Overlap Times in the G/G/1 Queue via Laplace Transforms

Onno Boxma
Department of Mathematics and Computer Science
Eindhoven University of Technology
o.j.boxma@tue.nl

Jamol Pender
School of Operations Research and Information Engineering
Cornell University
jjp274@cornell.edu

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Abstract

In this paper, we analyze the steady state maximum overlap time distribution in the G/G/1 queue. Our methodology exploits Laplace-Stieltjes transforms with a novel decomposition of the maximum overlap time. Explicit expressions are provided for the special cases of the M/G/1 and G/M/1 queues. We also study the steady state distribution of the minimum overlap time of a customer with its two adjacent customers. We show a novel relationship between the minimum, maximum and the steady state waiting time.

1 Introduction

The notion of overlap time has become a new metric for describing the complex dynamics of how customers interact in service systems. Overlap times were first developed in the work of Kang et al. [11], Palomo and Pender

[21] as an important metric in the context of COVID-19 and infectious disease spread. In fact, the work of Kang et al. [11], Palomo and Pender [21] provided a foundation for others to also explore overlap times in more complicated queueing models and has inspired a large number of recent papers on the topic, see for example Delasay et al. [7], Palomo and Pender [22], Hassin [9], Hassin et al. [10], Ko and Xu [13], Palomo and Pender [24], Ko et al. [14], Xu et al. [25], Gao and Pender [8], Boxma [2].

Recent work by Palomo and Pender [23] considers the M/M/1 queue and derives the steady state distribution of the maximum overlap time. The maximum overlap time, the longest amount of time a customer spends with any other customer in the system, serves as an important metric for understanding infectious disease spread as it quantifies the longest duration during which an individual interacts with another customer within a service system. This metric not only aids in identifying potential exposure periods but also plays a pivotal role in deriving lower bounds on infection probabilities within the M/M/1 queue.

In this work, we generalize the work of Palomo and Pender [23] to the G/G/1 queue. Our approach uses two steps. The first step is to leverage a new decomposition of the maximum waiting time into the waiting time plus an independent random variable. This important observation was not noticed in Palomo and Pender [23]. The second step is to use the Laplace-Stieltjes transform (LST) of the waiting time as given by Cohen [5] to compute the LST of the maximum overlap distribution. We are able to invert the distribution in some special cases to explicitly characterize the maximum overlap time distribution.

The *minimum* adjacent overlap time a customer experiences is also a useful metric for understanding infectious disease spread; it measures the least amount of time a customer is going to spend with its two adjacent customers in a service system. In this paper, we derive a novel relationship between the maximum overlap and minimum adjacent overlap to analyze the steady state distribution of the minimum adjacent overlap. This is quite straightforward in the M/M/1 queue, however, it is much more complicated in the G/G/1 queue. Thus, we develop a new methodology for analyzing the steady state maximum overlap distribution and the minimum adjacent overlap distribution. Our approach for the G/G/1 queue is to analyze the $G/\mathcal{PH}_{ME}/1$ and the $\mathcal{PH}_{ME}/G/1$ queues where the $\mathcal{PH}_{ME}$ is the class of finite mixtures of Erlang distributions with the same intensity (cf. Section 2.3). Since the class $\mathcal{PH}_{ME}$ is dense in the class of distributions of non-negative random variables, see for example Theorem 4.2 of [1], this serves as a good approximation to the G/G/1 queue.

1.1 Contributions of Our Work

In this paper, we make the following contributions to the literature:

- We derive the (Laplace-Stieltjes transform of the) steady state distribution of the maximum overlap time of any customer.
- We derive the (Laplace-Stieltjes transform of the) steady state distribution of the minimum adjacent customer overlap time of any customer.
- We also obtain moments for these two performance measures and provide inverse Laplace-Stieltjes transforms in some special cases.

1.2 Organization of the Paper

The remainder of the paper is organized as follows. Section 2 contains our main results for computing the tail distribution of the steady state maximum overlap time. The moments and LST are also calculated for the steady state maximum overlap time. In Section 3, we replicate the analytical approach employed in the previous section to investigate the minimum overlap among adjacent customers. Lastly, in Section 4, we conclude the paper and propose novel directions for future research.

1.3 Notation

First some key random variables:

- S_n represents the service time of the n^{th} customer, and S a generic service time.
- A_n represents the inter-arrival time between the n^{th} and $(n+1)^{st}$ customers, and A a generic inter-arrival time.
- W_n represents the waiting time of the n^{th} customer, and W_∞ the steady state waiting time.
- M_n represents the maximum overlap time of the n^{th} customer, and M_∞ the steady state maximum overlap time.

- M_n^* represents the minimum adjacent overlap time of the n^{th} customer, and M_∞^* the steady state minimum adjacent overlap time.
- $\{\cdot\}$ denotes an indicator function.
- $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$.
- λ is the arrival rate if the arrival process is given by a Poisson process as in the M_λ/G/1 queue.
- μ is the service rate if the service distribution is exponential as in the G/M_μ/1 queue.

Finally, $f_X(\cdot)$, $F_X(\cdot)$ and $\mathcal{L}_X(\theta)$ denote the density, distribution and Laplace-Stieltjes transform (LST) of random variable X. We shall in particular use this notation with A or S taken for X.

2 The Maximum Overlap Time

In this section, we derive our main results. First we derive a novel representation of the steady state maximum overlap time in terms of the steady state waiting time. Then, we use the representation to write the LST of the maximum overlap time as a product of the LST of the steady state waiting time and the LST of $(S - A)^+$. In Subsections 2.1 and 2.2 we work this out for the M/G/1 and G/M/1 queue. In Subsection 2.3 we obtain the LST of the steady state maximum overlap time for the $\mathcal{PH}_{ME}/G/1$ and $G/\mathcal{PH}_{ME}/1$ queue, the class $\mathcal{PH}_{ME}$ lying dense in the class of all distributions of non-negative random variables. We have chosen to treat M/G/1 and G/M/1 in separate subsections because these are important queueing models for which the waiting time LST is well known and quite tractable, while the results for $\mathcal{PH}_{ME}/G/1$ and $G/\mathcal{PH}_{ME}/1$ are mathematically more involved.

To make the paper self-contained, we start with a definition of an overlap time. The overlap time between customer n and customer $n + k$ is defined as $O_{n,n+k}$, and represents the time that both customers spend in the system simultaneously. Following [23], we can represent the overlap time between customer n and $n + k$ as

$$O_{n,n+k} = \left(D_n - \sum_{i=0}^{n+k-1} A_i \right)^+ \tag{1}$$

where D_n represents the departure time of the n^{th} customer. As proven in Palomo and Pender [21], we have the intuitively obvious result that $O_{n,n+k}$ is decreasing in k, and, accordingly, that also $O_{n-k,n}$ is decreasing in k. Hence the maximum overlap time of customer n equals the maximum of $O_{n-1,n}$ and $O_{n,n+1}$. As shown in Palomo and Pender [23],

$$O_{n,n+1} = \max(W_n + S_n - A_n, 0) = W_{n+1},$$

and hence

$$M_n = \max(W_n, W_{n+1}) = \max(W_n, W_n + S_n - A_n). \tag{2}$$

This has led Palomo and Pender [23] to the following result.

Proposition 2.1 ([23]). *Let M_n be the maximum overlap time for customer n in the G/G/1 queue. Then M_n is equal to*

$$M_n = W_n \cdot \{S_n \leq A_n\} + (W_n + S_n - A_n) \cdot \{S_n > A_n\}. \tag{3}$$

Note that we can write M_n as sum of two independent components W_n and $(S_n - A_n)^+$. This will be helpful in describing the LST of the maximum overlap in the sequel. We prove this in the Lemma below.

Lemma 2.2. *The maximum overlap time has the following decomposition into two independent terms, viz., the waiting time and max-plus difference of a service time and inter-arrival time:*

$$M_n = W_n + (S_n - A_n)^+ . \tag{4}$$

Proof. The proof follows by combining (2) with the obvious independence of W_n and (S_n, A_n). □

The independence of W_n and $(S_n - A_n)^+$ now implies:

Lemma 2.3. *The LST, mean and second moment of the maximum overlap time have the following expressions:*

$$\mathbb{E}\left[e^{-\theta M_n}\right] \equiv \mathcal{L}_{M_n}(\theta) = \mathcal{L}_{W_n}(\theta) \cdot \mathcal{L}_{(S_n - A_n)^+}(\theta); \tag{5}$$

furthermore,

$$\mathbb{E}[M_n] = \mathbb{E}[W_n] + \mathbb{E}[(S_n - A_n)^+], \tag{6}$$

and

$$\mathbb{E}\left[M_n^2\right] \equiv \mathbb{E}[W_n^2] + \mathbb{E}\left[((S_n - A_n)^+)^2\right] + 2\mathbb{E}[W_n] \cdot \mathbb{E}[(S_n - A_n)^+]. \tag{7}$$

With $f_A(\cdot)$ and $F_A(\cdot)$ the density and distribution of interarrival time A_n, and $\overline{F}_A(y) = 1-F_A(y)$, and with $f_S(\cdot)$ and $F_S(\cdot)$ the density and distribution of service time S_n, we can derive the following expression.

$$\mathbb{E}\left[e^{-\theta(S_n-A_n)^+}\right] = \int_{y=0}^{\infty} f_S(y) \int_{x=0}^{y} f_A(x)e^{-\theta(y-x)}\mathrm{d}x + \int_{y=0}^{\infty} f_S(y)\overline{F}_A(y)\mathrm{d}y. \tag{8}$$

We now specify these expressions for the M_λ/G/1 and G/M_μ/1 queues.

2.1 The M_λ/G/1 Queue

In this section, we specialize our results to the M_λ/G/1 queue. We start with two lemmas. The first calculates the LST of the max-plus difference of a service time and inter-arrival time. The second calculates the k^{th} moment of the max-plus difference of a service time and inter-arrival time.

Lemma 2.4. *For the M_λ/G/1 queue, we have that*

$$\mathbb{E}\left[e^{-\theta(S_n-A_n)^+}\right] = \frac{\theta}{\theta-\lambda}\mathcal{L}_S(\lambda) - \frac{\lambda}{\theta-\lambda}\mathcal{L}_S(\theta). \tag{9}$$

Proof. Using (8), we have:

$$\begin{aligned}
\mathbb{E}\left[e^{-\theta(S_n-A_n)^+}\right] &= \int_{y=0}^{\infty}\int_{x=0}^{y} \lambda \mathrm{e}^{-\lambda x}\mathrm{e}^{-\theta(y-x)}\mathrm{d}x + \int_{y=0}^{\infty} f_S(y)\mathrm{e}^{-\lambda y}\mathrm{d}y \\
&= \frac{\lambda}{\theta-\lambda}\left(\mathcal{L}_S(\lambda) - \mathcal{L}_S(\theta)\right) + \mathcal{L}_S(\lambda) \\
&= \frac{\theta}{\theta-\lambda}\mathcal{L}_S(\lambda) - \frac{\lambda}{\theta-\lambda}\mathcal{L}_S(\theta).
\end{aligned}$$

□

Lemma 2.5. *The k^{th} moment of the max-plus difference of the service time and inter-arrival time in the M_λ/G/1 queue is equal to*

$$\mathbb{E}\left[((S-A)^+)^k\right] = k!\left(\sum_{j=0}^{k} \frac{(-\lambda)^{j-k}}{j!}\mathbb{E}[S^j] - (-\lambda)^{-k}\mathcal{L}_S(\lambda)\right).$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[((S-A)^+)^k\right] &= \int_0^\infty \int_0^\infty ((x-y)^+)^k \lambda e^{-\lambda y} g(x)\mathrm{d}y\mathrm{d}x \\
&= \int_0^\infty \int_0^x (x-y)^k \lambda e^{-\lambda y} g(x)\mathrm{d}y\mathrm{d}x \\
&= \int_0^\infty \lambda e^{-\lambda x} g(x)\mathrm{d}x \int_{u=0}^x u^k e^{\lambda u}\mathrm{d}u \\
&= k! \int_0^\infty e^{-\lambda x} g(x) \left(\sum_{j=0}^k \frac{(-\lambda x)^j}{j!(-\lambda)^k} e^{\lambda x} - (-\lambda)^{-k} \right) \mathrm{d}x \\
&= k! \left(\sum_{j=0}^k \frac{(-\lambda)^{j-k}}{j!} \mathbb{E}[S^j] - (-\lambda)^{-k} \mathcal{L}_S(\lambda) \right).
\end{aligned}$$

□

Now with Lemmas 2.4 and 2.5, we provide our first main result, which is the LST of the steady state maximum overlap time of the $M_\lambda/G/1$ queue. Here and in the sequel, ρ denotes the load of the server, i.e., mean service time divided by mean inter-arrival time.

Theorem 2.6. *Let M_∞ be the steady state maximum overlap time for the $M_\lambda/G/1$ queue, then the LST of M_∞ is equal to*

$$\mathcal{L}_{M_\infty}(\theta) = \left(\frac{(1-\rho)\theta}{\theta - \lambda + \lambda \mathcal{L}_S(\theta)} \right) \cdot \left(\frac{\theta}{\theta-\lambda} \mathcal{L}_S(\lambda) - \frac{\lambda}{\theta-\lambda} \mathcal{L}_S(\theta) \right), \tag{10}$$

with mean

$$\mathbb{E}[M_\infty] = \frac{\lambda \mathbb{E}[S^2]}{2(1-\rho)} + \mathbb{E}[S] - \frac{1-\mathcal{L}_S(\lambda)}{\lambda} \tag{11}$$

and second moment

$$\begin{aligned}
\mathbb{E}[M_\infty^2] &= \frac{\lambda \mathbb{E}[S^3]}{3(1-\rho)} + \frac{\lambda^2 (\mathbb{E}[S^2])^2}{2(1-\rho)^2} \qquad (12) \\
&+ \frac{\lambda \mathbb{E}[S^2]}{1-\rho} (\mathbb{E}[S] - \frac{1-\mathcal{L}_S(\lambda)}{\lambda}) + \mathbb{E}[S^2] - \frac{2}{\lambda}\mathbb{E}[S] + \frac{2(1-\mathcal{L}_S(\lambda))}{\lambda^2}.
\end{aligned}$$

Proof. To derive (10), we first note that the steady state waiting time and the difference of the service and arrival random variables are independent. Thus, the LST of the sum is the product of the individual expressions. We then combine Lemmas 2.3 and 2.4, and use the Pollaczek-Khinchin formula for the LST of the steady state waiting time of the $M_\lambda/G/1$ queue (cf. p. 256 of [5]). The expressions for the moments follow from (6) and (7). It is important to observe that the first term in the righthand side of (11) equals $\mathbb{E}[W_\infty]$, while the two terms on the right in the first line of (12) equal $\mathbb{E}[W_\infty^2]$. □

Here we give three examples that provide more insight into our results for the $M_\lambda/G/1$ queue in special cases where the service times are Erlang, Deterministic, and Hyper-Exponential.

Example 2.1. If S is Erlang-k distributed with mean k/μ, so $\mathcal{L}_S(\theta) = (\frac{\mu}{\mu+\theta})^k$, then the expression in Theorem 2.6 reduces to

$$\mathcal{L}_{M_\infty}(\theta) = (1-\rho)\left(\frac{\mu}{\mu+\lambda}\right)^k \frac{\theta}{(\mu+\theta)^k(\theta-\lambda)+\lambda\mu^k}\frac{\theta(\mu+\theta)^k-\lambda(\mu+\lambda)^k}{\theta-\lambda}. \tag{13}$$

Inspection of (13) shows that (i) $\theta=\lambda$ is not a pole; (ii) $\theta=0$ is not a pole; and (iii) there are k poles $\theta_1,\dots,\theta_k$, and these are exactly the poles of $\mathcal{L}_{W_\infty}(\theta)$. Hence we know that all θ_i lie in the left-half θ-plane. Inversion yields that

$$\mathbb{P}(M_\infty \le x) = (1-\rho)\left(\frac{\mu}{\mu+\lambda}\right)^k + \sum_{i=1}^{k} C_i[1-\mathrm{e}^{\theta_i x}], \tag{14}$$

where the C_i can be readily determined by partial fraction decomposition.
For small k-values, the θ_i can be explicitly determined; e.g., for $k=2$ we have $\theta_{1,2} = \frac{1}{2}[\lambda - 2\mu \pm \sqrt{\{\lambda^2+4\lambda\mu\}}]$.
The above reasoning holds more generally, in the sense that $\theta=0$ and $\theta=\lambda$ are not poles of the right-hand side of (10), while poles of $\mathcal{L}_S(\theta)$ in the left-half θ-plane are compensated by the $\mathcal{L}_S(\theta)$ term in the first part of that right-hand side. Hence, for any service time distribution, the poles of $\mathcal{L}_{M_\infty}(\theta)$ and of $\mathcal{L}_{W_\infty}(\theta)$ in the left-half θ-plane coincide. Accordingly, the tail $\mathbb{P}(M_\infty > t)$, for large t, exhibits the same behavior as the tail $\mathbb{P}(W_\infty > t)$; both are determined by the pole in the left-half θ-plane with largest real part.

Example 2.2. If S is deterministic, i.e. $S_n \equiv D$, we have

$$\begin{aligned} \mathcal{L}_{M\infty}(\theta) &= \frac{(1-\rho)\theta}{\theta-\lambda+\lambda \mathrm{e}^{-\theta D}}\left(\frac{\theta}{\theta-\lambda}\mathrm{e}^{-\lambda D} - \frac{\lambda}{\theta-\lambda}\mathrm{e}^{-\theta D}\right) \\ &= \frac{1-\rho}{1-\rho\frac{1-e^{-\theta D}}{\theta}} \frac{\theta e^{-\lambda D} - \lambda e^{-\theta D}}{\theta-\lambda}. \end{aligned} \tag{15}$$

Example 2.3. If S is Hyper-exponential($\vec{\mu}, \vec{p}$) i.e. $M_\lambda/H_k/1$, a mixture of $\exp(\mu_i)$ with probabilities $p_i \in (0,1)$:

$$\begin{aligned} \mathcal{L}_{M_\infty}(\theta) &= \frac{(1-\rho)\theta}{\theta-\lambda+\lambda\sum_{i=1}^k \frac{p_i\mu_i}{\mu_i+\theta}}\left(\frac{\theta}{\theta-\lambda}\sum_{i=1}^k \frac{p_i\mu_i}{\mu_i+\lambda} - \frac{\lambda}{\theta-\lambda}\sum_{i=1}^k \frac{p_i\mu_i}{\mu_i+\theta}\right) \\ &= \frac{(1-\rho)\theta}{\prod_{i=1}^k(\mu_i+\theta)[\theta-\lambda] + \lambda\sum_{i=1}^k p_i\mu_i\prod_{j\neq i}^k(\mu_j+\theta)} \\ &\times \frac{\theta\sum_{i=1}^k \frac{p_i\mu_i}{\mu_i+\lambda}\prod_{i=1}^k(\mu_i+\theta) - \lambda\sum_{i=1}^k p_i\mu_i\prod_{j\neq i}^k(\mu_j+\theta)}{\theta-\lambda}. \end{aligned} \tag{16}$$

2.2 The G/M_μ/1 Queue

In this section, we specialize our results to the $G/M_\mu/1$ queue. We start with two lemmas. The first calculates the LST of the max-plus difference of a service time and inter-arrival time. The second calculates the k^{th} moment of the max-plus difference of a service time and inter-arrival time.

Lemma 2.7. *For the $G/M_\mu/1$ queue, we have that*

$$\mathbb{E}\left[\mathrm{e}^{-\theta(S_n-A_n)^+}\right] = 1 - \mathcal{L}_A(\mu) + \mathcal{L}_A(\mu)\frac{\mu}{\mu+\theta}. \tag{17}$$

Proof. Again using (8), we have:

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{-\theta(S_n-A_n)^+}] &= \int_{y=0}^{\infty} \mu \mathrm{e}^{-\mu y}\mathrm{d}y \int_{x=0}^{y} f_A(x)\mathrm{e}^{-\theta(y-x)}\mathrm{d}x + \int_{y=0}^{\infty}\mu\mathrm{e}^{-\mu y}(1-F_A(y))\mathrm{d}y \\ &= \int_{x=0}^{\infty} f_A(x)\mathrm{d}x\int_{y=x}^{\infty}\mu\mathrm{e}^{-\mu y}\mathrm{e}^{-\theta(y-x)}\mathrm{d}y - (1-F_A(y))\mathrm{e}^{-\mu y}|_0^\infty - \mathcal{L}_A(\mu) \\ &= \mathcal{L}_A(\mu)\frac{\mu}{\mu+\theta} + 1 - \mathcal{L}_A(\mu). \end{aligned}$$

□

Remark. Another proof simply uses the memoryless property twice to obtain

$$\mathbb{P}(S_n \le A_n) = 1 - \mathcal{L}_A(\mu)$$

and

$$\mathbb{E}\left[e^{-\theta(S_n - A_n)^+} \cdot \{S_n > A_n\}\right] = \mathcal{L}_A(\mu)\frac{\mu}{\mu+\theta}.$$

Lemma 2.8. *The k^{th} moment of the max-plus difference of the service time and inter-arrival time in the $G/M_\mu/1$ queue is equal to*

$$\mathbb{E}\left[((S-A)^+)^k\right] = \frac{k!}{\mu^k}\mathcal{L}_A(\mu).$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[((S-A)^+)^k\right] &= \int_0^\infty \int_0^\infty ((y-x)^+)^k \mu e^{-\mu y} g(x) \mathrm{d}y \mathrm{d}x \\
&= \int_0^\infty \int_x^\infty (y-x)^k \mu e^{-\mu y} g(x) \mathrm{d}y \mathrm{d}x \\
&= \int_0^\infty \mu e^{-\mu x} g(x) \mathrm{d}x \int_{u=0}^\infty u^k e^{-\mu u} \mathrm{d}u \\
&= \frac{k!}{\mu^k} \int_0^\infty e^{-\mu x} g(x) \mathrm{d}x \\
&= \frac{k!}{\mu^k} \mathcal{L}_A(\mu).
\end{aligned}$$

□

One should observe that this equals the k^{th} moment of an $\exp(\mu)$ random variable times $\mathcal{L}_A(\mu) = \mathbb{P}(S > A)$.

Now with Lemmas 2.7 and 2.8, we provide our second main result, which is the LST of the steady state maximum overlap time of the $G/M_\mu/1$ queue.

Theorem 2.9. *Let M_∞ be the steady state maximum overlap time for the $G/M_\mu/1$ queue, then the LST of M_∞ is equal to*

$$\mathcal{L}_{M_\infty}(\theta) = \left(1 - \rho^* + \rho^* \frac{\mu(1-\rho^*)}{\mu(1-\rho^*)+\theta}\right)\left(1 - \mathcal{L}_A(\mu) + \mathcal{L}_A(\mu)\frac{\mu}{\mu+\theta}\right), \quad (18)$$

where ρ^* is the unique solution to $\rho^* = \mathcal{L}_A(\mu - \mu\rho^*)$. Furthermore,

$$\mathbb{E}[M_\infty] = \frac{\rho^*}{\mu(1-\rho^*)} + \frac{\mathcal{L}_A(\mu)}{\mu}, \quad (19)$$

$$\mathbb{E}[M_\infty^2] = \frac{2\rho^*}{\mu^2(1-\rho^*)^2} + 2\frac{\rho^*}{\mu(1-\rho^*)}\frac{\mathcal{L}_A(\mu)}{\mu} + \frac{2\mathcal{L}_A(\mu)}{\mu^2}. \tag{20}$$

Proof. To derive (18), combine Lemmas 2.3 and 2.7, and use the known result for the LST of the steady state waiting time (cf. p. 230 of [5]; as is well known, the waiting time distribution is exponential, with an atom at zero). The moment expressions follow from (6) and (7); use that $\mathbb{E}[W_\infty] = \frac{\rho^*}{\mu(1-\rho^*)}$ and $\mathbb{E}[W_\infty^2] = \frac{2\rho^*}{\mu^2(1-\rho^*)^2}$. □

Remark. A careful inspection of (18) shows that $\theta = -\mu$ is *not* a pole; we can rewrite (18) as

$$\mathcal{L}_{M_\infty}(\theta) = (1-\rho^*)(1-\mathcal{L}_A(\mu)) + (\mathcal{L}_A(\mu) + (1-\mathcal{L}_A(\mu))\rho^*)\,\frac{\mu(1-\rho^*)}{\mu(1-\rho^*)+\theta}, \tag{21}$$

and inversion yields:

$$\begin{aligned}\mathbb{P}(M_\infty \le x) &= (1-\rho^*)(1-\mathcal{L}_A(\mu)) + (\mathcal{L}_A(\mu) + (1-\mathcal{L}_A(\mu))\rho^*)\,(1-\mathrm{e}^{-\mu(1-\rho^*)x}) \\ &= 1 - (\mathcal{L}_A(\mu) + (1-\mathcal{L}_A(\mu))\rho^*)\,\mathrm{e}^{-\mu(1-\rho^*)x}. \end{aligned} \tag{22}$$

Hence, in the $G/M_\mu/1$ queue, M_∞ is exponentially distributed (with the same exponent as W_∞) with an atom at zero. For $M_\lambda/M_\mu/1$, we have $\rho^* = \rho = \frac{\lambda}{\mu}$ and (22) reduces to Theorem 3.3 of [23]:

$$\mathbb{P}(M_\infty \le x) = 1 - \frac{2\rho}{1+\rho}\mathrm{e}^{-\mu(1-\rho)x}. \tag{23}$$

Remark. In the case where the arrival process is an Erlang(2,λ), yielding an $E_2/M_\mu/1$ queue, we have the fixed point for $\rho^* = \frac{\lambda}{\mu} + \frac{1}{2} - \sqrt{\frac{\lambda}{\mu} + \frac{1}{4}}$. Moreover, when the arrival process is deterministic, yielding a $D/M_\mu/1$ queue, we have the fixed point for $\rho^* = -\frac{W_{-1}(-e^{-\Delta\mu}\Delta\mu)}{\Delta\mu}$ where $W_{-1}(x)$ is defined as the principal branch of the Lambert W function when $-1/e \le x \le 0$. This is the case because

$$\begin{aligned}\rho^* &= \mathcal{L}_A(\mu - \mu\rho^*) \\ &= e^{-\Delta(\mu-\mu\rho^*)} \\ &= e^{-\Delta\mu}e^{\Delta\mu\rho^*}.\end{aligned}$$

Now by a generalization of the Lambert W function given in Leeuwaarden et al. [16], we have the above result for ρ^*.

2.3 The Maximum for the G/G/1 Queue

In this subsection we generalize Theorems 2.6 and 2.9 to the case of the $\mathcal{PH}_{ME}/G/1$ and $G/\mathcal{PH}_{ME}/1$ queues. $\mathcal{PH}_{ME}$ is the class of finite mixtures of Erlang distributions with the same intensity, i.e., their density has the form

$$f(x) \;=\; \sum_{j=1}^{k} p_j \zeta \frac{(\zeta x)^{n(j)}}{n(j)!} \mathrm{e}^{-\zeta x}, \quad x > 0, \tag{24}$$

where $n(j) \in \mathbb{N}$, $p_j > 0$ for all $j = 1, \ldots, k$ and $\sum_{j=1}^{k} p_j = 1$; we take $n(1) < n(2) < \cdots < n(k)$.

According to Theorem 4.2 of Asmussen [1], this class lies dense, in the sense of weak convergence, in the set of all probability distributions on $(0, \infty)$; the implication is that we can approximate any such probability distribution arbitrarily closely by a $\mathcal{PH}_{ME}$ distribution.

Our first observation, cf. Lemma 2.3, is that

$$\mathcal{L}_{M_\infty}(\theta) \;=\; \mathcal{L}_{W_\infty}(\theta) \mathbb{E}[\mathrm{e}^{-\theta(S-A)^+}]. \tag{25}$$

Our second observation is that the Laplace transform of the density given in (24) equals

$$\sum_{j=1}^{k} p_j \left(\frac{\zeta}{\zeta + \theta} \right)^{n(j)+1}. \tag{26}$$

Let us now successively determine the two components in the right-hand side of (25) for $\mathcal{PH}_{ME}/G/1$ and $G/\mathcal{PH}_{ME}/1$.

2.3.1 The $\mathcal{PH}_{ME}/G/1$ Queue

We take

$$\mathcal{L}_A(\theta) \;=\; \sum_{j=1}^{k} p_j \left(\frac{\lambda}{\lambda + \theta} \right)^{n(j)+1} \;=\; \frac{\sum_{j=1}^{k} p_j \lambda^{n(j)+1} (\lambda + \theta)^{n(k)-n(j)}}{(\lambda + \theta)^{n(k)+1}} =: \frac{\alpha_1(\theta)}{\alpha_2(\theta)}. \tag{27}$$

Furthermore, we consider the zeroes of

$$\alpha_2(-\theta) - \mathcal{L}_S(\theta) \alpha_1(-\theta).$$

As proven in p. 329 of Cohen [5] using Rouché's theorem, for $\rho < 1$ there are exactly $n(k) + 1$ such zeroes δ_j, with $\delta_1 = 0$ and Re $\delta_j > 0$ for $j = 2, \ldots, n(k)+1$. We can now use Formula (II.5.204) of [5] to obtain an explicit expression for the LST of W_∞.

Lemma 2.10. *For the $\mathcal{PH}_{ME}/G/1$ queue with inter-arrival time density given by (24) with $\zeta = \lambda$, the LST of the steady state waiting time is given by*

$$\mathcal{L}_{W_\infty}(\theta) \;=\; \frac{(\alpha_1'(0) - \alpha_2'(0))(1-\rho)\theta}{\alpha_2(-\theta) - \mathcal{L}_S(\theta)\alpha_1(-\theta)} \prod_{i=2}^{n(k)+1} \frac{\delta_i - \theta}{\delta_i}. \tag{28}$$

We next turn to the LST of $(S - A)^+$.

Lemma 2.11. *For the $\mathcal{PH}_{ME}/G/1$ queue,*

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\theta(S-A)^+}\right] \;&=\; \sum_{j=1}^{k} p_j \left(\frac{\lambda}{\lambda-\theta}\right)^{n(j)+1} \mathcal{L}_S(\theta) \\ &+\; \sum_{j=1}^{k} p_j \sum_{i=0}^{n(j)} \frac{(-\lambda)^i}{i!} \mathcal{L}_S^{(i)}(\lambda) \left[1 - \left(\frac{\lambda}{\lambda-\theta}\right)^{n(j)-i+1}\right],\end{aligned} \tag{29}$$

where $\mathcal{L}_S^{(i)}(\lambda)$ denotes the i^{th} derivative of $\mathcal{L}_S(\theta)$, evaluated at $\theta = \lambda$.

Proof. With $f_A(\cdot)$ the density of A, we can write:

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\theta(S-A)^+}\right] \;&=\; \int_{x=0}^{\infty} \mathrm{d}S(x) \int_{y=0}^{x} f_A(y)\mathrm{e}^{-\theta(x-y)}\mathrm{d}y + \int_{x=0}^{\infty} \mathrm{d}S(x) \int_{y=x}^{\infty} f_A(y)\mathrm{d}y \\ =\; & \sum_{j=1}^{k} p_j \int_{x=0}^{\infty} \mathrm{d}S(x) \left[\int_{y=0}^{x} \frac{(\lambda y)^{n(j)}}{n(j)!}\lambda \mathrm{e}^{-\lambda y}\mathrm{e}^{-\theta(x-y)}\mathrm{d}y + \int_{y=x}^{\infty} \frac{(\lambda y)^{n(j)}}{n(j)!}\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y\right] \\ =\; & \sum_{j=1}^{k} p_j \int_{x=0}^{\infty} \mathrm{d}S(x)[\mathrm{e}^{-\theta x} \left(\frac{\lambda}{\lambda-\theta}\right)^{n(j)+1} \int_{y=0}^{x} \frac{((\lambda-\theta)y)^{n(j)}}{n(j)!}(\lambda-\theta)\mathrm{e}^{-(\lambda-\theta)y}\mathrm{d}y \\ +\; & \sum_{i=0}^{n(j)} \frac{(\lambda x)^i}{i!}\mathrm{e}^{-\lambda x}] \\ =\; & \sum_{j=1}^{k} p_j \int_{x=0}^{\infty} \mathrm{d}S(x) \left[\left(\frac{\lambda}{\lambda-\theta}\right)^{n(j)+1} \left[\mathrm{e}^{-\theta x} - \sum_{i=0}^{n(j)} \frac{((\lambda-\theta)x)^i}{i!}\mathrm{e}^{-\lambda x}\right] + \sum_{i=0}^{n(j)} \frac{(\lambda x)^i}{i!}\mathrm{e}^{-\lambda x}\right].\end{aligned} \tag{30}$$

The result follows by observing that $\int_{x=0}^{\infty} x^i \mathrm{e}^{-\lambda x} \mathrm{d}S(x) = (-1)^i \mathcal{L}_S^{(i)}(\lambda)$. □

Combining (25) with Lemmata 2.10 and 2.11, we obtain the following result.

Theorem 2.12. *The steady state maximum overlapping time LST in the $\mathcal{PH}_{ME}/G/1$ queue is given by the product of the expressions in the right-hand sides of (28) and (29).*

2.3.2 The $G/\mathcal{PH}_{ME}/1$ Queue

We now take

$$\mathcal{L}_S(\theta) \;=\; \sum_{j=1}^{k} p_j \left(\frac{\mu}{\mu+\theta}\right)^{n(j)+1} \;=\; \frac{\sum_{j=1}^{k} p_j \mu^{n(j)+1} (\mu+\theta)^{n(k)-n(j)}}{(\mu+\theta)^{n(k)+1}} \;=:\; \frac{\sigma_1(\theta)}{\sigma_2(\theta)}. \tag{31}$$

Furthermore, consider the zeroes of

$$\sigma_2(\theta) - \mathcal{L}_A(-\theta)\sigma_1(\theta).$$

Cohen [5], p. 323, proves that it has exactly $n(k)+1$ zeroes ξ_i, $i = 1, \ldots, n(k)+1$, with Re $\xi_i < 0$. From (II.5.190) of [5], we now immediately conclude the following.

Lemma 2.13. *For the $G/\mathcal{PH}_{ME}/1$ queue with service time density given by (24) with $\zeta = \mu$, the LST of the steady state waiting time is given by*

$$\mathcal{L}_{W_\infty}(\theta) \;=\; \left(\frac{\mu+\theta}{\mu}\right)^{n(k)+1} \prod_{i=1}^{n(k)+1} \frac{\xi_i}{\xi_i - \theta}. \tag{32}$$

Notice that the waiting time distribution apparently is a mixture of $n(k)+1$ exponential terms, plus an atom at zero. For $G/M/1$, we have $n(k) = n(1) = 0$, resulting in the well-known exponential waiting time distribution with an atom at zero.

We next turn to the LST of $(S-A)^+$.

Lemma 2.14. *For the $G/\mathcal{PH}_{ME}/1$ queue,*

$$\mathbb{E}[\mathrm{e}^{-\theta(S-A)^+}] \;=\; 1 - \sum_{j=1}^{k} p_j \sum_{i=0}^{n(j)} \frac{(-\mu)^i}{i!} \mathcal{L}_A^{(i)}(\mu) \left[1 - \left(\frac{\mu}{\mu+\theta}\right)^{n(j)-i+1}\right]. \tag{33}$$

Proof. Start from the identity $\mathrm{e}^{-\theta(x)^+} + \mathrm{e}^{-\theta(x)^-} = \mathrm{e}^{-\theta x} + 1$, replacing x by $S - A$ and taking expectations on both sides. Subsequently, observe that $(S-A)^- = -(A-S)^+$. Finally, determine $\mathbb{E}[\mathrm{e}^{-\theta(S-A)^-}] = \mathbb{E}[\mathrm{e}^{\theta(A-S)^+}]$ from Lemma 2.11 by there replacing θ by $-\theta$, λ by μ and $\mathcal{L}_S(\cdot)$ by $\mathcal{L}_A(\cdot)$. □

Combining (25) with Lemmata 2.13 and 2.14 we obtain the following theorem.

Theorem 2.15. *The steady state maximum overlapping time LST in the $G/\mathcal{PH}_{ME}/1$ queue is given by the product of the expressions in the right-hand sides of (32) and (33).*

3 The Minimum Overlap Time

In this section we study the minimum of the overlap times of a tagged customer, say customer n, with the previous customer and the next customer. This clearly amounts to the minimum of the waiting times W_n and W_{n+1}. Hence we have

Lemma 3.1.

$$M_n^* = \min(W_n, (W_n + S_n - A_n)^+). \tag{34}$$

Below we give an expression for the LST of the steady state minimum overlap time M_∞^* and for its distribution.

Theorem 3.2. *The LST of the steady state minimum overlap time M_∞^* is given by*

$$\mathbb{E}[\mathrm{e}^{-\theta M_\infty^*}] = \mathbb{E}[\mathrm{e}^{-\theta W_\infty}]\left[1 + \mathbb{P}(S > A) - \mathbb{P}(S > A)\mathbb{E}[\mathrm{e}^{-\theta(S-A)}|S > A]\right]. \tag{35}$$

The distribution of M_∞^* can be expressed in the distribution of W_∞ in the following way:

$$\mathbb{P}(M_\infty^* \le t) = \mathbb{P}(S > A)\mathbb{P}(W_\infty \le t) + \mathbb{P}(S \le A) - \mathbb{P}(S \le A, W_\infty + S - A > t). \tag{36}$$

Proof. It follows from (34) that, in steady state,

$$\mathbb{E}[\mathrm{e}^{-\theta M_\infty^*}] = \mathbb{P}(S > A)\mathbb{E}[\mathrm{e}^{-\theta W_\infty}] + \mathbb{E}[\mathrm{e}^{-\theta(W_\infty + S - A)^+} \cdot \{S \le A\}]. \tag{37}$$

Now observe that

$$\mathbb{E}[\mathrm{e}^{-\theta(W_\infty+S-A)^+}\cdot\{S\le A\}]=\mathbb{E}[\mathrm{e}^{-\theta(W_\infty+S-A)^+}]-\mathbb{E}[\mathrm{e}^{-\theta(W_\infty+S-A)^+}\cdot\{S>A\}]. \tag{38}$$

Combining (34), (37) and (38), and using that

$$\mathbb{E}[\mathrm{e}^{-\theta(W_\infty+S-A)^+}]=\mathbb{E}[\mathrm{e}^{-\theta W_\infty}]$$

and

$$\mathbb{E}[\mathrm{e}^{-\theta(W+S-A)^+}|S>A]=\mathbb{E}[\mathrm{e}^{-\theta(W+S-A)}|S>A],$$

we obtain:

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\theta M^*_\infty}] &= \mathbb{P}(S>A)\mathbb{E}[\mathrm{e}^{-\theta W_\infty}]+\mathbb{E}[\mathrm{e}^{-\theta W_\infty}] \\ &- \mathbb{P}(S>A)\mathbb{E}[\mathrm{e}^{-\theta W_\infty}]\mathbb{E}[\mathrm{e}^{-\theta(S-A)}|S>A],\end{aligned} \tag{39}$$

and (35) follows.
Equation (36) follows from the reasoning below, where (34) is used to get the first equality:

$$\begin{aligned}&\mathbb{P}(M^*_\infty\le t)=\mathbb{P}(S>A)\mathbb{P}(W_\infty\le t)+\mathbb{P}(S\le A,(W_\infty+S-A)^+\le t)=\\ &\mathbb{P}(S>A)\mathbb{P}(W_\infty\le t)+\mathbb{P}(S\le A)-\mathbb{P}(S\le A,(W_\infty+S-A)^+>t)=\\ &\mathbb{P}(S>A)\mathbb{P}(W_\infty\le t)+\mathbb{P}(S\le A)-\mathbb{P}(S\le A,W_\infty+S-A>t).\end{aligned} \tag{40}$$

□

Remark. Notice that if we add the distribution of the minimum and the maximum, we obtain twice the distribution of the waiting time. In fact this is a remarkable result on its own, which we show directly below.

Lemma 3.3.

$$\mathbb{P}\left(M_\infty\le t\right)+\mathbb{P}\left(M^*_\infty\le t\right)=2\mathbb{P}\left(W_\infty\le t\right). \tag{41}$$

Proof.

$$\begin{aligned}\mathbb{P}\left(M_n\le t\right)+\mathbb{P}\left(M^*_n\le t\right) &= \mathbb{P}\left(\max(W_n,W_{n+1})\le t\right)+\mathbb{P}\left(\min(W_n,W_{n+1})\le t\right)\\ &= \mathbb{P}\left(W_n\le t\right)+\mathbb{P}\left(W_{n+1}\le t\right).\end{aligned} \tag{42}$$

Taking the limit $n\to\infty$ gives the result. □

We now specify these results for the $M_\lambda/G/1$ and $G/M_\mu/1$ queues.

Theorem 3.4. *The LST of the steady state minimum overlap time M_∞^* in the $M_\lambda/G/1$ queue is given by*

$$\mathcal{L}_{M_\infty^*}(\theta) \;=\; \frac{(1-\rho)\theta}{\theta-\lambda+\lambda\mathcal{L}_S(\theta)}\left[2-\frac{\lambda}{\lambda-\theta}\mathcal{L}_S(\theta)+\frac{\theta}{\lambda-\theta}\mathcal{L}_S(\lambda)\right]. \tag{43}$$

Proof. The term outside the brackets in (35) is given by the Pollaczek-Khinchin formula that appears outside the large brackets in (43). That the term inside the brackets in (35) is given by the expression inside the large brackets in (43) follows by observing that $\mathbb{P}(S > A) = 1 - \mathcal{L}_S(\lambda)$ and that

$$\mathbb{E}[\mathrm{e}^{-\theta(S-A)^+}\cdot\{S>A\}] \;=\; \int_{x=0}^{\infty}\mathrm{d}F_S(x)\int_{y=0}^{x}\lambda\mathrm{e}^{-\lambda y}\mathrm{e}^{-\theta(x-y)}\mathrm{d}y \tag{44}$$

$$\;=\; \frac{\lambda}{\lambda-\theta}[\mathcal{L}_S(\theta)-\mathcal{L}_S(\lambda)]. \tag{45}$$

Alternatively, we could have used (41), which implies that $\mathcal{L}_{M_\infty}(\theta)+\mathcal{L}_{M_\infty^*}(\theta) = 2\mathcal{L}_{W_\infty}(\theta)$. □

Remark. The moments of M_∞^* immediately follow from those of M_∞, in combination with Lemma 3.3.

Theorem 3.5. *The LST of the steady state minimum overlap time M_∞^* in the $G/M_\mu/1$ queue is given by*

$$\mathcal{L}_{M_\infty^*}(\theta) = \frac{\mu(1-\rho^*)+(1-\rho^*)(1+\mathcal{L}_A(\mu))\theta}{\mu(1-\rho^*)+\theta}. \tag{46}$$

Proof. The result follows from (35) by using the first part of the right-hand side of (18) for the LST of W_∞ in the $G/M_\mu/1$ queue, in combination with (use the memoryless property for S twice) $\mathbb{P}(S > A) = \mathcal{L}_A(\mu)$ and $\mathbb{E}[\mathrm{e}^{-\theta(S-A)^+}|S>A] = \frac{\mu}{\mu+\theta}$. Alternatively, we could again have used (41) and the result for $\mathcal{L}_{M_\infty}(\theta)$. □

Remark. Either by inversion of the LST expression in (46), or by using (36), we obtain the following expression for the distribution of M_∞^*:

$$\mathbb{P}(M_\infty^*\le t) \;=\; 1-[\rho^*-(1-\rho^*)\mathcal{L}_A(\mu)]\mathrm{e}^{-\mu(1-\rho^*)t}, \quad t\ge 0. \tag{47}$$

We only demonstrate the latter approach. From (36) and the fact that, in $G/M_\mu/1$, we have $\mathbb{P}(W \le t) = 1 - \rho^* \mathrm{e}^{-\mu(1-\rho^*)t}$, it follows that

$$\mathbb{P}(M^*_\infty \le t) = \mathcal{L}_A(\mu)[1 - \rho^* \mathrm{e}^{-\mu(1-\rho^*)t}] + (1 - \mathcal{L}_A(\mu)) \\ - \int_{x=0}^{\infty} \mathrm{d}F_A(x) \int_{y=0}^{x} \mu \mathrm{e}^{-\mu y} \rho^* \mathrm{e}^{-\mu(1-\rho^*)(t+x-y)} \mathrm{d}y. \tag{48}$$

The double integral can be evaluated in a fairly straightforward manner. It is not surprising that M^*_∞ in $G/M_\mu/1$ is $\exp(\mu(1-\rho^*))$ distributed with an atom at zero, because that also holds for W_∞ while, moreover, the sojourn time $W_\infty + S$ is purely $\exp(\mu(1-\rho^*))$ (cf. Chapter II.3 of Cohen [5]). The latter implies, in particular, that $\mathbb{P}(W_\infty + S - A > t | W_\infty + S > A) = \mathrm{e}^{-\mu(1-\rho^*)t}$. Incidentally, to prove that $\rho^* - (1-\rho^*)\mathcal{L}_A(\mu)$, featuring in (47), is positive, observe that we can replace the first ρ^* by $\mathcal{L}_A(\mu(1-\rho^*))$, while that term is larger than $\mathcal{L}_A(\mu)$.

Remark. For $M_\lambda/M_\mu/1$ it readily follows, from each of the two above theorems, that

$$\mathbb{P}(M^* \le t) \; = 1 - \frac{2\rho^2}{1+\rho} \mathrm{e}^{-\mu(1-\rho)t},$$

in agreement with Theorem 4.2 of [23].

We close this section by briefly outlining how the $M/G/1$ result can be generalized to $PH_{ME}/G/1$, and the $G/M/1$ result to $G/PH_{ME}/1$. Using Lemma 3.3 we can immediately conclude that

$$\mathcal{L}_{M^*_\infty}(\theta) = 2\mathcal{L}_{W_\infty}(\theta) - \mathcal{L}_{M_\infty}(\theta). \tag{49}$$

For $PH_{ME}/G/1$ we can now use Lemma 2.10 and Theorem 2.12 to obtain $\mathcal{L}_{M^*_\infty}(\theta)$. Alternatively, we can use (35) and observe that $\mathbb{E}[\mathrm{e}^{-\theta[S-A]^+} \cdot \{S > A\}]$ and $\mathbb{P}(S > A)$ are given by the first double integral in the right-hand side of (30) (taking $\theta = 0$ for the latter probability).
For $G/PH_{ME}/1$ we can use Lemma 2.13 and Theorem 2.15 to obtain $\mathcal{L}_{M^*_\infty}(\theta)$.

Alternatively, we can use (35) and derive that

$$
\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\theta(S-A)} \cdot \{S > A\}] \\
&= \int_{x=0}^{\infty} \mathrm{d}F_A(x) \int_{y=x}^{\infty} \mathrm{e}^{-\theta(y-x)} \sum_{j=1}^{k} p_j \mu \frac{(\mu y)^{n(j)}}{n(j)!} \mathrm{e}^{-\mu y} \mathrm{d}y \\
&= \int_{x=0}^{\infty} \mathrm{e}^{\theta x} \mathrm{d}F_A(x) \sum_{j=1}^{k} p_j \left(\frac{\mu}{\mu+\theta}\right)^{n(j)+1} \sum_{i=0}^{n(j)} \frac{((\mu+\theta)x)^i}{i!} \mathrm{e}^{-(\mu+\theta)x} \\
&= \sum_{j=1}^{k} p_j \left(\frac{\mu}{\mu+\theta}\right)^{n(j)+1} \sum_{i=0}^{n(j)} \frac{(\mu+\theta)^i}{i!} (-1)^i \mathcal{L}_A^{(i)}(\mu), \qquad (50)
\end{aligned}
$$

and (take $\theta = 0$ in the above formula)

$$
\mathbb{P}(S > A) = \sum_{j=1}^{k} p_j \sum_{i=0}^{n(j)} \frac{\mu^i}{i!} (-1)^i \mathcal{L}_A^{(i)}(\mu). \qquad (51)
$$

As an other alternative, we could have taken Equation (36) for the distribution of M_∞^*, and have used (see above (32)) that the distribution of W_∞ is a mixture of $n(k)+1$ exponentials plus an atom at zero. That allows us to explicitly evaluate the term $\mathcal{P}(W_\infty + S - A > t, S \leq A)$ in (36) by evaluating the double integral over the densities of S and A. Thus, from the LST expression, it is seen that M_∞^*, too, is distributed as a sum of $n(k)+1$ exponentials plus an atom at zero.

4 Conclusion

In this paper, we analyze the maximum overlap time in the G/G/1 queue. We first analyze the maximum overlap in the M_λ/G/1 and G/M_μ/1 queues and find the exact steady state distribution in some special cases. We also extend these results to the $G/\mathcal{PH}_{ME}/1$ and $\mathcal{PH}_{ME}/G/1$ queues using a mixture of Erlangs, which are dense in the space of distributions of non-negative random variables. We also leverage a new relationship between the maximum overlap time and the minimum adjacent overlap time to calculate the LST for the steady state distribution of the minimum adjacent overlap time. Certainly it would be great to obtain explicit LST formulas for the G/G/1, however, this seems to be out of reach until the LST of the waiting time of the G/G/1 queue can be computed explicitly.

A first extension of this work could be to analyze the $M_\lambda/M_\mu/1/k$ queue where there is finite waiting capacity in the queue and customers could be blocked from receiving service. The $M_\lambda/M_\mu/1/k$ queue is also interesting since the maximum overlap time of customer n is not necessarily equal to the maximum overlap time of the adjacent customers since the $(n-1)^{th}$ or $(n+1)^{th}$ customer might get blocked from service. Thus, the maximum overlap time of customer n could, e.g., involve customer $n-2$ or $n+2$ if they were not blocked. It is also of interest to compute the maximum overlap time of multi-server queues like the Erlang-A queue, see for example Massey and Pender [20], Ko et al. [14]. Finally, it would be interesting to analyze overlap times in queues with self exciting arrivals or time varying arrivals like in Massey [18], Mandelbaum and Massey [17], Massey [19], Ko and Pender [12], Daw and Pender [6], Koops et al. [15], Chen [3], Chen and Hong [4].

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