12), (4.13) and the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\mathbb{P}_{\bar\mu}\{\Delta_\phi \geq x\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\}} \\
&= \lim_{x\to\infty} \frac{\sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j \geq kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\}} \\
&= 1 + \sum_{k=2}^\infty \frac{\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}} \lim_{x\to\infty} \frac{\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j \geq kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\}} \\
&= 1 + \sum_{k=2}^\infty \frac{\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}} \lim_{x\to\infty} \frac{\mathbb{P}_{\bar\mu}\big\{ \sum_{j=1}^k d_j = kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}} = 1.
\end{aligned} \tag{4.14}$$

This, together with (4.12) and Stirling's formula, yields

$$\mathbb{P}_{\bar\mu}\big\{\Delta_\phi \geq x\big\} \sim \mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\} \sim \mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}$$
$$\sim \frac{1}{\sqrt{2\pi x}}\,\lambda\,\mathrm{e}^{-2\lambda} \exp\Big\{ -x\log\Big(\frac{x}{\lambda\mathrm{e}}\Big)\Big\}, \qquad x \to \infty.$$

4.2.2 Inhomogeneous Erdős-Rényi random graph

Proof of Theorem 3.4. (a) Let $c_k(z)$ be the probability that a random variable with a Poisson($\lambda\beta_1 z$)-distribution takes the value k. If $\nu_0(k)$ denotes the probability that the root has k children, then

$$\nu_0(k) = \int_{[0,1]} \mathrm{d}x\, c_k(f(x)).$$

Moreover, if Q_i is the type of the i-th child of the root, then

$$\sum_{j=1}^k d_j \;\Big|\; (Q_j)_{j=1}^k = (x_j)_{j=1}^k \overset{d}{=} \text{Poisson}\left(\lambda\beta_1 \sum_{j=1}^k f(x_j)\right),$$

and if $\nu_0^{\circledast k}$ is the convolution of the probability distributions of $(d_j)_{j=1}^k$, then

$$\nu_0^{\circledast k}\big([k(k-1),\infty)\big) = \beta_1^{-k} \prod_{j=1}^k \int_{[0,1]} \mathrm{d}x_j\, f(x_j) \sum_{l\geq k(k-1)} c_l \left(\sum_{j=1}^k f(x_j)\right).$$

Hence

$$\mu([0,\infty)) = \nu_0(0) + \sum_{k\in\mathbb{N}} \nu_0(k)\nu_0^{\circledast k}\big([k(k-1),\infty)\big) \geq \mathrm{e}^{-\lambda\beta_1 M_+} + \lambda\beta_1 M_- \mathrm{e}^{-\lambda\beta_1 M_+},$$

which implies $\lim_{\lambda\downarrow 0}\mu([0,\infty)) = 1$.

(b) For large λ we follow the strategy of using central limit theorem as in the proof of Theorem 3.1. Let $\lambda > \max\{1,\lambda'\}$ with λ' such that $\frac{\log\lambda}{\sqrt{\lambda}} < \frac{M_-}{2}$ for all $\lambda > \lambda'$. For $x,y\in[0,1]$, define

$$Z_\lambda = Z_\lambda(x) = \frac{d_\phi - \lambda\beta_1 f(x)}{\sqrt{\lambda\beta_1 f(x)}}, \qquad Z'_\lambda = Z'_\lambda(y) = \frac{\sum_{j=1}^{d_\phi} d_j - \lambda\beta_1 f(y) d_\phi}{\sqrt{\lambda\beta_1 f(y) d_\phi}}.$$

Then, putting

$$\begin{aligned}
\varphi_1 &= \varphi_1(x,y) = \beta_1^2 f(x)f(y),\\
\varphi_2(\lambda) &= \varphi_2(x,y,\lambda) = \beta_1^{3/2} f^{1/2}(x)\big(2f(x)-f(y)\big) - \lambda^{-1}\beta_1^{1/2} f^{1/2}(x),\\
\varphi_3(\lambda) &= \varphi_3(x,y,\lambda) = \beta_1^2 f(x)\big(f(x)-f(y)\big) - \lambda^{-1}\beta_1 f(x),\\
\varphi_4 &= \varphi_4(x) = \beta_1 f(x),
\end{aligned}$$

we have

$$\begin{aligned}
&\mu([0,\infty))\\
&= \nu_0(0) + \int_{[0,1]}\mathrm{d}x\int_{[0,1]}\mathrm{d}y\,\beta_1^{-1} f(y)\,\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^{d_\phi} d_j \geq d_\phi(d_\phi-1), d_\phi\geq 1 \;\Big|\; Q=x, Q'=y\Big\} \qquad (4.15)
\end{aligned}$$

with

$$\nu_0(0) = \int_{[0,1]}\mathrm{d}x\,\mathrm{e}^{-\lambda\beta_1 f(x)} \leq \mathrm{e}^{-\lambda M_-^2} \to 0, \qquad \lambda\to\infty. \qquad (4.16)$$

Let $\mathbb{P}_{\bar\mu,x,y}\{\cdot\} = \mathbb{P}_{\bar\mu}\{\cdot \mid Q=x, Q'=y\}$. Rewriting Δ_ϕ in terms of Z_λ, Z'_λ and $\{\varphi_j\}_{j=1}^4$, we have

$$\begin{aligned}
&\mathbb{P}_{\bar\mu,x,y}\Big\{\sum_{j=1}^{d_\phi} d_j \geq d_\phi(d_\phi-1), d_\phi\geq 1\Big\}\\
&= \mathbb{P}_{\bar\mu,x,y}\Big\{Z'_\lambda\Big(\varphi_1 + O\big(\tfrac{1}{\sqrt{\lambda}}\big)Z_\lambda\Big)^{1/2} \geq \sqrt{\lambda}\varphi_2(\lambda)Z_\lambda + \lambda\varphi_3(\lambda) + \varphi_4 Z_\lambda^2,\ d_\phi\geq 1\Big\}. \qquad (4.17)
\end{aligned}$$

Similarly to the proof of Theorem 3.1, taking the event $A_\lambda = \{|Z_\lambda|\leq\log\lambda, d_\phi\geq 1\}$, we have

$$\mathbb{P}_{\bar\mu}\big\{A_\lambda^c \mid Q=x\big\} \to 0, \qquad \lambda\to\infty. \qquad (4.18)$$

Equations (4.17)–(4.18) imply that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\mathbb{P}_{\bar\mu,x,y}\Big\{\sum_{j=1}^{d_\phi} d_j \geq d_\phi(d_\phi-1), d_\phi\geq 1\Big\}\\
&= \lim_{\lambda\to\infty}\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z'_\lambda\Big(\varphi_1 + O\big(\tfrac{1}{\sqrt{\lambda}}\big)Z_\lambda\Big)^{1/2} \geq \sqrt{\lambda}\varphi_2(\lambda)Z_\lambda + \lambda\varphi_3(\lambda) + \varphi_4 Z_\lambda^2\Big\}\cap A_\lambda\Big\}\\
&= \lim_{\lambda\to\infty}\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z'_\lambda\Big(\varphi_1 + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big)\Big)^{1/2} + O\big(\log^2\lambda\big) \geq \sqrt{\lambda}\varphi_2(\lambda)Z_\lambda + \lambda\varphi_3(\lambda)\Big\}\cap A_\lambda\Big\}. \qquad (4.19)
\end{aligned}$$

We investigate (4.19) for the following three sets:

$$
\begin{aligned}
I_f &= \{(x,y) \in [0,1] \times [0,1] \colon\ f(x) - f(y) > 0\}, \\
A_f &= \{(x,y) \in [0,1] \times [0,1] \colon\ f(x) - f(y) < 0\}, \\
B_f &= \{(x,y) \in [0,1] \times [0,1] \colon\ f(x) - f(y) = 0\}.
\end{aligned}
$$

Let $Z''_\lambda := Z'_\lambda \Big(\varphi_1 + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big)\Big)^{1/2}$. Using the continuous mapping theorem and the central limit theorem, we have

$$
\frac{1}{\lambda} Z''_\lambda + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big) \overset{\mathbb{P}_{\bar\mu,x,y}}{\longrightarrow} 0, \qquad \lambda \to \infty, \tag{4.20}
$$

Therefore, for $(x,y) \in I_f$,

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda + O\big(\log^2\lambda\big) \geq \sqrt{\lambda}\varphi_2(\lambda) Z_\lambda + \lambda\varphi_3(\lambda)\Big\} \cap A_\lambda\ \Big\} \\
&\leq \mathbb{P}_{\bar\mu,x,y}\Big\{Z''_\lambda + O\big(\log^2\lambda\big) \geq O\big(\sqrt{\lambda}\log\lambda\big) + \lambda\varphi_3(\lambda)\ \Big\} \\
&\leq \mathbb{P}_{\bar\mu,x,y}\Big\{\frac{1}{\lambda} Z''_\lambda + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big) \geq \beta_1^2 f(x)\big(f(x) - f(y)\big)\ \Big\} \\
&\quad \to 0, \qquad \lambda \to \infty.
\end{aligned} \tag{4.21}
$$

For $(x,y) \in A_f$, by (4.18),

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda + O\big(\log^2\lambda\big) \geq \sqrt{\lambda}\varphi_2(\lambda) Z_\lambda + \lambda\varphi_3(\lambda)\Big\} \cap A_\lambda\ \Big\} \\
&= \mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda + O\big(\log^2\lambda\big) \geq O\big(\sqrt{\lambda}\log\lambda\big) + \lambda\varphi_3(\lambda)\Big\} \cap A_\lambda\ \Big\} \\
&= \mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{\frac{1}{\lambda} Z''_\lambda + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big) \geq \beta_1^2 f(x)\big(f(x) - f(y)\big)\Big\} \cap A_\lambda\ \Big\} \\
&\geq \mathbb{P}_{\bar\mu,x,y}\Big\{\frac{1}{\lambda} Z''_\lambda + O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big) \geq \beta_1^2 f(x)\big(f(x) - f(y)\big)\ \Big\} \\
&\quad - \mathbb{P}_{\bar\mu}\big\{A_\lambda^c \mid Q = x\big\} \to 1, \qquad \lambda \to \infty.
\end{aligned} \tag{4.22}
$$

Next, let $(x,y) \in I_3$. Similarly as in the proof of Theorem 3.1, we consider the events

$$
B_{1,\lambda} = \Big\{Z_\lambda < -\frac{1}{\log\lambda}\Big\}, \quad B_{2,\lambda} = \Big\{Z_\lambda > \frac{1}{\log\lambda}\Big\}, \quad B_{3,\lambda} = \Big\{|Z_\lambda| \leq \frac{1}{\log\lambda}\Big\}.
$$

and note that there exists a standard Gaussian random variable N defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, by (4.18),

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu}\big\{A_\lambda \cap B_{1,\lambda} \mid Q = x\big\} \\
&= \mathbb{P}_{\bar\mu}\big\{B_{1,\lambda} \mid Q = x\big\} - \mathbb{P}_{\bar\mu}\big\{A_\lambda^c \cap B_{1,\lambda} \mid Q = x\big\} \to \mathbb{P}\big\{N < 0\big\} = \tfrac{1}{2}, \qquad \lambda \to \infty,
\end{aligned} \tag{4.23}
$$

and

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu}\big\{B_{2,\lambda} \mid Q = x\big\} \to \mathbb{P}\big\{N > 0\big\} = \tfrac{1}{2}, \\
&\mathbb{P}_{\bar\mu}\big\{B_{3,\lambda} \mid Q = x\big\} \to \mathbb{P}\big\{|N| \leq 0\big\} = 0.
\end{aligned} \tag{4.24}
$$

Similar to (4.20) we have

$$\frac{1}{\lambda}Z'_\lambda\left(\varphi_1+O\big(\tfrac{\log\lambda}{\sqrt{\lambda}}\big)\right)^{1/2}+O\big(\tfrac{\log^3\lambda}{\sqrt{\lambda}}\big)\overset{\mathbb{P}_{\bar\mu,x,y}}{\longrightarrow}0,\qquad\lambda\to\infty.\tag{4.25}$$

Since $\frac{\log\lambda}{\sqrt{\lambda}}<\frac12 M_-$, we have

$$\begin{aligned}&\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda+O\big(\log^2\lambda\big)\ge\sqrt{\lambda}\varphi_2(\lambda)Z_\lambda+\lambda\varphi_3(\lambda)\Big\}\cap B_{2,\lambda}\Big\}\\&\le\mathbb{P}_{\bar\mu,x,y}\Big\{Z''_\lambda\frac{\log\lambda}{\sqrt{\lambda}}+O\big(\tfrac{\log^3\lambda}{\sqrt{\lambda}}\big)>\beta_1 f(x)\left(\sqrt{\beta_1}-\frac{\log\lambda}{\sqrt{\lambda}}\right)\Big\}\\&\le\mathbb{P}_{\bar\mu,x,y}\Big\{Z''_\lambda\frac{\log\lambda}{\sqrt{\lambda}}+O\big(\tfrac{\log^3\lambda}{\sqrt{\lambda}}\big)>\tfrac12 M_-^3\Big\}\to0,\qquad\lambda\to\infty.\end{aligned}\tag{4.26}$$

Moreover,

$$\begin{aligned}&\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda+O\big(\log^2\lambda\big)<\sqrt{\lambda}\varphi_2(\lambda)Z_\lambda+\lambda\varphi_3(\lambda)\Big\}\cap B_{1,\lambda}\Big\}\\&\le\mathbb{P}_{\bar\mu,x,y}\Big\{Z''_\lambda\frac{\log\lambda}{\sqrt{\lambda}}+O\big(\tfrac{\log^3\lambda}{\sqrt{\lambda}}\big)<-\beta_1^{3/2}f^{3/2}(x)\Big\}\to0,\qquad\lambda\to\infty.\end{aligned}$$

Therefore by (4.23),

$$\begin{aligned}&\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda+O\big(\log^2\lambda\big)\ge\sqrt{\lambda}\varphi_2(\lambda)Z_\lambda+\lambda\varphi_3(\lambda)\Big\}\cap A_\lambda\cap B_{1,\lambda}\Big\}\\&=\mathbb{P}_{\bar\mu}\big\{A_\lambda\cap B_{1,\lambda}\mid Q=x\big\}\\&-\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda+O\big(\log^2\lambda\big)<\sqrt{\lambda}\varphi_2(\lambda)Z_\lambda+\lambda\varphi_3(\lambda)\Big\}\cap A_\lambda\cap B_{1,\lambda}\Big\}\to\tfrac12,\qquad\lambda\to\infty.\end{aligned}\tag{4.27}$$

It follows from (4.24)–(4.27) that, for $(x,y)\in B_f$,

$$\mathbb{P}_{\bar\mu,x,y}\Big\{\Big\{Z''_\lambda+O\big(\log^2\lambda\big)\ge\sqrt{\lambda}\varphi_2(\lambda)Z_\lambda+\lambda\varphi_3(\lambda)\Big\}\cap A_\lambda\Big\}\to\tfrac12,\qquad\lambda\to\infty.\tag{4.28}$$

Finally, from (4.19)–(4.22) and (4.28) we get

$$\lim_{\lambda\to\infty}\mathbb{P}_{\bar\mu,x,y}\Big\{\sum_{j=1}^{d_\phi}d_j\ge d_\phi(d_\phi-1),d_\phi\ge1\Big\}=\begin{cases}0,&\text{if }(x,y)\in I_f,\\1,&\text{if }(x,y)\in A_f,\\\frac12,&\text{if }(x,y)\in B_f,\end{cases}$$

and so, by (4.15)–(4.16) and the dominated convergence theorem, we arrive at

$$\lim_{\lambda\to\infty}\mu([0,\infty))=\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in A_f\big\}+\tfrac12\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in B_f\big\}.\tag{4.29}$$

(c) By Fubini's theorem, we have

$$\begin{aligned}
\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in A_f\big\} &= \iint\limits_{A_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(y) \geq \iint\limits_{A_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(x) \\
&= 1-\iint\limits_{I_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(x) - \iint\limits_{B_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(x) \\
&= 1-\iint\limits_{\substack{\{(y,x)\in[0,1]\times[0,1]:\\ f(x)-f(y)>0\}}} \mathrm{d}y\,\mathrm{d}x\,\beta_1^{-1}f(x) - \iint\limits_{B_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(x) \\
&= 1-\iint\limits_{A_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(y) - \iint\limits_{B_f} \mathrm{d}x\,\mathrm{d}y\,\beta_1^{-1}f(x) \\
&= 1-\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in A_f\big\}-\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in B_f\big\}.
\end{aligned}$$

Hence

$$\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in A_f\big\} \geq \tfrac12 - \tfrac12\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in B_f\big\}, \tag{4.30}$$

which together with (4.29) implies that

$$\lim_{\lambda\to\infty}\mu([0,\infty)) \geq \tfrac12.$$

(d) If $\ell_2(A_f)>0$, then $\geq$ in (4.30) can be replaced by $>$. Following the same approach as in the proof of (c), we get $\lim_{\lambda\to\infty}\mu([0,\infty))>\frac12$. But, if $\ell_2(A_f)=0$, then it follows from (4.29) that

$$\lim_{\lambda\to\infty}\mu([0,\infty)) = \tfrac12\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in B_f\big\}.$$

This, together with part (c), leads to the conclusion that $\lim_{\lambda\to\infty}\mu([0,\infty))=\frac12$.

(e) If f is injective, then $B_f=\{(x,y)\in[0,1]\times[0,1]\colon\ x=y\}$ and so

$$\mathbb{P}_{\bar\mu}\big\{(Q,Q')\in B_f\big\} = \iint\limits_{B_f} \mathrm{d}x\mathrm{d}y\ \beta_1^{-1}f(y)=0.$$

Therefore, by (4.29),

$$\lim_{\lambda\to\infty}\mu([0,\infty)) = \mathbb{P}_{\bar\mu}\big\{(Q,Q')\in A_f\big\}.$$

Proof of Theorem 3.5. Let $\nu_0(k)$ denote the probability that the root has k children. Since

$$\begin{aligned}
\mathbb{E}_{\bar\mu}\big[d_\phi\big] &= \int_{[0,1]} \mathrm{d}x\,\mathbb{E}_{\bar\mu}\big[d_\phi \mid Q=x\big] = \int_{[0,1]} \mathrm{d}x\,\lambda\beta_1 f(x) = \lambda\beta_1^2, \\
\mathbb{E}_{\bar\mu}\big[d_1\big] &= \int_{[0,1]} \mathrm{d}x\,\beta_1^{-1}f(x)\mathbb{E}_{\bar\mu}\big[d_1 \mid Q'=x\big] = \int_{[0,1]} \mathrm{d}x\,\lambda f^2(x) = \lambda\beta_2,
\end{aligned}$$

similarly as in the proof of Theorem 3.2 we obtain

$$\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] = \sum_{k\in\mathbb{N}} \nu_0(k)\,\mathbb{E}_{\bar\mu}\Big[\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big] = \mathbb{E}_{\bar\mu}\big[d_1\big]\big(1-\nu_0(0)\big) + 1 - \nu_0(0) - \mathbb{E}_{\bar\mu}\big[d_\phi\big]$$
$$= 1 - (\lambda\beta_2 + 1)\int_{[0,1]} \mathrm{d}x\, \mathrm{e}^{-\lambda\beta_1 f(x)} + \lambda(\beta_2 - \beta_1^2).$$

Since

$$1 - (\lambda M_+^2 + 1)\,\mathrm{e}^{-\lambda M_-^2} + \lambda(M_-^2 - M_+^2) \le \mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] \le 1 - (\lambda M_-^2 + 1)\,\mathrm{e}^{-\lambda M_+^2} + \lambda(M_+^2 - M_-^2),$$

it follows that $\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] \to 0$ as $\lambda \downarrow 0$. On the other hand, if U is a uniform $[0,1]$ random variable, then

$$\beta_2 - \beta_1^2 = \mathbb{V}ar(f(U)) > 0$$

by noting that f is a non-constant function. Since

$$\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] \ge 1 - (\lambda M_+^2 + 1)\mathrm{e}^{-\lambda M_-^2} + \lambda(\beta_2 - \beta_1^2),$$

it follows that $\mathbb{E}_{\bar\mu}[\Delta_\phi] \to \infty$ as $\lambda \to \infty$. Similarly, since

$$\mathbb{E}_{\bar\mu}\big[d_\phi^2\big] = \int_{[0,1]} \mathrm{d}x\, \mathbb{E}_{\bar\mu}\big[d_\phi^2 \mid Q = x\big] = \int_{[0,1]} \mathrm{d}x\, \big(\lambda\beta_1 f(x) + \lambda^2\beta_1^2 f^2(x)\big) = \lambda\beta_1^2 + \lambda^2\beta_1^2\beta_2,$$
$$\mathbb{E}_{\bar\mu}\big[d_1^2\big] = \int_{[0,1]} \mathrm{d}x\, \beta_1^{-1} f(x)\,\mathbb{E}_{\bar\mu}\big[d_1^2 \mid Q' = x\big] = \int_{[0,1]} \mathrm{d}x\, \big(\lambda f^2(x) + \lambda^2\beta_1 f^3(x)\big) = \lambda\beta_2 + \lambda^2\beta_1\beta_3,$$

it follows that

$$\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] = \sum_{k\in\mathbb{N}} \nu_0(k)\,\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2 \mid d_\phi = k\big] = \sum_{k\in\mathbb{N}} \nu_0(k)\,\mathbb{E}_{\bar\mu}\Big[\Big(\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big)^2\Big]$$
$$= \sum_{k\in\mathbb{N}} \nu_0(k)\,\mathbb{E}_{\bar\mu}\Big[\frac{1}{k^2}\sum_{j=1}^k d_j^2 + \frac{1}{k^2}\sum_{i=1}^k\sum_{\substack{j=1\\ j\ne i}}^k d_i d_j + 1 + k^2 + \frac{2}{k}\sum_{j=1}^k d_j - 2\sum_{j=1}^k d_j - 2k\Big]$$
$$= \mathbb{V}ar_{\bar\mu}\big(d_1\big)\,\mathbb{E}_{\bar\mu}\Big[d_\phi^{-1}\mathbb{1}_{\{d_\phi\ne 0\}}\Big] + \big(\mathbb{E}_{\bar\mu}\big[d_1\big]\big)^2\big(1-\nu_0(0)\big) + 1 - \nu_0(0)$$
$$+ \mathbb{E}_{\bar\mu}\big[d_\phi^2\big] + 2\mathbb{E}_{\bar\mu}\big[d_1\big]\big(1-\nu_0(0)\big) - 2\mathbb{E}_{\bar\mu}\big[d_1\big]\,\mathbb{E}_{\bar\mu}\big[d_\phi\big] - 2\mathbb{E}_{\bar\mu}\big[d_\phi\big]$$
$$= (\lambda\beta_2 + \lambda^2\beta_1\beta_3 - \lambda^2\beta_2^2)\,\mathbb{E}_{\bar\mu}\big[d_\phi^{-1}\mathbb{1}_{\{d_\phi\ne 0\}}\big] - (1 + \lambda^2\beta_2^2 + 2\lambda\beta_2)\int_0^1 \mathrm{d}x\, \mathrm{e}^{-\lambda\beta_1 f(x)}$$
$$+ \lambda^2(\beta_2^2 - \beta_1^2\beta_2) + \lambda(2\beta_2 - \beta_1^2) + 1$$
$$= \lambda\big(\beta_2 + \lambda(\beta_1\beta_3 - \beta_2^2)\big)\int_{[0,1]} \mathrm{d}x \sum_{k\in\mathbb{N}} k^{-1}\frac{\mathrm{e}^{-\lambda\beta_1 f(x)}(\lambda\beta_1 f(x))^k}{k!}$$
$$- (1 + \lambda^2\beta_2^2 + 2\lambda\beta_2)\int_{[0,1]} \mathrm{d}x\, \mathrm{e}^{-\lambda\beta_1 f(x)} + \lambda^2(\beta_2^2 - \beta_1^2\beta_2) + \lambda(2\beta_2 - \beta_1^2) + 1.$$

Again, since

$$-(1 + \lambda^2 M_+^4 + 2\lambda M_+^2)\,\mathrm{e}^{-\lambda M_-^2} + \lambda^2(M_-^4 - M_+^4) + \lambda(2M_-^2 - M_+^2) + 1$$
$$\le \mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] \le (\lambda M_+^2 + \lambda^2 M_+^4) - \mathrm{e}^{-\lambda M_+^2} + \lambda^2 M_+^4 + 2\lambda M_+^2 + 1,$$

it follows that $\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] \to 0$ as $\lambda \downarrow 0$. In addition, $\lim_{\lambda\to\infty}\mathbb{E}_{\bar\mu}[\Delta_\phi^2] \ge \lim_{\lambda\to\infty}(\mathbb{E}_{\bar\mu}[\Delta_\phi])^2 = \infty$.

Proof of Theorem 3.6. Fix $\lambda \in (0,\infty)$. First, we show that for each $k \in \mathbb{N}$,

$$\mathbb{P}_{\bar{\mu}}\Big\{\sum_{j=1}^{k} d_j \geq kx + k(k-1)\Big\} \sim \mathbb{P}_{\bar{\mu}}\Big\{\sum_{j=1}^{k} d_j = kx + k(k-1)\Big\}, \qquad x \to \infty. \tag{4.31}$$

Indeed, without loss of generality we may assume that $x \in \mathbb{N}$. With the same notations and methods used in the proof of Theorem 3.4, we have, for $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbb{P}_{\bar{\mu}}\Big\{\sum_{j=1}^{k} d_j = kx + k(k-1) + m\Big\} \\
&= \beta_1^{-k} \prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, f(y_j)\, \frac{\mathrm{e}^{-\lambda\beta_1 \sum_{j=1}^{k} f(y_j)} (\lambda\beta_1 \sum_{j=1}^{k} f(y_j))^{kx+k(k-1)+m}}{(kx+k(k-1)+m)!}.
\end{aligned}$$

Therefore, for $k, l \in \mathbb{N}$,

$$\begin{aligned}
&\frac{\mathbb{P}_{\bar{\mu}}\{\sum_{j=1}^{k} d_j = kx + k(k-1) + l\}}{\mathbb{P}_{\bar{\mu}}\{\sum_{j=1}^{k} d_j = kx + k(k-1)\}} \\
&\leq \frac{(\lambda\beta_1)^l \mathrm{e}^{-\lambda\beta_1 k(M_- - M_+)}}{(kx+k(k-1)+1) \times \cdots \times (kx+k(k-1)+l)} \\
&\qquad \times \frac{\prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, f(y_j) (\sum_{j=1}^{k} f(y_j))^{kx+k(k-1)+l}}{\prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, f(y_j) (\sum_{j=1}^{k} f(y_j))^{kx+k(k-1)}} \\
&\leq \frac{(\lambda\beta_1 k M_+)^l \mathrm{e}^{-\lambda\beta_1 k(M_- - M_+)}}{(kx+k(k-1)+1) \times \cdots \times (kx+k(k-1)+l)} \to 0, \qquad x \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{l \in \mathbb{N}} \frac{\mathbb{P}_{\bar{\mu}}\{\sum_{j=1}^{k} d_j = kx + k(k-1) + l\}}{\mathbb{P}_{\bar{\mu}}\{\sum_{j=1}^{k} d_j = kx + k(k-1)\}} \\
&\leq \mathrm{e}^{-\lambda\beta_1 k(M_- - M_+)} \sum_{l \in \mathbb{N}} \frac{(\lambda\beta_1 k M_+)^l}{l!} \leq \mathrm{e}^{-\lambda\beta_1 k(M_- - 2M_+)} < \infty.
\end{aligned}$$

Therefore, (4.31) follows by the dominated convergence theorem.

For $k \in \mathbb{N} \setminus \{1\}$, since

$$\begin{aligned}
&\prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, f(y_j) \Big(\sum_{j=1}^{k} f(y_j)\Big)^{kx} \leq k^{kx} \prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, f(y_j) \sum_{j=1}^{k} f^{kx}(y_j) \\
&\leq k^{kx} M_+^{kx-x+k-1} \prod_{j=1}^{k} \int_{[0,1]} \mathrm{d}y_j\, \sum_{j=1}^{k} f^{x+1}(y_j) = k^{kx+1} M_+^{kx-x+k-1} \beta_{x+1},
\end{aligned}$$

by Stirling's formula we have

$$\begin{aligned}
&\frac{\mathbb{P}_{\bar\mu}\big\{\sum_{j=1}^k d_j = kx + k(k-1)\big\}}{\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}} \\
&\leq \frac{x!}{(kx)!} \times \frac{\mathrm{e}^{-\lambda\beta_1(kM_- - M_+)}\beta_1^{-k+1}(\lambda\beta_1 kM_+)^{k(k-1)}(\lambda\beta_1)^{kx-x}}{(kx+1)\times\cdots\times(kx+k(k-1))} \\
&\qquad \times \frac{\prod_{j=1}^k \int_{[0,1]}\mathrm{d}y_j\, f(y_j)(\sum_{j=1}^k f(y_j))^{kx}}{\int_{[0,1]}\mathrm{d}y_1\,(f(y_1))^{x+1}} \\
&\leq \frac{x!}{(kx)!} \times \frac{\mathrm{e}^{-\lambda\beta_1(kM_- - M_+)}\beta_1^{-k+1}(\lambda\beta_1 kM_+)^{k(k-1)}(\lambda\beta_1)^{kx-x}}{(kx+1)\times\cdots\times(kx+k(k-1))} \times k^{kx+1}M_+^{kx-x+k-1} \\
&\sim \left(\frac{x}{\mathrm{e}M_+}\right)^{-x(k-1)} \times \frac{\mathrm{e}^{-\lambda\beta_1(kM_- - M_+)}\beta_1^{-k+1}(\lambda\beta_1 kM_+)^{k(k-1)}(\lambda\beta_1)^{kx-x}}{(kx+1)\times\cdots\times(kx+k(k-1))} \times \sqrt{k}M_+^{k-1} \\
&\to 0, \qquad x\to\infty.
\end{aligned} \tag{4.32}$$

Therefore, taking the same approach as in (4.14), we get from (4.31)–(4.32) and the dominated convergence theorem that

$$\mathbb{P}_{\bar\mu}\{\Delta_\phi \geq x\} \sim \mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\}, \qquad x\to\infty.$$

Hence, from (4.31),

$$\mathbb{P}_{\bar\mu}\big\{\Delta_\phi \geq x\big\} \sim \mathbb{P}_{\bar\mu}\big\{d_\phi = 1\big\}\mathbb{P}_{\bar\mu}\big\{d_1 = x\big\}, \qquad x\to\infty,$$

and so Stirling's formula yields

$$\left(\frac{\lambda M_-^2\mathrm{e}^{-2\lambda\beta_1 M_+}}{\sqrt{2\pi}}\right)\frac{1}{\sqrt{x}}\beta_x\mathrm{e}^{-x\log(\frac{x}{\lambda\beta_1\mathrm{e}})} \lesssim \mathbb{P}_{\bar\mu}\big\{\Delta_\phi \geq x\big\} \lesssim \left(\frac{\lambda M_+^2\mathrm{e}^{-2\lambda\beta_1 M_-}}{\sqrt{2\pi}}\right)\frac{1}{\sqrt{x}}\beta_x\mathrm{e}^{-x\log(\frac{x}{\lambda\beta_1\mathrm{e}})},$$
$$x\to\infty,$$

where we recall that $\beta_x = \int_{[0,1]}\mathrm{d}y\, f^x(y)$.

(a) Suppose that $\ell_1(\{y\in[0,1]\colon\ f(y) = M_+\}) > 0$ with ℓ_1 the Lebesgue measure. Then

$$\ell_1\big(\{y\in[0,1]\colon\ f(y) = M_+\}\big)M_+^x \leq \beta_x \leq M_+^x,$$

and hence $\beta_x \asymp M_+^x$ as $x\to\infty$.

(b) Suppose that $f(y_\star) = M_+$ and $f(y_\star) - f(y) \asymp |y_\star - y|^\alpha$, $y\to y_\star$, for some $y_\star\in[0,1]$ and $\alpha\in(0,\infty)$, and f is bounded away from M_+ outside any neighbourhood of $y_\star$. Then

$$\begin{aligned}
\frac{\beta_x}{M_+^x} &= \int_{[0,1]}\mathrm{d}y\,\left(\frac{f(y)}{f(y_\star)}\right)^x = \int_{[0,1]}\mathrm{d}y\,\left(1 - \frac{f(y_\star)-f(y)}{f(y_\star)}\right)^x \\
&\asymp \int_{[0,1]}\mathrm{d}y\,\exp\left(-x\,\frac{|y_\star - y|^\alpha}{M_+}\right) \asymp \int_{y_\star - x^{-1/\alpha}}^{y_\star + x^{-1/\alpha}}\mathrm{d}y \asymp x^{-1/\alpha}, \qquad x\to\infty,
\end{aligned}$$

where the first $\asymp$ uses that only the neighbourhood of $y_\star$ contributes, and the second $\asymp$ uses that the integral is dominated by the neighbourhood where the exponent is of order 1. Hence $\beta_x \asymp x^{-1/\alpha}\, M_+^x$ as $x\to\infty$.

4.2.3 Configuration model

Proof of Theorem 3.7. Write

$$\mu([0,\infty)) = \sum_{k\in\mathbb{N}} p_k \nu_1^{\circledast k}\big([k(k-1),\infty)\big),$$

where $\nu_1^{\circledast k}$ is the k-fold convolution of $\nu_1 = p^\star$. We look at the special case where $p_k = k^{-\tau}/\zeta(\tau)$ with $\tau \in (2,\infty)$ and ζ the Riemann function. Then,

$$\mu([0,\infty)) \geq p_1 = \frac{1}{\zeta(\tau)} \geq \frac{1}{\zeta(2)} = \frac{6}{\pi^2} > \frac{1}{2}.$$

For $m \in \mathbb{N}$ and $s \in (1,\infty)$, let $\zeta_m(s) = \sum_{j=m}^{\infty} j^{-s}$ be the truncated Riemann function. Then[2]

$$\frac{m^{-s+1}}{s-1} = \int_m^\infty \mathrm{d}x\, x^{-s} \leq \zeta_m(s) \leq 1 + \int_m^\infty \mathrm{d}x\, x^{-s} = \frac{s-1+m^{-s+1}}{s-1}. \tag{4.33}$$

For each $k \in \mathbb{N}$,

$$\nu_1^{\circledast k}\big([k(k-1),\infty)\big) \geq \mathbb{P}_{\bar\mu}\big\{d_1 \geq k(k-1)\big\} = \frac{\zeta_{k(k-1)+1}(\tau-1)}{\zeta_1(\tau-1)} \geq \frac{\zeta_{k^2}(\tau-1)}{\zeta_1(\tau-1)} \geq \frac{k^{-2\tau+4}}{\tau-1}.$$

Consequently,

$$\mu([0,\infty)) \geq \frac{1}{\tau-1}\sum_{k\in\mathbb{N}} p_k\, k^{-2\tau+4} \geq \frac{1}{\tau-1}\frac{\zeta_1(3\tau-4)}{\zeta_1(\tau)}.$$

By the dominated convergence theorem, $\zeta_1(3\tau-4) \to \zeta_1(2)$ as $\tau \downarrow 2$, from which we obtain that $\mu([0,\infty)) \to 1$ as $\tau \downarrow 2$, and

$$\mu([0,\infty)) \geq \frac{1}{\zeta(\tau)} \to 1, \qquad \tau \to \infty,$$

from which we obtain that $\mu([0,\infty)) \to 1$ as $\tau \to \infty$.

Proof of Theorem 3.8. Let $\mathbb{E}[D] < \infty$. Since

$$\mathbb{E}_{\bar\mu}[d_\phi] = \mathbb{E}[D], \qquad \mathbb{E}_{\bar\mu}[d_1] = (\mathbb{E}[D])^{-1}(\mathbb{E}[D^2] - \mathbb{E}[D]),$$

the first moment equals

$$\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] = \sum_{k\in\mathbb{N}} p_k \left(\frac{1}{k}\sum_{j=1}^{k}\mathbb{E}_{\bar\mu}[d_j] + 1 - k\right) = \mathbb{E}_{\bar\mu}[d_1] + 1 - \mathbb{E}_{\bar\mu}[d_\phi] = \frac{\mathbb{V}ar(D)}{\mathbb{E}[D]},$$

which is strictly positive whenever D is a non-degenerate random variable, i.e., the limit is not a regular tree. Moreover, $\mathbb{E}_{\bar\mu}[d_\phi^{-1}] = \mathbb{E}[D^{-1}]$, $\mathbb{E}_{\bar\mu}[d_\phi^2] = \mathbb{E}[D^2]$, and if also $\mathbb{E}[D^2] < \infty$, then

$$\mathbb{E}_{\bar\mu}\big[d_1^2\big] = \frac{\sum_{k\in\mathbb{N}}(k-1)^2 k p_k}{\mathbb{E}[D]} = \frac{\mathbb{E}[D^3] + \mathbb{E}[D] - 2\mathbb{E}[D^2]}{\mathbb{E}[D]}.$$

[2]It is worthwhile to note here that $\zeta_1(s) > 1$, and for $m > 1$, there is a sharper upper bound which will be presented in (4.34).

Therefore, if $\mathbb{E}[D^2] < \infty$, then

$$
\begin{aligned}
&\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] \\
&= \sum_{k\in\mathbb{N}} p_k \mathbb{E}_{\bar\mu}\big[\Delta_\phi^2 \mid d_\phi = k\big] = \sum_{k\in\mathbb{N}} p_k \mathbb{E}_{\bar\mu}\Big[\Big(\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big)^2\Big] \\
&= \sum_{k\in\mathbb{N}} p_k \mathbb{E}_{\bar\mu}\Big[\frac{1}{k^2}\sum_{j=1}^k d_j^2 + \frac{1}{k^2}\sum_{i=1}^k \sum_{\substack{j=1\\ j\neq i}}^k d_i d_j + 1 + k^2 + \frac{2}{k}\sum_{j=1}^k d_j - 2\sum_{j=1}^k d_j - 2k\Big] \\
&= \mathbb{V}ar_{\bar\mu}\big(d_1\big)\mathbb{E}_{\bar\mu}\big[d_\phi^{-1}\big] + \big(\mathbb{E}_{\bar\mu}\big[d_1\big]\big)^2 + 1 + \mathbb{E}_{\bar\mu}\big[d_\phi^2\big] + 2\mathbb{E}_{\bar\mu}\big[d_1\big] - 2\mathbb{E}_{\bar\mu}\big[d_1\big]\mathbb{E}_{\bar\mu}\big[d_\phi\big] - 2\mathbb{E}_{\bar\mu}\big[d_\phi\big] \\
&= \frac{\big(\mathbb{E}[D^3]\mathbb{E}[D] - (\mathbb{E}[D^2])^2\big)\,\mathbb{E}[D^{-1}] + \mathbb{E}[D^2]\mathbb{V}ar(D)}{(\mathbb{E}[D])^2},
\end{aligned}
$$

and so using the Cauchy-Schwarz inequality,

$$
\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] \geq \frac{\mathbb{E}[D^2]\mathbb{V}ar(D)}{(\mathbb{E}[D])^2},
$$

which implies that

$$
\mathbb{V}ar_{\bar\mu}(\Delta_\phi) \geq \mathbb{V}ar(D)\frac{\mathbb{E}[D^2] - \mathbb{V}ar(D)}{(\mathbb{E}[D])^2} = \mathbb{V}ar(D).
$$

Proof of Theorem 3.9. Let $x \in (0,\infty)$ and $\tau \in (2,\infty)$. For $m \in \mathbb{N}$ and $s \in (1,\infty)$,

$$
\zeta_m(s) = \sum_{k=m}^\infty k^{-s} \leq \int_{m-1}^\infty \mathrm{d}x\, x^{-s} = \frac{(m-1)^{-s+1}}{s-1}. \tag{4.34}
$$

Therefore

$$
\mathbb{P}_{\bar\mu}\{d_1 \geq x\} = \frac{1}{\zeta(\tau)\,\mathbb{E}[D]}\sum_{k=\lceil x\rceil}^\infty (k+1)^{-\tau+1} \leq \frac{1}{\mathbb{E}[D]}\sum_{k=\lceil x\rceil+1}^\infty k^{-\tau+1} \leq \frac{1}{\mathbb{E}[D]}\,\frac{x^{-\tau+2}}{\tau-2}
$$

and

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu}\{\Delta_\phi \geq x\} = \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^k d_j \geq kx + k(k-1)\Big\} \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^k d_j \geq kx\Big\} \leq \sum_{k\in\mathbb{N}} k\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq x\big\} \leq (\tau-2)^{-1}x^{-\tau+2}.
\end{aligned}
$$

Moreover, by (4.33),

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu}\{\Delta_\phi \geq x\} \geq \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\big\{d_1 \geq k^2(x+1)\big\} \\
&= \frac{1}{\zeta(\tau)\zeta(\tau-1)}\sum_{k\in\mathbb{N}} k^{-\tau}\zeta_{\lceil k^2(x+1)\rceil+1}(\tau-1) \\
&\geq \frac{1}{(\tau-2)\zeta(\tau)\zeta(\tau-1)}\sum_{k\in\mathbb{N}} k^{-\tau}\big(k^2(x+1)+2\big)^{-\tau+2} \\
&\geq \frac{3^{-\tau+2}\zeta(3\tau-4)}{(\tau-2)\zeta(\tau)\zeta(\tau-1)}(x+1)^{-\tau+2} \geq \frac{3^{-\tau+2}}{\tau(3\tau-5)}(x+1)^{-\tau+2} \geq 3^{-\tau+1}\tau^{-2}(x+1)^{-\tau+2}.
\end{aligned}
$$

4.2.4 Preferential attachment model

Proof of Theorem 3.12. First note that

$$d_\phi = N^{(\mathsf{old})}(\phi) + N^{(\mathsf{young})}(\phi) = 1 + N^{(\mathsf{young})}(\phi).$$

Let d_i denote the number of children of $\phi_i = \phi i$. We have $d_1 = 1 + N^{(\mathsf{young})}(\phi_1)$ and, when $N^{(\mathsf{young})}(\phi) \geq 1$,

$$d_i = N^{(\mathsf{young})}(\phi_i), \qquad i \in \{2, \dots, d_\phi\}.$$

For $a \in [0,1]$, define

$$\kappa(a) = \int_{[a,1]} \mathrm{d}x \, \frac{1}{2+\delta} \frac{x^{-(1+\delta)/(2+\delta)}}{a^{1/(2+\delta)}} = a^{-\frac{1}{2+\delta}} - 1.$$

For $b \in (0,\infty)$, let $Y(a,b) \stackrel{d}{=} \mathrm{Poisson}(b\kappa(a))$. Then

$$\mu([0,\infty)) = \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar{\mu}}\{d_\phi = k\} \mathbb{P}_{\bar{\mu}} \left\{ \sum_{j=1}^{k} d_j \geq k(k-1) \ \mid \ d_\phi = k \right\}.$$

Recall that $\Gamma_\phi \stackrel{d}{=} \mathrm{Gamma}(1+\delta, 1)$, hence

$$\begin{aligned}
\mathbb{P}_{\bar{\mu}}\{d_\phi = k \mid A_\phi = a\} &= \int_0^\infty \mathrm{d}b \, \mathbb{P}_{\bar{\mu}}\big\{N^{(\mathsf{young})}(\phi) = k-1 \mid A_\phi = a, \Gamma_\phi = b\big\} \frac{b^\delta \mathrm{e}^{-b}}{\Gamma(1+\delta)} \\
&= \int_0^\infty \mathrm{d}b \, \mathbb{P}\big\{Y(a,b) = k-1\big\} \frac{b^\delta \mathrm{e}^{-b}}{\Gamma(1+\delta)} \\
&= \frac{\Gamma(k+\delta)}{(k-1)!\,\Gamma(1+\delta)} \left(1 - a^{\frac{1}{2+\delta}}\right)^{k-1} a^{\frac{1+\delta}{2+\delta}}, \qquad (4.35)
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{P}_{\bar{\mu}}\{d_\phi = k\} &= \frac{\Gamma(k+\delta)}{(k-1)!\,\Gamma(1+\delta)} \int_{[0,1]} \mathrm{d}a \left(1 - a^{\frac{1}{2+\delta}}\right)^{k-1} a^{\frac{1+\delta}{2+\delta}} \\
&= \frac{(2+\delta)\Gamma(k+\delta)}{(k-1)!\,\Gamma(1+\delta)} \int_{[0,1]} \mathrm{d}u \, (1-u)^{k-1} u^{2+2\delta} \\
&= \frac{(2+\delta)\Gamma(3+2\delta)\Gamma(k+\delta)}{\Gamma(1+\delta)\Gamma(k+3+2\delta)}. \qquad (4.36)
\end{aligned}$$

Hence

$$\mu([0,\infty)) \geq \mathbb{P}_{\bar{\mu}}\{d_\phi = 1\} = \frac{2+\delta}{3+2\delta}, \qquad (4.37)$$

and so $\mu([0,\infty)) \to 1$ as $\delta \downarrow -1$. Moreover, since $\delta \mapsto \frac{2+\delta}{3+2\delta}$ is decreasing and tends to $\frac{1}{2}$ as $\delta \to \infty$, it follows from (4.37) that, for all $\delta \in (-1,\infty)$,

$$\mu([0,\infty)) > \frac{1}{2}.$$

Proof of Theorem 3.13. Let $\kappa(\cdot)$ and $Y(\cdot,\cdot)$ be as in the proof of Theorem 3.12. Then $N^{(\mathsf{young})}(\phi)$ is distributed as $Y(A_\phi,\Gamma_\phi)$, and hence

$$\mathbb{E}_{\bar\mu}\big[d_\phi\big] = 1+\mathbb{E}_{\bar\mu}\big[N^{(\mathsf{young})}(\phi)\big] = 1+\mathbb{E}_{\bar\mu}[\Gamma_\phi]\mathbb{E}_{\bar\mu}[\kappa(A_\phi)] = 2. \tag{4.38}$$

Abbreviate

$$\gamma^\dagger_{k,\delta} = \frac{\Gamma(k+3+2\delta)}{(k-1)!\Gamma(3+2\delta)}.$$

For $k\in\mathbb{N}$, using (4.35)–(4.36), we get

$$\begin{aligned}\mathbb{E}_{\bar\mu}\big[A_\phi^{-\frac{1}{2+\delta}} \mid d_\phi = k\big] &= \int_{[0,1]} \mathrm{d}a\, \frac{\mathbb{P}_{\bar\mu}\{d_\phi = k \mid A_\phi = a\}}{\mathbb{P}_{\bar\mu}\{d_\phi=k\}} a^{-\frac{1}{2+\delta}} \\ &= \frac{\gamma^\dagger_{k,\delta}}{2+\delta}\int_{[0,1]} \mathrm{d}a\,(1-a^{\frac{1}{2+\delta}})^{k-1} a^{\frac{\delta}{2+\delta}} \\ &= \gamma^\dagger_{k,\delta}\int_{[0,1]}\mathrm{d}u\,(1-u)^{k-1}u^{1+2\delta} = \frac{k+2+2\delta}{2+2\delta}.\end{aligned}$$

Hence, considering ϕ_1, we have

$$\mathbb{E}_{\bar\mu}\big[\kappa(A_{\phi_1})\mid d_\phi = k\big] = \mathbb{E}_{\bar\mu}\big[U_{\phi_1}^{-\frac{1}{1+\delta}}\big]\mathbb{E}_{\bar\mu}\big[A_\phi^{-\frac{1}{2+\delta}}\mid d_\phi = k\big]-1 = \begin{cases}\frac{k+2}{2\delta}, & \text{if } \delta>0,\\ \infty, & \text{if } \delta\le 0.\end{cases}$$

Thus, observing that $N^{(\mathsf{young})}(\phi_1)$ is distributed as $Y(A_{\phi_1},\Gamma_{\phi_1})$, we have

$$\begin{aligned}\mathbb{E}_{\bar\mu}\big[d_1\mid d_\phi = k\big] &= 1+\mathbb{E}_{\bar\mu}\big[N^{(\mathsf{young})}(\phi_1)\mid d_\phi=k\big] \\ &= 1+\mathbb{E}_{\bar\mu}[\Gamma_{\phi_1}]\,\mathbb{E}_{\bar\mu}\big[\kappa(A_{\phi_1})\mid d_\phi = k\big] = \begin{cases} 1+\frac{2+\delta}{2\delta}(k+2), & \text{if } \delta>0,\\ \infty, & \text{if } \delta\le 0.\end{cases}\end{aligned} \tag{4.39}$$

Next, let $k\ge 2$. For $2\le j\le k$, $d_j = N^{(\mathsf{young})}(\phi_j)$ and is distributed as $Y(A_{\phi_j},\Gamma_{\phi_j})$. Hence

$$\begin{aligned}\mathbb{E}_{\bar\mu}\Bigg[\sum_{j=2}^k d_j \mid d_\phi = k\Bigg] &= \mathbb{E}_{\bar\mu}\Bigg[\sum_{j=2}^k N^{(\mathsf{young})}(\phi_j)\mid d_\phi = k\Bigg] \\ &= \sum_{j=2}^k \mathbb{E}_{\bar\mu}[\Gamma_{\phi_j}]\,\mathbb{E}_{\bar\mu}\big[\kappa(A_{\phi_j})\mid d_\phi = k\big] \\ &= (1+\delta)\mathbb{E}_{\bar\mu}\Bigg[\sum_{j=2}^k \kappa(A_{\phi_j})\mid d_\phi = k\Bigg] \\ &= (1+\delta)\Bigg(\mathbb{E}_{\bar\mu}\Bigg[\sum_{j=2}^k A_{\phi_j}^{-\frac{1}{2+\delta}}\mid d_\phi = k\Bigg]-k+1\Bigg).\end{aligned} \tag{4.40}$$

There exists $k-1$ non-ordered i.i.d. uniform $[A_\phi,1]$ random variables $\tilde U_1,\dots,\tilde U_{k-1}$ such that the random variable $\sum_{j=2}^k A_{\phi_j}^{-\frac{1}{2+\delta}}$ given the event $\{d_\phi = k\}$ is distributed as $\sum_{j=1}^{k-1}\tilde U_j^{-\frac{1}{2+\delta}} \mid d_\phi = k$. Therefore, using (4.40), we can write

$$\begin{aligned}\mathbb{E}_{\bar\mu}\Bigg[\sum_{j=2}^k d_j\mid d_\phi = k\Bigg] &= (1+\delta)\Bigg(\mathbb{E}_{\bar\mu}\Bigg[\sum_{j=1}^{k-1}\tilde U_j^{-\frac{1}{2+\delta}}\mid d_\phi = k\Bigg]-k+1\Bigg) \\ &= (1+\delta)(k-1)\Bigg(\mathbb{E}_{\bar\mu}\big[\tilde U_1^{-\frac{1}{2+\delta}}\mid d_\phi = k\big]-1\Bigg).\end{aligned} \tag{4.41}$$

Using (4.35) and (4.36), we compute

$$
\begin{aligned}
&\mathbb{E}_{\bar\mu}\big[\tilde U_1^{-\frac{1}{2+\delta}} \mid d_\phi = k\big] \\
&= \int_{[0,1]} \mathrm{d}u \int_{[0,u]} \mathrm{d}a\, \frac{\mathbb{P}_{\bar\mu}\{d_\phi = k \mid A_\phi = a\}}{\mathbb{P}_{\bar\mu}\{d_\phi = k\}} (1-a)^{-1} u^{-\frac{1}{2+\delta}} \\
&= \frac{\gamma^\dagger_{k,\delta}}{2+\delta} \int_{[0,1]} \mathrm{d}a\,(1-a^{\frac{1}{2+\delta}})^{k-1} a^{\frac{1+\delta}{2+\delta}} (1-a)^{-1} \int_{[a,1]} \mathrm{d}u\, u^{-\frac{1}{2+\delta}} \\
&= \frac{\gamma^\dagger_{k,\delta}}{1+\delta} \int_{[0,1]} \mathrm{d}a\,(1-a^{\frac{1}{2+\delta}})^{k-1} a^{\frac{1+\delta}{2+\delta}} (1-a)^{-1} (1-a^{\frac{1+\delta}{2+\delta}}) \\
&\le \frac{\gamma^\dagger_{k,\delta}}{1+\delta} \int_{[0,1]} \mathrm{d}a\,(1-a^{\frac{1}{2+\delta}})^{k-1} a^{\frac{1+\delta}{2+\delta}} \\
&= \frac{2+\delta}{1+\delta}\, \gamma^\dagger_{k,\delta} \int_{[0,1]} \mathrm{d}u\,(1-u)^{k-1} u^{2+2\delta} = \frac{2+\delta}{1+\delta}.
\end{aligned}
$$

Therefore, from (4.41) we get

$$
0 \le \mathbb{E}_{\bar\mu}\left[\sum_{j=2}^{k} d_j \mid d_\phi = k\right] \le k-1. \tag{4.42}
$$

Observe that

$$
\mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] = \sum_{k=1}^{\infty} \mathbb{P}_{\bar\mu}\{d_\phi = k\} \bigg(\frac{1}{k} \sum_{j=1}^{k} \mathbb{E}_{\bar\mu}\big[d_j \mid d_\phi = k\big] \bigg) - 1.
$$

Hence from (4.38), (4.39) and (4.42) we get with $\mathbb{E}_{\bar\mu}[\Delta_\phi] = \infty$ for $\delta \le 0$, and

$$
\frac{2+\delta}{\delta}\Big(\frac{1}{2} + \mathbb{E}_{\bar\mu}[d_\phi^{-1}]\Big) - \big(1 - \mathbb{E}_{\bar\mu}[d_\phi^{-1}]\big) \le \mathbb{E}_{\bar\mu}\big[\Delta_\phi\big] \le \frac{2+\delta}{\delta}\Big(\frac{1}{2} + \mathbb{E}_{\bar\mu}[d_\phi^{-1}]\Big), \qquad \delta > 0,
$$

where, by (4.36),

$$
\mathbb{E}_{\bar\mu}[d_\phi^{-1}] = \sum_{k\in\mathbb{N}} \frac{(2+\delta)\Gamma(3+2\delta)\Gamma(k+\delta)}{k\Gamma(1+\delta)\Gamma(k+3+2\delta)}.
$$

This completes the proof for the first moment. Now we derive the second moment. For $\delta \le 0$, we have $\mathbb{E}_{\bar\mu}[\Delta_\phi^2] = \infty$ because $\mathbb{E}_{\bar\mu}[\Delta_\phi] = \infty$. Let us therefore consider $\delta > 0$. First we observe that

$$
\mathbb{E}_{\bar\mu}\big[d_\phi^2\big] = \mathbb{E}_{\bar\mu}\big[\big(1 + N^{(\mathsf{young})}(\phi)\big)^2\big] = 1 + \mathbb{E}_{\bar\mu}[\Gamma_\phi^2]\mathbb{E}_{\bar\mu}[\kappa^2(A_\phi)] + 3\mathbb{E}_{\bar\mu}[\Gamma_\phi]\mathbb{E}_{\bar\mu}[\kappa(A_\phi)] = \tfrac{4}{\delta} + 6.
$$

Also, since

$$
\mathbb{E}_{\bar\mu}\Big[U_{\phi_1}^{-\frac{2}{1+\delta}}\Big] = \begin{cases} \frac{1+\delta}{-1+\delta}, & \text{if } \delta > 1, \\ \infty, & \text{if } 0 < \delta \le 1, \end{cases}
$$

for $k \in \mathbb{N}$ we have

$$
\begin{aligned}
\mathbb{E}_{\bar\mu}\big[\kappa^2(A_{\phi_1}) \mid d_\phi = k\big] &= \mathbb{E}_{\bar\mu}\big[U_{\phi_1}^{-\frac{2}{1+\delta}}\big]\ E_{\bar\mu}\big[A_\phi^{-\frac{2}{2+\delta}} \mid d_\phi = k\big] \\
&\quad - 2\mathbb{E}_{\bar\mu}\big[U_{\phi_1}^{-\frac{1}{1+\delta}}\big]\mathbb{E}_{\bar\mu}\big[A_\phi^{-\frac{1}{2+\delta}} \mid d_\phi = k\big] + 1 \\
&\quad \begin{cases} < \infty, & \text{if } \delta > 1, \\ = \infty, & \text{if } 0 < \delta \le 1, \end{cases}
\end{aligned}
$$

which implies that

$$\mathbb{E}_{\bar\mu}\big[d_1^2 \mid d_\phi = k\big] \begin{cases} < \infty, & \text{if } \delta > 1, \\ = \infty, & \text{if } 0 < \delta \le 1. \end{cases}$$

Moreover, if $k \ge 2$, then from (4.40) and (4.42) we get that, for every $2 \le j \le k$,

$$\begin{aligned} \mathbb{E}_{\bar\mu}\big[d_j^2 \mid d_\phi = k\big] &\le \mathbb{E}_{\bar\mu}\Big[\sum_{j=2}^k d_j^2 \mid d_\phi = k\Big] \\ &= \sum_{j=2}^k \mathbb{E}_{\bar\mu}[\Gamma_{\phi_j}]\, \mathbb{E}_{\bar\mu}\big[\kappa(A_{\phi_j}) \mid d_\phi = k\big] + \mathbb{E}_{\bar\mu}[\Gamma_{\phi_2}^2]\, \mathbb{E}_{\bar\mu}\Big[\sum_{j=2}^k \kappa^2(A_{\phi_j}) \mid d_\phi = k\Big] \\ &\le k - 1 + (k-1)\, \mathbb{E}_{\bar\mu}[\Gamma_{\phi_2}^2]\, \mathbb{E}_{\bar\mu}\Big[\tilde U_1^{-\frac{2}{2+\delta}} - 2\tilde U_1^{-\frac{1}{2+\delta}} + 1 \mid d_\phi = k\Big] < \infty. \end{aligned}$$

Consequently, if $\delta > 1$, then by the Cauchy-Schwarz inequality we have $\mathbb{E}_{\bar\mu}[d_i d_j \mid d_\phi = k] < \infty$ for every $k \ge 1$ and $1 \le i, j \le k$. Writing $\nu_2(k) = \mathbb{P}_{\bar\mu}\{d_\phi = k\}$, we obtain

$$\begin{aligned} \mathbb{E}_{\bar\mu}\big[\Delta_\phi^2\big] &= \sum_{k\in\mathbb{N}} \nu_2(k)\mathbb{E}_{\bar\mu}\big[\Delta_\phi^2 \mid d_\phi = k\big] = \sum_{k\in\mathbb{N}} \nu_2(k)\mathbb{E}_{\bar\mu}\Big[\Big(\frac{1}{k}\sum_{j=1}^k d_j + 1 - k\Big)^2 \;\Big|\; d_\phi = k\Big] \\ &= \sum_{k\in\mathbb{N}} \nu_2(k)\mathbb{E}_{\bar\mu}\Big[\frac{1}{k^2}\sum_{j=1}^k d_j^2 + \frac{1}{k^2}\sum_{i=1}^k \sum_{\substack{j=1\\ j\ne i}}^k d_i d_j + 1 + k^2 + \frac{2}{k}\sum_{j=1}^k d_j - 2\sum_{j=1}^k d_j - 2k \;\Big|\; d_\phi = k\Big] \\ &= \sum_{k\in\mathbb{N}} \sum_{j=1}^k \frac{1}{k^2}\nu_2(k)\mathbb{E}_{\bar\mu}\big[d_j^2 \mid d_\phi = k\big] + \sum_{k\in\mathbb{N}} \sum_{i=1}^k \sum_{\substack{j=1\\ j\ne i}}^k \frac{1}{k^2}\nu_2(k)\mathbb{E}_{\bar\mu}\big[d_i d_j \mid d_\phi = k\big] + 1 + \mathbb{E}_{\bar\mu}\big[d_\phi^2\big] \\ &\quad + \sum_{k\in\mathbb{N}} \sum_{j=1}^k \frac{2}{k}\nu_2(k)\mathbb{E}_{\bar\mu}\big[d_j \mid d_\phi = k\big] - 2\sum_{k\in\mathbb{N}} \sum_{j=1}^k \nu_2(k)\mathbb{E}_{\bar\mu}\big[d_j \mid d_\phi = k\big] - 2\mathbb{E}_{\bar\mu}\big[d_\phi\big]. \end{aligned}$$

Therefore $\mathbb{E}_{\bar\mu}[\Delta_\phi^2]$ is finite if and only if $\delta > 1$.

Proof of Theorem 3.14. Let $\kappa(\cdot)$ and $Y(\cdot,\cdot)$ be as in the proof of Theorem 3.12. For $j, k \in \mathbb{N}$, we first derive an upper and lower bound for $\mathbb{P}_{\bar\mu}\big\{d_1 = j \mid d_\phi = k\big\}$. Abbreviate

$$\gamma_{k,\delta}^* = \frac{\Gamma(k+3+2\delta)}{\Gamma(k)\Gamma(4+2\delta)\Gamma(1+\delta)}.$$

We show that

$$\begin{aligned} &\frac{\gamma_{k,\delta}^*\Gamma(j+1+\delta)\Gamma(j+k-1)\Gamma(5+3\delta)}{\Gamma(j)\Gamma(j+k+4+3\delta)} \le \mathbb{P}_{\bar\mu}\big\{d_1 = j \mid d_\phi = k\big\} \\ &\le \frac{(1+\delta)\Gamma(j+1+\delta)\Gamma(k+3+2\delta)}{\Gamma(j+3+2\delta)\Gamma(k+2+\delta)} \end{aligned} \tag{4.43}$$

For $j \in \mathbb{N}$ and $a_1 \in [0,1]$, we have

$$\begin{aligned}
\mathbb{P}_{\bar\mu}\big\{d_1 = j \mid A_{\phi_1} = a_1\big\} &= \int_0^\infty \mathrm{d}b\, \mathbb{P}_{\bar\mu}\big\{N^{(\mathsf{young})}(\phi_1) = j-1 \mid A_{\phi_1} = a_1, \Gamma_{\phi_1} = b\big\} \frac{b^{1+\delta}\mathrm{e}^{-b}}{\Gamma(2+\delta)} \\
&= \int_0^\infty \mathrm{d}b\, \mathbb{P}\big\{Y(a_1,b) = j-1\big\} \frac{b^{1+\delta}\mathrm{e}^{-b}}{\Gamma(2+\delta)} \\
&= \frac{\Gamma(j+1+\delta)}{\Gamma(j)\Gamma(2+\delta)} \Big(1 - a_1^{\frac{1}{2+\delta}}\Big)^{j-1} a_1.
\end{aligned}$$

Also, the joint density of (A_{ϕ_1}, A_ϕ) is

$$f_{(A_\phi, A_{\phi_1})}(a, a_1) = \frac{1+\delta}{2+\delta}\, a^{-\frac{1+\delta}{2+\delta}}\, a_1^{-\frac{1}{2+\delta}}, \qquad a \in [0,1],\ a_1 \in [0,a].$$

Therefore, for $j \in \mathbb{N}$ and $a \in [0,1]$,

$$\begin{aligned}
&\int_{[0,a]} \mathrm{d}a_1\, \mathbb{P}_{\bar\mu}\big\{d_1 = j \mid A_{\phi_1} = a_1\big\} f_{(A_\phi, A_{\phi_1})}(a, a_1) \\
&= \frac{\Gamma(j+1+\delta)}{(2+\delta)\Gamma(j)\Gamma(1+\delta)} a^{-\frac{1+\delta}{2+\delta}} \int_{[0,a]} \mathrm{d}a_1 \Big(1 - a_1^{\frac{1}{2+\delta}}\Big)^{j-1} a_1^{\frac{1+\delta}{2+\delta}} \\
&\geq \frac{\Gamma(j+1+\delta)}{(2+\delta)\Gamma(j)\Gamma(1+\delta)} \Big(1 - a^{\frac{1}{2+\delta}}\Big)^{j-1} a^{-\frac{1+\delta}{2+\delta}} \int_{[0,a]} \mathrm{d}a_1\, a_1^{\frac{1+\delta}{2+\delta}} \\
&= \frac{\Gamma(j+1+\delta)}{(3+2\delta)\Gamma(j)\Gamma(1+\delta)} \Big(1 - a^{\frac{1}{2+\delta}}\Big)^{j-1} a
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[0,a]} \mathrm{d}a_1\, \mathbb{P}_{\bar\mu}\big\{d_1 = j \mid A_{\phi_1} = a_1\big\} f_{(A_\phi, A_{\phi_1})}(a, a_1) \\
&\leq \frac{\Gamma(j+1+\delta)}{(2+\delta)\Gamma(j)\Gamma(1+\delta)} a^{-\frac{1+\delta}{2+\delta}} \int_{[0,1]} \mathrm{d}a_1 \Big(1 - a_1^{\frac{1}{2+\delta}}\Big)^{j-1} a_1^{\frac{1+\delta}{2+\delta}} \\
&= \frac{\Gamma(j+1+\delta)}{\Gamma(j)\Gamma(1+\delta)} a^{-\frac{1+\delta}{2+\delta}} \int_{[0,1]} \mathrm{d}u\, \big(1-u\big)^{j-1} u^{2+2\delta} \\
&= \frac{\Gamma(j+1+\delta)\Gamma(3+2\delta)}{\Gamma(1+\delta)\Gamma(j+3+2\delta)} a^{-\frac{1+\delta}{2+\delta}}.
\end{aligned}$$

Using (4.35) and (4.36), we get that, for $j, k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_{\bar\mu}\big\{d_1 = j \mid d_\phi = k\big\} \\
&= \int_{[0,1]} \mathrm{d}a\, \frac{\mathbb{P}_{\bar\mu}\big\{d_\phi = k \mid A_\phi = a\big\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}} \int_{[0,a]} \mathrm{d}a_1\, \mathbb{P}_{\bar\mu}\big\{d_1 = j \mid A_{\phi_1} = a_1\big\} f_{(A_\phi, A_{\phi_1})}(a, a_1) \\
&\geq \frac{\gamma^*_{k,\delta}\Gamma(j+1+\delta)}{(2+\delta)\Gamma(j)} \int_{[0,1]} \mathrm{d}a \Big(1 - a^{\frac{1}{2+\delta}}\Big)^{j+k-2} a^{\frac{3+2\delta}{2+\delta}} \\
&= \frac{\gamma^*_{k,\delta}\Gamma(j+1+\delta)}{\Gamma(j)} \int_{[0,1]} \mathrm{d}u\, \big(1-u\big)^{j+k-2} u^{4+3\delta} \\
&= \frac{\gamma^*_{k,\delta}\Gamma(j+1+\delta)\Gamma(j+k-1)\Gamma(5+3\delta)}{\Gamma(j)\Gamma(j+k+4+3\delta)}
\end{aligned}$$

and

$$\mathbb{P}_{\bar\mu}\big\{d_1 = j \mid d_\phi = k\big\} \le \frac{\gamma^*_{k,\delta}\Gamma(4+2\delta)\Gamma(j+1+\delta)}{(2+\delta)\Gamma(j+3+2\delta)} \int_{[0,1]} \mathrm{d}a \left(1 - a^{\frac{1}{2+\delta}}\right)^{k-1}$$
$$= \frac{\gamma^*_{k,\delta}\Gamma(4+2\delta)\Gamma(j+1+\delta)}{\Gamma(j+3+2\delta)} \int_{[0,1]} \mathrm{d}u \left(1-u\right)^{k-1} u^{1+\delta}$$
$$= \frac{(1+\delta)\Gamma(j+1+\delta)\Gamma(k+3+2\delta)}{\Gamma(j+3+2\delta)\Gamma(k+2+\delta)}.$$

This proves (4.43).

Lower bound. We show that, for x large enough,

$$\mathbb{P}_{\bar\mu}\big\{d_1 \ge k^2(x+1) \mid d_\phi = k\big\} \tag{4.44}$$
$$\ge \begin{cases} \dfrac{\gamma_{k,\delta}}{5+3\delta}\, x^{-(5+3\delta)}, & \delta \in (-1,0) \\ \dfrac{\gamma_{k,\delta}}{5+2\delta}\left(\frac{1}{2}\right)^{1+\delta} (3k^2+5+3\delta)^{-(5+2\delta)}\, x^{-(5+2\delta)}, & \delta \in [0,\infty). \end{cases}$$

where we abbreviate

$$\gamma_{k,\delta} = \frac{\Gamma(k+3+2\delta)\Gamma(5+3\delta)}{\Gamma(k)\Gamma(4+2\delta)\Gamma(1+\delta)}.$$

Using (4.36) and (4.44) we get, for x large enough,

$$\mathbb{P}_{\bar\mu}\{\Delta_\phi \ge x\} = \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\Big\{\sum_{j=1}^{k} d_j \ge kx + k(k-1) \mid d_\phi = k\Big\}$$
$$\ge \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}\mathbb{P}_{\bar\mu}\Big\{d_1 \ge k^2(x+1) \mid d_\phi = k\Big\}$$
$$\ge \begin{cases} C_{1,\delta}\, x^{-(5+3\delta)}, & \text{if } \delta \in (-1,0), \\ C_{2,\delta}\, x^{-(5+2\delta)}, & \text{if } \delta \in [0,\infty), \end{cases}$$

where

$$C_{1,\delta} = \frac{(2+\delta)\Gamma(5+3\delta)}{(3+2\delta)(5+3\delta)(\Gamma(1+\delta))^2} \sum_{k\in\mathbb{N}} \frac{\Gamma(k+\delta)}{\Gamma(k)} (3k^2+5+3\delta)^{-(5+3\delta)} \in (0,\infty),$$
$$C_{2,\delta} = \frac{(2+\delta)\Gamma(5+3\delta)}{(3+2\delta)(5+2\delta)(\Gamma(1+\delta))^2} \left(\frac{1}{2}\right)^{1+\delta} \sum_{k\in\mathbb{N}} (3k^2+5+3\delta)^{-(5+2\delta)} \in (0,\infty).$$

The lower bound in Theorem 3.14 follows once we prove (4.44). For $k \in \mathbb{N}$ and $x > 0$,

$$\mathbb{P}_{\bar\mu}\big\{d_1 \ge k^2(x+1) \mid d_\phi = k\big\} \ge \gamma_{k,\delta} \sum_{j=\lceil k^2(x+1)\rceil}^{\infty} \frac{\Gamma(j+1+\delta)\Gamma(j+k-1)}{\Gamma(j)\Gamma(j+k+4+3\delta)}$$
$$\ge \gamma_{k,\delta} \sum_{j=\lceil k^2(x+1)\rceil}^{\infty} \frac{\Gamma(j+\lfloor 1+\delta\rfloor)\Gamma(j+k-1)}{\Gamma(j)\Gamma(j+k+1+\lceil 3+3\delta\rceil)}$$
$$\ge \gamma_{k,\delta} \sum_{j=\lceil k^2(x+1)\rceil}^{\infty} \frac{\Gamma(j+\lfloor 1+\delta\rfloor)}{\Gamma(j)(j+k+\lceil 3+3\delta\rceil)^{\lceil 3+3\delta\rceil+2}},$$

which, via (4.33), implies that, for $\delta \in (-1, 0)$ and x large enough,

$$\mathbb{P}_{\bar{\mu}}\big\{d_1 \ge k^2(x+1) \mid d_\phi = k\big\} \ge \gamma_{k,\delta} \sum_{j=\lceil k^2(x+1)\rceil + k + \lceil 3+3\delta\rceil}^{\infty} j^{-(6+3\delta)} \frac{\gamma_{k,\delta}}{(5+3\delta)} x^{-(5+3\delta)}. \tag{4.45}$$

Noting that, for $j \ge \lceil k^2(x+1)\rceil$ and x large enough,

$$\frac{j}{j+k+\lceil 3+3\delta\rceil} \ge \frac{1}{1+\frac{k+\lceil 3+3\delta\rceil}{\lceil k^2(x+1)\rceil}} \ge \frac{1}{2},$$

we find that, for $\delta \in [0, \infty)$ and x large enough,

$$\begin{aligned}
&\mathbb{P}_{\bar{\mu}}\big\{d_1 \ge k^2(x+1) \mid d_\phi = k\big\} \\
&\ge \gamma_{k,\delta} \sum_{j=\lceil k^2(x+1)\rceil}^{\infty} \frac{j^{\lfloor 1+\delta\rfloor}}{(j+k+\lceil 3+3\delta\rceil)^{\lceil 3+3\delta\rceil+2}} \\
&\ge \gamma_{k,\delta}\Big(\frac{1}{2}\Big)^{1+\delta} \sum_{j=\lceil k^2(x+1)\rceil+k+\lceil 3+3\delta\rceil}^{\infty} j^{-(6+2\delta)} \\
&\ge \frac{\gamma_{k,\delta}}{(5+3\delta)}\Big(\frac{1}{2}\Big)^{1+\delta}(3k^2+5+3\delta)^{-(5+2\delta)} x^{-(5+2\delta)}.
\end{aligned} \tag{4.46}$$

This completes the proof of (4.44).

Upper bound. The proof of the upper bound requires more intricate bounds. First we provide an upper bound for $\mathbb{P}_{\bar{\mu}}\big\{d_1 \ge x \mid d_\phi = k\big\}$ in three different cases, $\delta \in [-\frac{1}{2}, 0)$, $\delta = 0$ and $\delta \in (0, \infty)$.

Abbreviate

$$\tilde{\gamma}_{k,\delta} = \frac{(1+\delta)\Gamma(k+3+2\delta)}{\Gamma(k+2+\delta)}$$

Using (4.34) and the fact that $\frac{\Gamma(m+s)}{\Gamma(m+1)} \le m^{s-1}$ for $m \in \mathbb{N}$ and $0 \le s \le 1$ [8], we see that, for $k \in \mathbb{N}$, $\delta \in [-\frac{1}{2}, 0)$ and $x > 0$,

$$\begin{aligned}
\mathbb{P}_{\bar{\mu}}\big\{d_1 \ge x \mid d_\phi = k\big\} &\le \tilde{\gamma}_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} \frac{\Gamma(j+1+\delta)}{\Gamma(j+3+2\delta)} \le \tilde{\gamma}_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} j^{\delta} \frac{\Gamma(j+1)}{\Gamma(j+1+\lfloor 2+2\delta\rfloor)} \\
&\le \tilde{\gamma}_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} j^{-1+\delta} \le \frac{\tilde{\gamma}_{k,\delta}}{(-\delta)}(x-1)^{\delta},
\end{aligned} \tag{4.47}$$

and, for $k \in \mathbb{N}$, $\delta = 0$ and $x > 0$,

$$\mathbb{P}_{\bar{\mu}}\big\{d_1 \ge x \mid d_\phi = k\big\} \le (k+2) \sum_{j=\lceil x\rceil}^{\infty} \frac{1}{(j+1)(j+2)} \le (k+2) \sum_{j=\lceil x\rceil+1}^{\infty} j^{-2} \le (k+2)\, x^{-1}, \tag{4.48}$$

and, for $k \in \mathbb{N}$, $\delta \in (0,\infty)$ and $x > 0$,

$$
\begin{aligned}
&\mathbb{P}_{\bar\mu}\big\{d_1 \geq x \mid d_\phi = k\big\} \\
&\leq \tilde\gamma_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} \frac{\Gamma(j+1+\delta)}{\Gamma(j+2+\delta+\lfloor 1+\delta\rfloor)} \leq \tilde\gamma_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} \frac{1}{(j+1+\delta)\times\cdots\times(j+1+\delta+\lfloor 1+\delta\rfloor)} \\
&\leq \tilde\gamma_{k,\delta} \sum_{j=\lceil x\rceil}^{\infty} (j+\lfloor 1+\delta\rfloor)^{-(1+\lfloor 1+\delta\rfloor)} \leq \tilde\gamma_{k,\delta} \sum_{j=\lceil x\rceil+1}^{\infty} j^{-(1+\delta)} \leq \frac{\tilde\gamma_{k,\delta}}{\delta} x^{-\delta}.
\end{aligned}
\tag{4.49}
$$

Next, we provide an upper bound for $\mathbb{P}_{\bar\mu}\Big\{\sum_{i=2}^k d_i \geq (k-1)x \mid d_\phi = k\Big\}$ in three different cases. To that end, fix $k \geq 2$, $2 \leq i \leq k$ and $j \geq 0$. For $a_i \in [0,1]$,

$$
\begin{aligned}
\mathbb{P}_{\bar\mu}\big\{d_i = j \mid A_{\phi_i} = a_i\big\} &= \int_0^\infty \mathrm{d}b_i\, \mathbb{P}_{\bar\mu}\big\{N^{(\mathsf{young})}(\phi_i) = j \mid A_{\phi_i} = a_i, \Gamma_{\phi_i} = b_i\big\} \frac{b_i^\delta \mathrm{e}^{-b_i}}{\Gamma(1+\delta)} \\
&= \int_0^\infty \mathrm{d}b_i\, \mathbb{P}\big\{Y(a_i,b_i) = j\big\} \frac{b_i^\delta \mathrm{e}^{-b_i}}{\Gamma(1+\delta)} \\
&= \frac{\Gamma(j+1+\delta)}{\Gamma(j+1)\Gamma(1+\delta)} \Big(1 - a_i^{\frac{1}{2+\delta}}\Big)^j a_i^{\frac{1+\delta}{2+\delta}}.
\end{aligned}
$$

Also, for $a \in [0,1]$, conditioned on $d_\phi = k, A_\phi = a$, A_{ϕ_i} is distributed as the $(i-1)$-th order statistic of a sample of $k-1$ i.i.d. uniform$[a,1]$ random variables. The conditional density of A_{ϕ_i} given $d_\phi = k$ and $A_\phi = a$ is given by

$$
f_{A_{\phi_i}|d_\phi=k,A_\phi=a}(a_i) = \frac{\Gamma(k)}{\Gamma(i-1)\Gamma(k-i+1)} a_i^{i-2}(1-a-a_i)^{k-i}(1-a)^{-(k-1)}, \qquad a_i \in [a,1].
$$

Since

$$
\frac{\Gamma(k)}{\Gamma(i-1)\Gamma(k-i+1)} = \frac{i(i-1)}{k}\binom{k}{i} \leq k\sum_{m=0}^{k}\binom{k}{m} = k\,2^k,
$$

it follows that, for $a \in [0,1]$,

$$
\begin{aligned}
&\int_{[a,1]} \mathrm{d}a_i\, \mathbb{P}_{\bar\mu}\big\{d_i = j \mid A_{\phi_i} = a_i\big\} f_{A_{\phi_i}|d_\phi=k,A_\phi=a}(a_i) \\
&\leq \frac{k\,2^k\,\Gamma(j+1+\delta)}{\Gamma(j+1)\Gamma(1+\delta)}(1-a)^{-i+1} \int_{[a,1]} \mathrm{d}a_i\, \Big(1 - a_i^{\frac{1}{2+\delta}}\Big)^j a_i^{\frac{1+\delta}{2+\delta}} \\
&\leq \frac{(2+\delta)k\,2^k\,\Gamma(j+1+\delta)}{\Gamma(j+1)\Gamma(1+\delta)}(1-a)^{-i+1} \int_{[0,1]} \mathrm{d}u\, \big(1-u\big)^j u^{2+2\delta} \\
&\leq \frac{(2+\delta)k\,2^k\,\Gamma(j+1+\delta)\Gamma(3+2\delta)}{\Gamma(1+\delta)\Gamma(j+4+2\delta)}(1-a)^{-k+1}.
\end{aligned}
$$

Abbreviate

$$
\hat\gamma_{k,\delta} = \frac{2^k\,\Gamma(k+3+2\delta)}{\Gamma(1+\delta)\Gamma(k)}.
$$

Using (4.35) and (4.36), we obtain

$$\begin{aligned}
&\mathbb{P}_{\bar\mu}\big\{d_i = j \mid d_\phi = k\big\}\\
&= \int_{[0,1]} \mathrm{d}a\, \frac{\mathbb{P}_{\bar\mu}\big\{d_\phi = k \mid A_\phi = a\big\}}{\mathbb{P}_{\bar\mu}\big\{d_\phi = k\big\}} \int_{[a,1]} \mathrm{d}a_i\, \mathbb{P}_{\bar\mu}\big\{d_i = j \mid A_{\phi_i} = a_i\big\} f_{A_{\phi_i}|d_\phi=k, A_\phi=a}(a_i)\\
&\le \frac{k\,\hat\gamma_{k,\delta}\,\Gamma(j+1+\delta)}{\Gamma(j+4+2\delta)} \int_{[0,1]} \mathrm{d}a\, a^{\frac{1+\delta}{2+\delta}} \le \frac{(2+\delta)k\,\hat\gamma_{k,\delta}\,\Gamma(j+1+\delta)}{(3+2\delta)\,\Gamma(j+4+2\delta)}.
\end{aligned}$$

Hence, using (4.34), for $k \ge 2$, $\delta \in (-1,0)$ and x large enough, we obtain

$$\begin{aligned}
&\mathbb{P}_{\bar\mu}\Big\{\sum_{i=2}^{k} d_i \ge (k-1)x \mid d_\phi = k\Big\} \le \sum_{i=2}^{k} \mathbb{P}_{\bar\mu}\big\{d_i \ge x \mid d_\phi = k\big\}\\
&\le \sum_{i=2}^{k} \mathbb{P}_{\bar\mu}\big\{d_i \ge \lceil x^{-\delta} k^2\, 2^{2k}\rceil + 1 \mid d_\phi = k\big\}\\
&\le k(k-1)\,\hat\gamma_{k,\delta} \sum_{j=\lceil x^{-\delta}k^2\,2^{2k}\rceil+1}^{\infty} \frac{\Gamma(j+1+\delta)}{\Gamma(j+3+\delta)}\\
&= k(k-1)\,\hat\gamma_{k,\delta} \sum_{j=\lceil x^{-\delta}k^2\,2^{2k}\rceil+1}^{\infty} \frac{1}{(j+1+\delta)(j+2+\delta)}\\
&\le k(k-1)\,\hat\gamma_{k,\delta} \sum_{j=\lceil x^{-\delta}k^2\,2^{2k}\rceil+1}^{\infty} j^{-2} \le \frac{\Gamma(k+3+2\delta)}{\Gamma(1+\delta)\Gamma(k)} 2^{-k}\, x^{\delta},
\end{aligned} \tag{4.50}$$

and, for $k \ge 2$, $\delta = 0$ and x large enough,

$$\begin{aligned}
&\mathbb{P}_{\bar\mu}\Big\{\sum_{i=2}^{k} d_i \ge (k-1)x \mid d_\phi = k\Big\}\\
&\le \sum_{i=2}^{k} \mathbb{P}_{\bar\mu}\big\{d_i \ge \lceil x^{\frac12}\,k\,2^k\rceil + 1 \mid d_\phi = k\big\}\\
&\le \frac{k(k-1)\,2^k\,\Gamma(k+3)}{\Gamma(k)} \sum_{j=\lceil x^{\frac12}\,k\,2^k\rceil+1}^{\infty} \frac{1}{(j+1)(j+2)(j+3)}\\
&\le \frac{k(k-1)\,2^k\,\Gamma(k+3)}{\Gamma(k)} \sum_{j=\lceil x^{\frac12}\,k\,2^k\rceil+1}^{\infty} j^{-3} \le \frac{\Gamma(k+3)}{\Gamma(k)} x^{-1} 2^{-k},
\end{aligned} \tag{4.51}$$

and, for $k \geq 2$, $\delta \in (0, \infty)$ and x large enough,

$$\begin{aligned}
&\mathbb{P}_{\bar{\mu}}\Big\{ \sum_{i=2}^{k} d_i \geq (k-1)x \mid d_\phi = k \Big\} \leq \sum_{i=2}^{k} \mathbb{P}_{\bar{\mu}}\big\{ d_i \geq \lceil (x^\delta k^2\, 2^{2k})^{\frac{1}{1+\delta}} \rceil + 1 \mid d_\phi = k \big\} \\
&\leq k(k-1)\, \hat{\gamma}_{k,\delta} \sum_{j=\lceil (x^\delta k^2\, 2^{2k})^{\frac{1}{1+\delta}} \rceil + 1}^{\infty} \frac{\Gamma(j+1+\delta)}{\Gamma(j+3+\delta+\lfloor 1+\delta \rfloor)} \\
&= k(k-1)\, \hat{\gamma}_{k,\delta} \sum_{j=\lceil (x^\delta k^2\, 2^{2k})^{\frac{1}{1+\delta}} \rceil + 1}^{\infty} \frac{1}{(j+1+\delta) \times \cdots \times (j+2+\delta+\lfloor 1+\delta \rfloor)} \\
&\leq k(k-1)\, \hat{\gamma}_{k,\delta} \sum_{j=\lceil (x^\delta k^2\, 2^{2k})^{\frac{1}{1+\delta}} \rceil + 1}^{\infty} j^{-(2+\lfloor 1+\delta \rfloor)} \\
&\leq k(k-1)\, \hat{\gamma}_{k,\delta} \sum_{j=\lceil (x^\delta k^2\, 2^{2k})^{\frac{1}{1+\delta}} \rceil + 1}^{\infty} j^{-(2+\delta)} \leq \frac{\Gamma(k+3+2\delta)}{\Gamma(2+\delta)\Gamma(k)} x^{-\delta} 2^{-k}. \qquad (4.52)
\end{aligned}$$

Similarly, abbreviating

$$\begin{aligned}
C_{3,\delta} &= \frac{(2+\delta)\Gamma(3+2\delta)}{\Gamma(1+\delta)} \bigg(\frac{1+\delta}{-\delta} \sum_{k\in\mathbb{N}} \frac{\Gamma(k+\delta)}{\Gamma(k+2+\delta)} + \frac{1}{\Gamma(1+\delta)} \sum_{k\in\mathbb{N}} 2^{-k} \bigg) \in (0,\infty), \\
C_4 &= 4 \bigg(\sum_{k\in\mathbb{N}} \frac{\Gamma(k)}{\Gamma(k+2)} + \sum_{k\in\mathbb{N}} 2^{-k} \bigg) \in (0,\infty), \\
C_{5,\delta} &= \frac{(2+\delta)\Gamma(3+2\delta)}{\Gamma(1+\delta)} \bigg(\frac{1+\delta}{\delta} \sum_{k\in\mathbb{N}} \frac{\Gamma(k+\delta)}{\Gamma(k+2+\delta)} + \frac{1}{\Gamma(2+\delta)} \sum_{k\in\mathbb{N}} \frac{\Gamma(k+\delta)}{\Gamma(k)} 2^{-k} \bigg) \in (0,\infty),
\end{aligned}$$

and using (4.36), (4.47)–(4.52), we get, for x large enough,

$$\begin{aligned}
&\mathbb{P}_{\bar{\mu}}\{\Delta_\phi \geq x\} \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar{\mu}}\big\{ d_\phi = k \big\} \mathbb{P}_{\bar{\mu}}\Big\{ \sum_{j=1}^{k} d_j \geq kx \mid d_\phi = k \Big\} \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar{\mu}}\big\{ d_\phi = k \big\} \mathbb{P}_{\bar{\mu}}\Big\{ d_1 \geq x \mid d_\phi = k \Big\} \\
&+ \sum_{k\in\mathbb{N}} \mathbb{P}_{\bar{\mu}}\big\{ d_\phi = k \big\} \mathbb{P}_{\bar{\mu}}\Big\{ \sum_{j=2}^{k} d_j \geq (k-1)x \mid d_\phi = k \Big\} \leq \begin{cases} C_{3,\delta}\, (x-1)^\delta, & \text{if } \delta \in [-\frac{1}{2}, 0), \\ C_4\, x^{-1}, & \text{if } \delta = 0, \\ C_{5,\delta}\, x^{-\delta}, & \text{if } \delta \in (0,\infty). \end{cases}
\end{aligned}$$

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